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EXISTENCE AND STABILITY OF GLOBAL CLASSICAL SOLUTIONS FOR A FIRST FOURIER PROBLEM

1. Introduction

Let Ω be a bounded domain in the n-dimensional Euclidean space R^n, $n \geq 2$, with C^∞-boundary Γ. Let A be an uniformly elliptic operator of the form

$$A = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right) - a(x), \qquad a_{ij}(x) = a_{ji}(x)$$

for $x \in \Omega$, $a, a_{ij} \in C^\infty(\Omega)$, $i,j = 1,\dots,n$, $a(x) \geq 0$, and let $D = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$.

This paper deals with the existence and the stability of global bounded classical solutions of the equation

$$(1) \qquad u_t - Au = f(t,x,u,Du), \qquad (x,t) \in \Omega \times (0,\infty),$$

with the initial condition

$$(2) \qquad u(x,0) = \Phi(x), \qquad x \in \Omega,$$

and the boundary condition

$$(3) \qquad u(x,t) = 0, \qquad t \in R^+ = \langle 0,\infty), \quad x \in \Gamma.$$

J.Havlová [1] has investigated the existence of the periodic solution for this type problem. She has used a fixed-point-theorem to the equivalent system of nonlinear integral equations. The existence and the stability of solution of the initial-boundary value problem for the nonlinear hyperbolic equations have been proved in [6] and [2].

Our method of the proof is similar to [6]. At first the existence of the time global solution for linear problem with $f = f(t,x)$ will be proven and the Fourier method will be used. In the case $f = f(t,x,u,Du)$ the Picard iteration method will be applied.

Let $L^q(\Omega)$, $0 \leqslant q \leqslant \infty$, and $H^p(\Omega)$, $H_0^p(\Omega)$ be Lebesgue and Sobolev spaces with the norms $\|\cdot\|_{L^q}$ and $\|\cdot\|_{H^p}$, respectively.

Define the subspace

$$(4) \qquad K^p(\Omega) = \left\{u \in H^p(\Omega),\ A^{\alpha}u \in H_0^1(\Omega);\quad 0 \leqslant \alpha \leqslant \left[\tfrac{p-1}{2}\right],\ p \geqslant 1\right\}$$

of the space $H^p(\Omega)$ which is connected with the boundary condition (3).

It follows from the trace theory, the continuity of the operator A and the properties of the space $H^p(\Omega)$ that $K^p(\Omega)$ is a Banach space with a norm $\|\cdot\|_{H^p}$.

Let X be a Banach space with a norm $\|\cdot\|_X$. We denote by $C^p(R^+,X)$ the space of p-times continuously differentiable mappings $u: R^+ \to X$ and introduce a norm $\sup\limits_{t\in R^+} \sum\limits_{k=0}^{p} \|D_t^k u(t)\|_X < \infty$ (D_t^k is a k-order differential operator in t). Denote by $F^p(\Omega)$ a space of functions from $C^0(R^+,K^p(\Omega))$ with the norm $\|\cdot\|_{F^p} = \sup\limits_{t\in R^+} \|u(t,.)\|_{H^p}$ and let $G^p(\Omega) = \bigcap\limits_{i=0}^{1} C^i(R^+,K^{p-2i}(\Omega))$ be a space with the norm $\|u\|_{G^p} = \sup\limits_{t\in R^+} \sum\limits_{k=0}^{1} \left\|\frac{d^k u(t,.)}{dt^k}\right\|_{H^{p-2k}}$. The space $F^p(\Omega)$ is a Banach space with the norm $\|\cdot\|_{F^p}$. We remark that $G^p(\Omega) \subset \bigcap\limits_{i=0}^{1} C^i(R^+ \times \bar{\Omega})$ for $p = \left[\frac{n}{2}\right] + 3$ (see [5]).

