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ON THE EXISTENCE OF SOLUTIONS OF A SYSTEM OF DIFFERENTIAL INEQUALITIES

Introduction

Let C and D be matrices of dimensions kxm. Let v(t) be continuous function for $t \in [t_0, t_1] =: T$, with values in R^k.

We will be dealing with the problem of existence of a solution of the following system of differential inequalities:

$$(1) \qquad C\dot{x}(t) + Dx(t) \leqslant v(t),$$

$$(2) \qquad x(0) = \overline{x}, \quad \text{for} \quad t \in T.$$

In this paper we show that under some natural conditions, system (1)-(2) admits a solution.

The proof will be based on some recently obtained results for differential inclusions.

The purpose of this paper is to obtain the following theorem.

T h e o r e m . If the rows of matrix C are non-negative and nonzero (see Application), then there exists a function x(t) absolutely continuous on the time interval T such, that (1) is satisfied almost everywhere on this interval and (2) holds.

The proof of this theorem consists of several lemmas, which we shall be dealing with in the sequel.

Main Result

Let E^m be the m-dimensional vector space composed of vectors $x = (x_1,\ldots,x_m)$ with the norm $\|x\| = \sqrt{\sum_{i=1}^{m} x_i^2}$ and let $P(E^m)$ be the metric space of all nonempty closed subsets of E^m with the Hausdorff metric:

$$h(W,Z) = \min\left\{d : W \subset S_d(Z), \quad Z \subset S_d(W)\right\},$$

where $W,Z \in P(E^m)$ and $S_d(M)$ denotes a d-neighbourhood of the set M in the space E^m.

For a given multimapping $H : E^1 \times E^m \longrightarrow P(E^m)$ we consider the relation:

$$(3) \qquad \dot{x} \in H(t,x), \quad x(0) = \bar{x}.$$

An absolutely continuous function $x(t)$ is said to be a solution of (3) on the time interval T if the condition $\dot{x}(t) \in H(t,x(t))$ holds almost everywhere on this interval.

We consider a function H given in the form:

$$(4) \qquad H(t,x) = \left\{z \in E^m : Cz \leqslant -Dx + v(t)\right\},$$

where the function $v(t)$, $t \in T$ is given.

Let us remark that a function $x(t)$ which is defined on the time interval T is a solution of the differential inequalities (1) - (2) if and only if it satisfies (3) with the function $H(t,x)$ defined by (4).

Now we reformulate the inequalities (1) in terms of so called differential inclusions.

In connection with the above, the problem of existence of any solution of differential inequalities (1) - (2) can be reduced to the problem of existence of a solution of the differential inclusion (3), where $H(t,x)$ is a function defined by (4).

L e m m a 1. If the rows $r_i(C)$ of the matrix C are non-negative and nonzero, then the set $H(t,x)$ is nonempty and closed for all $x \in E^m$ and $t \in T$.

P r o o f . From the assumption of the lemma, the set $H(t,x)$ is a common part of k closed half-spaces defined by the inequalities:

(i) $(r_i(C), z) \leq (r_i(-D), x) + v_i(t) =: \gamma_i(t,x)$,

where $i = 1,2,\dots,k$ and (a,b) denotes the scalar product of vectors a and b.

Because each vector $r_i(C)$ is non-negative and nonzero, the intersection of these half-spaces is non-empty for each γ_i, $i = 1,\dots,k$.

Let λ_i be the smallest positive number such that the vector $-\lambda_i$ $(1,1,\dots,1)$ belongs to the half-space defined by inequality (i). Then $-\mu(1,1\dots,1) \in H(t,x)$, where

$$\mu = \max\left\{\lambda_1,\dots,\lambda_k\right\} .$$

It follows that for each $t \in T$ and for each $x \in E^m$ $H(t,x)$ is a nonempty set, independent of the value of the function $v(t)$. This completes the proof of Lemma 1.

