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ORBITS OF K^+-ACTION ON NORMAL VARIETIES

Let K be an algebraically closed field of characteristic zero. In this paper K^+ denotes the additive group of K and P^n the projective space of dimension n. Let K^+ acts on a normal variety X. We will prove the following theorem.

Theorem. Let X be a normal variety with K^+-action on it. If $\overline{K^+(a)}$ is an orbit of a point a, then $K^+(a)$ is a smooth curve.

Proof. By [1], there exists a K^+-invariant quasi-projective neighbourhood of $\overline{K^+(a)}$, also, by [1], we can embed it equivariantly in some P^{n-1} with a linear action on it. Hence, it is sufficient to prove the theorem in the case of $X = P^{n-1}$.

Let K^+ act algebraically on P^{n-1} by an algebraic morphism φ i.e. $\varphi : K^+ \times P^{n-1} \to P^{n-1}$. Then there exists a nilpotent matrix A such that $\varphi(t,x) = e^{At}x$ for any $t \in K^+$ and $x \in P^{n-1}$, [2]. We may assume that the matrix A is in its Jordan form and dimension of blocks of A decreases and the highest dimension of blocks is m and number of the biggest blocks is k, [2]. Let $a = [a_1, \dots, a_n] \in P^{n-1}$. We will show that $K^+(a)$ is a smooth curve. We consider two cases:

1. There exists i, $1 \leq i \leq k$, such that $a_{m(k-i+1)} \neq 0$ i.e. there exists a non zero coordinate which lies on the last place in one of the biggest blocks.

2. For any i, $1 \leq i \leq k$, there is $a_{m(k-i+1)} = 0$.

In the first case we may assume that $a_m \neq 0$ (in the opposite case we change the base on P^{n-1}). The second case can be reduced to the first one, since the set $\{a \in P^{n-1} : a_{m(k-i+1)} = 0$ for any i, $1 \leqslant i \leqslant k\}$ is K^+-invariant and isomorphic to the projective space P^{n-k-1}. If the point a in the space P^{n-k-1} is not of the form as in the point 1, we continue the same method. At the end we find the projective space P^j in which the point a is of the form as in the case 1 and P^j is embedded in P^{n-1} as a K^+-invariant subspace.

So to complete the proof it remains to prove that if $a_m \neq 0$, then $\overline{K^+(a)}$ is smooth. We shall assume $a_m = 1$.

Let $x \in \overline{K^+(a)}$, then there exists such $t \in K^+$ that $x_1 = f(t)$, $x_2 = f'(t), \dots, x_m = f^{(m-1)}(t)$, where $f(t) = t^{m-1} + b_{m-2}t^{m-2} + \dots + b_1 t + b_0$ and $b_i = \frac{a_{i+1}(m-1)!}{i!}$ for $i = 0,1,\dots,m-1$.

By the smoothness of the orbit $K^+(a)$, it is sufficient to prove that the point $\lim_{t \to \infty} ta$ (which lies in the closure of the orbit $K^+(a)$) is nonsingular in $\overline{K^+(a)}$.

Since $a_m \neq 0$, the point $\lim_{t \to \infty} ta$ has the first coordinate not equal to zero.

We will consider the equations of $\overline{K^+(a)}$ in some neighbourhood of the point $\lim_{t \to \infty} ta$. Let $U = \{x \in P^{n-1} : x_1 \neq 0\}$. Then $\lim_{t \to \infty} ta \in U \cap \overline{K^+(a)}$. The set $U \cap \overline{K^+(a)}$ contains almost all points of $\overline{K^+(a)}$; it does not contain only these points for which $f(t) = 0$. So $U \cap \overline{K^+(a)}$ is a neighbourhood of the point $\lim_{t \to \infty} ta$ in $\overline{K^+(a)}$.

Let $x = [x_1,\dots,x_n] \in U \cap \overline{K^+(a)}$, then $x_1 = 1$, $x_2 = \frac{f'}{f}, \dots, x_m = \frac{f^{(m-1)}}{f}$. Putting $u = \frac{1}{t}$, we get $\lim_{t \to \infty} ta = \lim_{u \to 0} \frac{a}{u}$.

We will show that there exist constants $c_1,\dots,c_m$ such that: $f(t) = c_1 t\, f'(t) + c_2 t\, f''(t) + \dots + c_{m-1} t\, f^{(m-1)}(t) + c_m f^{(m-1)}(t)$. Coefficients at the same power of t of the left and right-hand side of this equation are equal, hence:

$$1 = c_1(m-1),$$

$$b_{m-2} = b_{m-2}(m-2)c_1 + c_2(m-2)(m-1),$$
$$\vdots$$
$$b_1 = b_1c_1 + 2\, b_2c_2 + 2\cdot 3\, b_3c_3 + \dots + (m-1)!\, c_{m-1},$$
$$b_o = c_m(m-1)!.$$

We get m equations in m variables. The determinant of this system of equations is not equal to zero, so there exists a unique solution of it. Hence, we get $1 = c_1t\, x_2 + \dots + c_{m-1}t\, x_m + c_mx_m$ implying

$$t = \frac{1 - c_mx_m}{c_1x_2 + \dots + c_{m-1}x_m} \text{ and } u = \frac{c_1x_2 + \dots + c_{m-1}x_m}{1 - c_mx_m}.$$

Let $U_1 = \{[x_1,\dots,x_m] \in U : c_mx_m \neq 1\}$, U_1 is an open neighbourhood of $\lim\limits_{t\to\infty} ta$. This gives the injection $u : U_1 \cap \overline{K^+(a)} \to K$, where K denotes affine space of dimension 1. It is easy to check that the inverse is also regular. Hence, it is an isomorphism of $U_1 \cap \overline{K^+(a)}$ with an open subset in K, so $\overline{K^+(a)}$ is a smooth curve.

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