O.A.Ładyżenska [3], [4] has considered the eigenvalues problem $A\Phi = \lambda\Phi$ in Ω, $\Phi = 0$ on $\Gamma = \partial\Omega$ and proved that if the following assumptions hold:

A1. $\Gamma \in C^{\infty}$, $a, a_{ij} \in C^{\infty}(\Omega)$, $i,j = 1,\dots,n$,

A2. $a(x) \geq 0$ and A is an uniformly elliptic operator in Ω, then

(i) there exists a negative decreasing eigenvalues sequence $\{-\lambda_k^2\}$ which has no accumulating point and $\lambda_1 \neq 0$,

(ii) the system $\{\phi_k\}$ of the corresponding eigenfunctions is complete and orthonormal in $L^2(\Omega)$, each ϕ_k belongs to $K^{\infty}(\Omega)$,

(iii) any $u \in K^p(\Omega)$, $p \geq 1$, is expanded into the Fourier series $u = \sum_{k=1}^{\infty} \phi_k u_k$, converging to u in $K^p(\Omega)$.

We introduce now in the space $K^p(\Omega)$ the new norm

$$|u|_{K^p} := \Big(\sum_{k=1}^{\infty} \lambda_k^{2p} u_k^2 \Big)^{\frac{1}{2}}$$

equivalent to the norm $\|\cdot\|_{H^p}$ for $u \in K^p(\Omega)$ i.e. there exist positive constants δ_1, δ_2 depending on Ω and such that

$$\delta_1 \|u\|_{H^p} \leq |u|_{K^p} \leq \delta_2 \|u\|_{H^p} \quad \text{for any} \quad u \in K^p(\Omega), \quad p \geq 1. \tag{5}$$

The solution u of the problem (1)-(3) is searched in the space $G^p(\Omega)$.

2. <u>The linear case with $f = f(t,x)$</u>

Let us assume that $p \geq 2$ and

A3. $f = f(t,x)$, $f \in C^0(R^+, K^{p-1}(\Omega))$, $\sup_{t \in R^+} |f(t,\cdot)|_{K^{p-1}} = H < \infty$,

A4. $\Phi \in K^p(\Omega)$,

and consider the equation

$$u_t - Au = f(t,x). \tag{1'}$$

T h e o r e m 1. If the assumptions A1 - A4 hold, then the problem (1'), (2), (3) has a unique global classical solution $u \in G^p(\Omega)$ satisfying

(6) $$\sup_{t\in R^+} \|u(t,.)\|_{H^p} \leq C_0\left(\|\Phi\|_{H^p} + \frac{1}{|\lambda_1|}\sup_{t\in R^+}\|f(t,.)\|_{H^{p-1}}\right).$$

P r o o f . By A3 and (iii), the function f can be expanded into the Fourier series

(7) $$f(t,.) = \sum_{k=1}^{\infty} f_k(t)\Phi_k$$

which converge in $K^{p-1}(\Omega)$ with respect to $t\in R^+$ and has a norm given by

(7′) $$|f(t,.)|^2_{K^{p-1}} = \sum_{k=1}^{\infty} \lambda_k^{2(p-1)} f_k^2(t).$$

Similarly, by A4 and (iii), for the function Φ we have the series

(8) $$\Phi = \sum_{k=1}^{\infty} p_k\Phi_k$$

converging in $K^p(\Omega)$ and with norm given by

(8′) $$|\Phi_k|^2_{K^p} = \sum_{k=1}^{\infty} \lambda_k^{2p} p_k^2.$$

We set up a solution u in a formal Fourier series $u(t,.) = \sum_{k=1}^{\infty} u_k(t)\Phi_k$ and at first we show that this formal series converges in $F^p(\Omega)$. Next, we prove that $u\in G^p(\Omega)$. Substituting (7), (8) into (1′), (2) and taking an inner product with Φ_k in $L^2(\Omega)$, due to the property (ii) of eigenfunctions, we obtain an infinite ordinary differential system