Let us observe that if one of the rows of the matrix C, for example $r_i(C)$, is zero for some i, then the condition $(r_i(C), z) \leq \gamma_i$ determines a nonempty set (the whole space E^m) only if $\gamma_i \geq 0$. So, in the case where some rows of matrix C are zero, $H(t,x)$ is nonempty with additional limitations of x and $v(t)$. We observe that $H(t,x)$ is an unbounded set. Thus to show the existence of a solution of the problem (3) we may apply the theorem on the existence of solutions for differential inclusions, when the right side is unbounded.

The following theorem is included in the work of J.Himmelberg and F.S. Van Vleck [1].

T h e o r e m (Himmelberg, Van Vleck). Let a multimapping $H : T \times E^m \to P(E^m)$ be such that:

a) $H(t,x)$ is a closed set for each $(t,x) \in T \times E^m$;

b) $H(\cdot,x)$ is measurable for each $x \in E^m$;

c) there exists a function φ integrable on the time interval T, such that $h\ (H(t,x), H(t,y)) \leq \varphi(t)\|x-y\|$ for each x, y, t;

d) there exists an absolutely continuous function $w:T \times E^m$ such that: $\sup\{d(\dot{w}(t), H(t,w(t)))| t \in T\} < \infty$.

Then the problem (3) has a solution on the time interval T (in the sense of absolutely continuous functions).

We shall prove the assumptions of this theorem are satisfied by the function defined by (4).

We know that H(t,x) is a closed and nonempty subset E^m. So the condition a) of the Himmelberg and Van Vleck's theorem is satisfied.

Now we shall prove that H(.,x) is a measurable function for each $x \in E^m$ in the sense of the following definition:

D e f i n i t i o n . A multivalued function $F:E^m \to \Omega(E^m)$ is called measurable if for any closed set $P \subset E^m$ the set $\{x : F(x) \cap P \neq \emptyset\}$ is Lebesque measurable.

Continuity and lipschitzianity of a multivalued function F(x) are defined in the usual way. For example a function F(x) satisfies a Lipschitz condition with constant I if for any points $x, x' \in E^m$ the inequality

$$h(F(x), F(x')) \leq l \cdot \|x-x'\|$$

holds. The number $|F| = h(\{0\}, F)$ is called the modulus of the set F.

L e m m a 2. If Z is a closed set of E^m, $H(t) = \{z \in E^m : (c,z) \leq \gamma(t)\}$, $\gamma(t)$ is continuous on the time interval T,c is a nonzero vector with non-negative coordinates, then $t(Z) = \{t \in T : H(t) \cap Z \neq \emptyset\}$ is a measurable set.

P r o o f . Let $\underline{\gamma} = \inf\{\gamma(t) : t \in T\}$, $\bar{\gamma} = \sup\{\gamma(t) : t \in T\}$. There exists $\underline{t} \in T$ and $\bar{t} \in T$ such that $\gamma(\underline{t}) = \underline{\gamma}$ and $\gamma(\bar{t}) = \bar{\gamma}$.

If $H(\bar{t}) \cap Z = \emptyset$, then $t(Z) = \emptyset$.

If $H(\underline{t}) \cap Z \neq \emptyset$, then $t(Z) = T$.

If $H(\underline{t}) \cap Z = \emptyset$ and $H(\bar{t}) \cap Z \neq \emptyset$, then there exists a number γ_z such that $t(Z) = \{t \in T : \gamma(t) \geq \gamma_z\}$.

From Lemma 2 it follows that assumption b) is satisfied because the intersection of measurable sets is also a measurable set.

L e m m a 3. There exists a constant δ such that $h(H(t,x),\ H(t,y)) \leq \delta \|x-y\|$ for all x and y.

The distance ε_i between the following parallel hyperplanes $(r_i(C),z) = \gamma_i(t,x)$ and $r_i(C),\ z = \gamma_i(t,y)$, bounding respectively the sets $H(t,x)$ and $H(t,y)$ is equal to:

$$\varepsilon_i = \frac{|\gamma_i(t,x) - \gamma_i(t,y)|}{\|r_i(C)\|} = \frac{|(r_i(-D),\ x-y)|}{\|r_i(C)\|} \leq$$

$$\leq \frac{\|r_i(-D)\|}{\|r_i(C)\|}\ \|x-y\| = \mu_i\|x-y\|, \quad i=1,\dots,k.$$

$H(t,y)$ is a polyhedron which is obtained from the polyhedron $H(t,x)$ by a parallel displacement of its faces.