(9) $$\ddot{u}_k + \lambda_k^2 u_k = f_k(t), \quad k=1,2,\ldots,$$

with the initial conditions

(10) $$u_k(0) = p_k, \quad k=1,2,\ldots .$$

The solution u_k of the problem (9), (10) has a form

$$(11)\qquad u_k(t) = p_k e^{-\lambda_k^2 t} + \int_0^t e^{-\lambda_k^2(t-\tau)} f_k(\tau)d\tau,$$

$$t \in R^+,\ k = 1,2,\ldots .$$

Then the formal solution of the problem (1′), (2), (3) is

$$(12)\qquad u(x,t) = \sum_{k=1}^{\infty}\left[p_k e^{-\lambda_k^2 t} + \int_0^t e^{-\lambda_k^2(t-\tau)} f_k(\tau)d\tau\right]\Phi_k(x).$$

In order to show that $u \in F^p(\Omega)$ we must examine the convergence of the following series

$$\sum_{k=1}^{\infty}\left[p_k e^{-\lambda_k^2 t} + \int_0^t e^{-\lambda_k^2(t-\tau)} f_k(\tau)d\tau\right].$$

By applying the Schwarz inequality,it is easy to obtain the estimate

$$(13)\qquad u_k^2(t) \leqslant 2\left[p_k^2 e^{-2\lambda_k^2 t} + \frac{1}{\lambda_k^2}\int_0^t e^{-\lambda_k^2(t-\tau)} f_k^2(\tau)d\tau\right].$$

Thus, by (i), we have

$$(14)\qquad \sum_{k=1}^{m} u_k^2(t)\lambda_k^{2p} \leqslant$$

$$\leqslant 2\left[\sum_{k=1}^{m} p_k^2\lambda_k^{2p}\, e^{-2\lambda_1^2 t} + \sum_{k=1}^{m}\int_0^t e^{-\lambda_1^2(t-\tau)}\,\lambda_k^{2p-2} f_k^2(\tau)d\tau\right] =$$

$$= 2\left[e^{-2\lambda_1^2 t}\sum_{k=1}^{m} p_k^2\lambda_k^{2p} + \int_0^t\left(\sum_{k=1}^{m}\lambda_k^{2p-2} f_k^2(\tau)\right)e^{-\lambda_1^2(t-\tau)}d\tau\right].$$

Since $\Phi \in K^p(\Omega)$ and $f \in C^0(R^+,K^{p-1}(\Omega))$, the right-hand side of (14) tends to zero uniformly in $t \in R^+$, as $m,l \to \infty$. This means that the series $\sum_{k=1}^{\infty} \lambda_k^{2p} u_k^2(t)$ converges uniformly in $t \in R^+$. Due to (7), (7′), (8), (8′), it follows from (14) that

$$(15) \quad |u(t,.)|^2_{K^p} \leq 2\left[e^{-2\lambda_1^2 t} |\phi|^2_{K^p} + \int_0^t |f(t,.)|^2_{K^{p-1}} e^{-\lambda_1^2(t-\tau)} d\tau\right].$$

Using the equivalence of the norms $\|\cdot\|_{H^p}$, $|\cdot|_{K^p}$ and the inequality (5), we have the desire estimate

$$(15') \quad \|u(t,.)\|^2_{H^p} \leq$$

$$\leq C_0^2 \left[e^{-2\lambda_1^2 t} \|\phi\|^2_{H^p} + \int_0^t \|f(t,.)\|^2_{H^{p-1}} e^{-\lambda_1^2(t-\tau)} d\tau\right],$$

with $C_0 = \frac{\sqrt{2}\delta_2}{\delta_1}$. From (15′) we obtain the inequality (6).

In order to show that $u \in C^1(R^+, K^{p-2}(\Omega))$ it is enough to prove that $\sum_{k=1}^{m} \lambda_k^{2(p-2)} (\dot{u}_k(t))^2$ converges uniformly to zero, as $1, m \to \infty$. From the equation (9), after some calculations, we have

$$\sum_{k=1}^{m} \lambda_k^{2(p-2)} \dot{u}_k^2(t) \leq 2\left[\sum_{k=1}^{m} \lambda_k^{2(p-2)} f_k^2(t) + \sum_{k=1}^{m} \lambda_k^{2p} u_k^2\right].$$

By the assumptions A3, A4, the right-hand side of the previous inequality tends to zero uniformly for $t \in R^+$ and this implies that $\dot{u} \in C(R^+, K^{p-2}(\Omega))$. Thus

$$|\dot{u}(t,.)|^2_{K^{p-2}} \leq 2\left[|f(t,.)|^2_{K^{p-2}} + |u(t,.)|^2_{K^p}\right].$$

Having $|f(t,.)|_{K^{p-2}} \leq \tilde{C}_1 |f(t,.)|_{K^{p-1}}$ for some positive constant $\tilde{C}_1$, we get

$$|\dot{u}(t,.)|^2_{K^{p-2}} \leq \tilde{C}_2 \left[|f(t,.)|^2_{K^{p-1}} + |u(t,.)|^2_{K^p}\right],$$

with $\tilde{C}_2 = \max(2, 2\tilde{C}_1)$.

Uniqueness follows from (6). Thus Theorem 1 has been proved. For $p = \left[\frac{n}{2}\right] + 3$, due to Sobolev's imbedding theory, u is a classical solution i.e. $u \in C^1(R^+ \times \bar{\Omega})$ and $u_{xx} \in C(R^+ \times \bar{\Omega})$

3. The nonlinear case

Due to Moser's theorem (see [6] Appendix), the following lemma holds.

L e m m a 1. Let function $f(t,x,a)$, $a = (a_1, a_2^1, \dots, a_2^n) \in R^{n+1}$, be defined on $R^+ \times \Omega \times R^{n+1}$. Let f be of the class C^{s+1} in (x,a) and $(D^k D^l_a f)(t,x,a)$, $0 \leqslant k+l \leqslant s+1$, be continuous in $t \in R^+$. Let $B(Q)$ be any bounded domain in R^{n+1} of the form

$$B(Q) = \left\{a:\ a \in R^{n+1};\ |a_i| \leqslant Q,\ i = 1,2,\ a_2 = (a_2^1, \dots, a_2^n)\right\}.$$

Set

$$h_Q(t) = \max_{0 \leqslant k+l \leqslant s+1} \ \sup_{x \in \bar{\Omega}, a \in B(Q)} (D^k D_a^l f)(t,x,a),$$

and denote

$$F_Q = \left\{u:\ u \in F^s(\Omega);\ |u| \leqslant Q,\ |u_{x_\mu}| \leqslant Q,\ \mu = 1, \dots, n\right\}.$$

Then the following assertions hold:

(I) For any function $u \in F_Q$ we have

$$\|f(t,.,u(t,.),Du(t,.))\|_{H^s} \leqslant C_1 h_Q(t) \ (\|u(t,.)\|_{H^s} + 1),$$

where C_1 is a constant depending on f, Q and n.

(II) For any functions $u_j \in F_Q$ satisfying $\|u_j(t,.)\|_{H^s} \leqslant M$, $j = 1,2$, with some constant M, it holds

$$\|f(t,.,u_1(t,.),Du_1(t,.)) - f(t,.,u_2(t,.),Du_2(t,.))\|_{H^{s-1}} \leqslant$$
$$\leqslant C_2 h_Q(t) \|u_1(t,.) - u_2(t,.)\|_{H^s},$$

where C_2 is a constant depending on f, Q, M and n. Both C_1, C_2 monotonically increase as Q does.

Now, assume that:

A1. g satisfies the assumptions of Lemma 1,

$\overline{A2}$. for any $u \in F^p(\Omega)$ the function $g(t,.,u(t,.),Du(t,.))$ belongs to $C^0(R^+, K^{p-1}(\Omega))$,

$\overline{A3}$. $\sup_{t \in R^+} h_Q(t) = \tilde{H}$ for every $Q > 0$,

$\overline{A4}$. $\Phi \in K^p(\Omega)$.

Consider the equation

$$u_t - Au = \varepsilon g(t,x,u(t,x),Du(t,x)) \tag{1''}$$

with the conditions (2) and (3).