From geometrical considerations it follows that $h(H(t,x),\ H(t,y)) \leq \sqrt{m}\ \max\{\varepsilon_1,\dots,\varepsilon_k\}$. Thus we have

$$h(H(t,x),\ H(t,y)) \leq \sqrt{m}\ \max\{\mu_1,\dots,\mu_m\}\ \|x-y\|$$

which shows that the multifunction $H(t,x)$ satisfies the assumption c) of Himmelberg and Van Vleck's theorem.

Now we shall prove that there exists an absolutely continuous function $w : T \to E^m$ such that:

$$\sup\{d(\dot{w}(t),\ H(t,w(t)))\ |\ t \in T\} < \infty .$$

Let $v(t)$ be continuous function on the time interval T. Then the functions $\gamma_i(t,x)$ reach a minimum on the interval T for each fixed $x \in E^m$. Let

$$\tilde{\gamma}_i(x) = \min_T \gamma_i(t,x).$$

Then the inequalities (1) for $\gamma_i = \tilde{\gamma}_i(x)$ define a nonempty set $G(x)$ such that $G(x) \subset H(t,x)$ for each $t \in T$ and $x \in E^m$.

Thus $d(0,\ H(t,x)) \leq d(0,\ G(x))$ for $t \in T$.

Let $w(t) = \text{const} = w_0 \in E^m$, then we have:

$d(\dot{w}(t),\ H(t,w(t))) = d(0,\ H(t,w_0)) \leq d(0,\ G(w_0)) < \infty,$

because $G(w_0)$ is a nonempty set.

Thus the assumptions of Himmelberg and Van Vleck's theorem are satisfied and there exists a solution of the problem $\dot{x} \in H(t,x)$, $x(0) = x_0$ for $t \in T$, where $H(t,x)$ is defined by (4).

In this way we have completed the proof of our theorem.

Application

A dynamical open economic model of Leontief's type (see [2]) is given by a system of linear differential equations of the first order:

$$\begin{cases} x(t) = Ax(t) + B\dot{x}(t) + u(t) \\ y(t) = Fx(t) + G\dot{x}(t). \end{cases}$$

We want to find such vector-functions $u(t)$ and $y(t)$ of final output and of consumption of primary factors that satisfy inequalities

$$u(t) \geqslant \bar{u}(t) \quad \text{and} \quad y(t) \leqslant \bar{y}(t) \quad \text{for} \quad t \in T$$

when $\bar{u}(t)$ and $\bar{y}(t)$ are given continuous vector-functions, and $x(0) = x_0$.

The problem of existence of a plan of total output $x(t)$ given constraints $\bar{u}(t)$ and $\bar{y}(t)$ reduces to a problem of existence of a solution of the system of differential inequalities

$$(5) \qquad \begin{cases} \begin{bmatrix} B \\ G \end{bmatrix} \dot{x}(t) + \begin{bmatrix} A - I \\ F \end{bmatrix} x(t) \leqslant \begin{bmatrix} -\bar{u}(t) \\ \bar{y}(t) \end{bmatrix}, \\ x(0) = x_0, \quad t \in T. \end{cases}$$

We have proved that if the rows of matrices B and G (which are non-negative by definition) are non-zero then (5) has a solution $\dot{x}(t)$. But there is not to ensured that $x(t)$ be non-negative for all $t \in T$.

R e m a r k . If $\dot{x}(0) = 0$, $x(0) = \bar{x} > 0$ and $v(t) = v(0) = \bar{v} > 0$, where $\bar{v} = D\bar{x}$, then $x(t) = \bar{x}$, $t \in T$, is a positive solution of the problem $\dot{x} \in H(t,x)$, $x(0) = x_0$ for $t \in T$, where $H(t,x)$ is defined by (4).

The question whether there exists a non-negative solution of this problem when v(t) is not constant has not been solved so far.

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