T h e o r e m 2. If the assumptions $\overline{A1}$ - $\overline{A4}$ are satisfied, then for any $M > C_0 L$ (C_0 being the constant from Theorem 1) and $L = \|\Phi\|_{H^p}$ there exists a positive constant

$$(*) \qquad \varepsilon_0 := \min\left(\frac{(M - C_0 L)|\lambda_1|}{C_0 \tilde{H} C_1 (M+1)}, \frac{|\lambda_1|\theta}{C_0 C_2 \tilde{H}}\right), \quad 0 < \theta < 1.$$

such that for any $\varepsilon \in (0, \varepsilon_0)$, the problem (1″), (2), (3) has a unique bounded clássical solution in $G^p(\Omega)$ satisfying $\|u(t,.)\|_{H^p} \leq M$ for $t \in R^+$ and $p = \left[\frac{n}{2}\right] + 3$.

P r o o f . Following [6], we will apply the Picard iteration method. The sequence $\{u_n(t,.)\}$ is constructed by an iteration scheme in the following way

$$\dot{u}_n - Au_n = \varepsilon g(t,.,u_{n-1}(t,.),Du_{n-1}(t,.)), \quad n=1,2,..., \tag{16}$$

$$\dot{u}_0 - Au_0 = \varepsilon g(t,.,0,0) \tag{17}$$

with the initial conditions

$$u_n(0) = 0, \quad n = 0,1,2,..., \tag{18}$$

and the boundary conditions

$$u_n(t,x)|_{\Gamma} = 0 \quad \text{for} \quad t \in R^+, \; n = 0,1,2,... . \tag{19}$$

By the assumption $\overline{A1}$ - $\overline{A4}$, it follows from the Theorem 1 that the problem (16)-(19) has a solution $u_n \in G^p(\Omega)$, $n = 0,1,...$.

We must show now that $\{u_n\}$ converges to u in $G^p(\Omega)$ and $\|u\|_{F^p} \leqslant M$. Boundedness of u_n will be proved by induction. For $n = 0$ it follows from the equation (17) and Theorem 1 that $\|u_0\|_{F^p} \leqslant M$. Assume that $\|u_n\|_{F^p} \leqslant M$ for $n = 1,2,\dots,k$. Applying Theorem 1 to (16) for $n = k+1$ and using the estimate (6) we have

$$\text{(20)} \qquad \|u_{k+1}(t,\cdot)\|_{H^p} \leqslant$$

$$\leqslant C_0\left[\|\phi\|_{H^p} + \varepsilon \frac{1}{|\lambda_1|} \sup_{t\in R^+} \|g(t,\cdot,u_k(t,\cdot),Du_k(t,\cdot))\|_{H^{p-1}}\right].$$

Since $\|u_k\|_{F^p} \leqslant M$, by the Sobolev inequality, it follows that $|u_k(t,x)| \leqslant C(p)M$, $\left|\frac{\partial u_k(t,x)}{\partial x_\mu}\right| \leqslant C(p-1)M$, where $C(p), C(p-1)$ are the Sobolev constants. Let $Q = C(p)M$. From Lemma 1, the assumption $\overline{A3}$ and induction assumption $\|u_k\|_{F^p} \leqslant M$ we obtain

$$\text{(21)} \qquad \|g(t,\cdot,u_k(t,\cdot),Du_k(t,\cdot))\|_{H^{p-1}} \leqslant$$

$$\leqslant h_Q(t)C_1(\|u_k(t,\cdot)\|_{H^p} + 1) \leqslant \tilde{H}C_1(M+1).$$

The inequalities (20), (21) and the assumption $\overline{A4}$ imply the desired estimate

$$\|u_{k+1}\|_{F^p} \leqslant C_0\left[L + \frac{1}{|\lambda_1|} \varepsilon\tilde{H}C_1(M+1)\right] \leqslant M$$

for $\varepsilon \in (0,\varepsilon_0)$ with ε_0 given by (*). Thus $\|u_n\|_{F^p} \leqslant M$ for any $n = 0,1,\dots$.

In order to prove the convergence of $\{u_n\}$ to u in Banach space $F^p(\Omega)$, we must show that $\{u_n\}$ is a Cauchy sequence. Setting $v_n = u_{n+1} - u_n$, from (16), (17), (19) we have

$$\dot{v}_n - Av_n = \varepsilon\left[g(t,\cdot,u_n(t,\cdot),Du_n(t,\cdot)) - \right.$$

$$\left. - g(t,\cdot,u_{n-1}(t,\cdot),Du_{n-1}(t,\cdot))\right],$$

$$v_n(0) = 0, \quad v_n|_\Gamma = 0 \quad \text{for} \quad t \in R^+, \; n = 1,2,\dots .$$

Applying Theorem 1 and Lemma 1 to the above problems and using the assumption $\overline{A3}$, we obtain the estimate

$$\|v_n\|_{H^p} \leqslant \frac{\varepsilon}{|\lambda_1|} C_0 C_2 h_Q(t) \|u_n(t,\cdot) - u_{n-1}(t,\cdot)\|_{H^p} \leqslant \frac{\varepsilon}{|\lambda_1|} C_0 C_2 \tilde{H} \|v_{n-1}\|_{H^p}.$$

The above estimate guarantees, by (*), that $\{u_n\}$ is a Cauchy sequence in $F^p(\Omega)$ with $\theta \in (0,1)$, i.e.

$$\|u_{n+1} - u_n\|_{F^s} \leqslant \theta \|u_n - u_{n-1}\|_{F^s} \quad \text{for} \quad n = 1,2,\dots .$$

Now we will prove that $\{u_n\}$ converges to u in $G^p(\Omega)$. Using the triangle inequality of the norm and the properties of the operator A and of the Sobolev space $H^p(\Omega)$, $p \geqslant 2$, we obtain, by (16),

$$\|\dot{u}_m(t,\cdot) - \dot{u}_n(t,\cdot)\|_{H^{p-2}} \leqslant \text{const}\Big\{\|u_m(t,\cdot) - u_n(t,\cdot)\|_{H^p} +$$

$$+ \varepsilon \|f(t,\cdot,u_{m-1}(t,\cdot),Du_{m-1}(t,\cdot)) - f(t,\cdot,u_{n-1}(t,\cdot),Du_{n-1}(t,\cdot))\|_{H^{p-2}}\Big\} \leqslant$$

$$\leqslant \text{const}\Big\{\|u_m(t,\cdot)-u_n(t,\cdot)\|_{H^p} + \varepsilon C_2 h_Q(t) \|u_{m-1}(t,\cdot)-u_{n-1}(t,\cdot)\|_{H^{p-1}} \leqslant$$

$$\leqslant \text{const}\Big\{\|u_m(t,\cdot)-u_n(t,\cdot)\|_{H^p} + \varepsilon C_2 \tilde{H} \|u_{m-1}(t,\cdot)-u_{n-1}(t,\cdot)\|_{H^{p-1}}\Big\}.$$

The right-hand side tends to zero uniformly in $t \in R^+$, as $m,n \to \infty$. This means that $\{u_n\}$ converges in $C^0(R^+,K^{p-2}(\Omega))$. Hence $u_n \in G^p(\Omega)$ converges in $\bigcap_{i=0}^{1} C^i(R^+,K^{p-2i}(\Omega))$ to some element u, satisfying $\|u\|_{F^s} \leqslant M$ and being a solution of the problem (1″), (2), (3). Similarly as in [6] and [2] we can prove that u is a unique solution of the problem (1″), (2), (3) in $G^p(\Omega)$.

T h e o r e m 3. If the assumptions $\overline{A1}$ - $\overline{A4}$ are fulfilled, then for any $M > C_0 L$ there exists a positive constant

$$(**)\qquad \varepsilon_1 = \min\left(\frac{(M-C_0L)|\lambda_1|}{C_0\tilde{H}C_1(M+1)}, \frac{|\lambda_1|\theta}{C_0\tilde{C}_2\tilde{H}}, \frac{|\lambda_1|}{C_0C_2\tilde{H}}\right), \qquad 0<\theta<1,$$

such that any solution of the problem (1″), (2), (3) for any $\varepsilon = (0,\varepsilon_1)$ is stable and asymptotically stable i.e. for any two solutions $u_{1\varepsilon}$ and $u_{2\varepsilon}$ of the problem (1″), (2), (3) with the initial data ϕ_1 and ϕ_2, respectively, the following estimate holds for $t \in \mathbb{R}^+$

$$(23)\qquad \|u_{1\varepsilon}(t,.) - u_{2\varepsilon}(t,.)\|_{H^p} \leq$$

$$\leq C_0 \|\phi_1 - \phi_2\|_{H^p} \exp\left(-\frac{1}{2}(\lambda_1^2 - \tilde{H}^2\varepsilon^2C_0^2C_2^2)\ t\right).$$

P r o o f . The assumptions $\overline{A1}$ - $\overline{A4}$ guarantee the existence and uniqueness of a solution of the problem (1″), (2), (3). In virtue of the estimate (15′), we have for the difference of two solutions $u_{1\varepsilon}$, $u_{2\varepsilon}$ of the problem (1″), (2), (3) the following estimate

$$\|u_{1\varepsilon}(t,.) - u_{2\varepsilon}(t,.)\|^2_{H^p} \leq C_0^2 e^{-2\lambda_1^2 t} \|\phi_1 - \phi_2\|^2_{H^p} +$$

$$+ C_0^2\varepsilon^2C_2^2\tilde{H}^2\left(\int_0^t \|u_{1\varepsilon} - u_{2\varepsilon}\|^2_{H^p} e^{\lambda_1^2\tau}\, d\tau\right) e^{-\lambda_1^2 t}.$$

By the Gronwall inequality, it follows

$$\|u_{1\varepsilon}(t,.) - u_{2\varepsilon}(t,.)\|^2_{H^p} e^{\lambda_1^2 t} \leq$$

$$\leq C_0^2 \|\phi_1 - \phi_2\|^2_{H^p} \exp((\varepsilon C_0C_2\tilde{H})^2 t).$$

Hence, we obtain the estimate

$$\|u_{1\varepsilon} - u_{2\varepsilon}\|_{H^p} \leq C_0 \|\phi_1 - \phi_2\|_{H^p} \exp\left[-\frac{1}{2}(\lambda_1^2 - \varepsilon^2C_0^2C_2^2\tilde{H}^2)t\right],$$

which implies stability and asymptotical stability of any solution of the problem (1″), (2), (3).

R e m a r k 1. If the assumptions A1, A2, A4 and besides

$$A3'.\quad f = f(t,x),\ f \in C^0(R^+, K^{p-1}(\Omega)),\ \int_0^\infty |f(t,.)|_{K^{p-1}}dt = H$$

hold, then the problem (1′), (2), (3) has a unique global solution $u \in G^p(\Omega)$.

R e m a r k 2. If the assumptions $\overline{A1}$, $\overline{A2}$, $\overline{A4}$ and besides

$$\overline{A3}'.\qquad \int_0^\infty h_Q(t)dt = \tilde{H} \quad \text{for every} \quad Q > 0$$

hold, then the problem (1″), (2), (3) has a unique global solution $u \in G^p(\Omega)$.

R e m a r k 3. For the boundary condition of the third type i.e.

$$(3')\qquad \frac{\partial u(x,t)}{\partial N} + h(x)u(x,t))\Big|_\Gamma = 0, \qquad t \in R^+,$$

similarly as in [2] we can construct the Banach space $K^p(\Omega)$ in the following way

$$K^p = \left\{u \in H^p,\ \frac{\partial A^\alpha u}{\partial N} + hA^\alpha u\Big|_\Gamma = 0; \quad 0 \leqslant \alpha \leqslant \left[\frac{p-2}{2}\right],\ p \geqslant 2\right\}.$$

In the space $G^p(\Omega) = \bigcap_{i=0}^{1} C^i(R^+, K^{p-2i}(\Omega))$ we can prove the existence and stability of solutions of the problem (1), (2), (3′).

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