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COMMON BEHAVIOUR OF SOLUTIONS OF SOME COLLECTIONS OF SECOND ORDER LINEAR ORDINARY DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS

1. Consider the following collection of equations

$$r'' + a(t)\,r + \left[\lambda_n^2 + b(t)\right] r = 0, \quad n = 1,2,\ldots \tag{1}$$

where $0 < \lambda_1 < \ldots < \lambda_n < \ldots$, $\lambda_n \to \infty$ as $n \to \infty$, $a(t+\pi) = a(t)$, $a \in H^1(0,\pi)$, $a(0) = a(\pi)$ ($a(0)$ and $a(\pi)$ are understood in the sense of trace), $b(t+\pi) = b(t)$ and $b \in L^2(0,\pi)$. As in [2] solutions of (1) are understood in the sense of Carathéodory.
Let

$$\alpha = \frac{1}{2\pi}\int_0^\pi a(t)dt, \quad a_1(t) = \frac{1}{2}\left[\frac{t}{\pi}\int_0^\pi a(t)dt - \int_0^t a(s)ds\right].$$

The equations (1) can be transformed into

$$y'' + \left[\lambda_n^2 + Q(t)\right] y = 0, \quad n = 1,2,\ldots \tag{2}$$

by putting

$$y(t) = r(t)\exp\left[\alpha t - a_1(t)\right], \tag{3}$$

where

$$Q(t) = b(t) - \frac{1}{2}a'(t) - \frac{1}{4}a^2(t). \tag{4}$$

Let Q be bounded and

$$N = \sup\{|Q(t)| : t\in(0,\pi)\},$$
$$N_s = \sup\{Q(t) : t\in(0,\pi)\},$$
$$N_i = \inf\{Q(t) : t\in(0,\pi)\}.$$

T h e o r e m 1. Let the function Q defined by (4) be bounded and if either

$$(5)\qquad \lambda_1^2 + N_s > 0,\quad \frac{1}{2\pi\sqrt{\lambda_1^2+N_s}}\int_0^{\pi}(N_s - Q(t))dt = q_1 < \alpha,$$

or

$$(6)\qquad \lambda_1^2 + N_i > 0,\quad \frac{1}{2\pi\sqrt{\lambda_1^2+N_i}}\int_0^{\pi}(Q(t) - N_i)dt = q_2 < \alpha,$$

then there exists $D>0$ independent of n such that the solutions r_{n1}, r_{n2} of the equations (1) satisfying

$$(7)\qquad \begin{cases} r_{n1}(0) = 1, & r'_{n1}(0) = 0,\\ r_{n2}(0) = 0, & r'_{n2}(0) = 1\end{cases}$$

can be estimated as follows

$$(8)\qquad \begin{cases} |r_{n1}(t)| \leqslant D\, e^{-\varepsilon t}, & |r'_{n1}(t)| \leqslant D\,\lambda_n\, e^{-\varepsilon t}\\ |r_{n2}(t)| \leqslant D\,\lambda_n^{-1}\, e^{-\varepsilon t}, & |r'_{n2}(t)| \leqslant D\, e^{-\varepsilon t},\end{cases}$$

with $\varepsilon = \alpha - q_1$ (or $\varepsilon = \alpha - q_2$).

P r o o f . Let the function Q satisfy (5) and y_{n1}, y_{n2} be the solutions of (2) satisfying

$$(9)\qquad \begin{cases} y_{n1}(0) = 1, & y'_{n1}(0) = 0,\\ y_{n2}(0) = 0, & y'_{n2}(0) = 1.\end{cases}$$

By (3), (4) in [2] we have

$$(10)\quad \begin{cases} |y_{n1}(t)| \leqslant \exp N\lambda_n^{-1}t, & |y'_{n1}(t)| \leqslant \lambda_n \exp N\lambda_n^{-1}t, \\ |y_{n2}(t)| \leqslant \lambda_n^{-1}\exp K\lambda_n^{-1}t, & |y'_{n2}(t)| \leqslant \exp K\lambda_n^{-1}t. \end{cases}$$

Choose n_0 so that $N\lambda_{n_0}^{-1} < q_1$. Then for $n > n_0$ we have

$$|y_{n1}(t)| \leqslant \exp q_1 t, \qquad |y'_{n1}(t)| \leqslant \lambda_n \exp q_1 t$$

$$|y_{n2}(t)| \leqslant \lambda_n^{-1} \exp q_1 t, \quad |y'_{n2}(t)| \leqslant \exp q_1 t.$$

Let α_n be characteristic exponents of the equations (1). By (5) we get $\operatorname{Re}\alpha_n \leqslant q_1$ for all n (see [3], p.126). Hence there exist constants D_n such that

$$|y_{n1}(t)| \leqslant D_n \exp(q_1 t), \qquad |y'_{n1}(t)| \leqslant \lambda_n D_n \exp(q_1 t),$$

$$|y_{n2}(t)| \leqslant \lambda_n^{-1} D_n \exp(q_1 t), \quad |y'_{n2}(t)| \leqslant D_n \exp(q_1 t).$$

Putting $D_0 = \max\left\{1, K_j;\ j \leqslant n_0\right\}$ we have for all n

$$|y_{n1}(t)| \leqslant D_0 \exp(q_1 t), \qquad |y'_{n1}(t)| \leqslant \lambda_n D_0 \exp(q_1 t),$$

$$|y_{n2}(t)| \leqslant \lambda_n^{-1} D_0 \exp(q_1 t), \quad |y'_{n2}(t)| \leqslant D_0 \exp(q_1 t).$$

It is easy to see that

$$(11)\quad \begin{cases} r_{n1}(t) = \exp\left(-\frac{1}{2}\int_0^t a(s)\,ds\right)\left[y_{n1}(t) + \frac{a(0)}{2}\, y_{n2}(t)\right], \\ r_{n2}(t) = \exp\left(-\frac{1}{2}\int_0^t a(s)\,ds\right) y_{n2}(t) \end{cases}$$

are the solutions of (1) satisfying (7). From this we get (8) with $D = D_0\left(1 + \frac{a(0)}{2\lambda_1}\right)\max\left\{\exp a_1(t) : t \in [0,\pi]\right\}$.

For the function φ satisfying (6) the proof is similar.

R e m a r k . This theorem is still valid for the case $\lambda_1 = 0$, λ_n in (8) is replaced by $\lambda_n + \lambda_0$ with λ_0 being a positive constant.

2. Consider the following collection of nonhomogeneous equations corresponding to (1)

$$s'' + a(t)\, s' + \left[\lambda_n^2 + b(t)\right] s = c_n(t), \quad n=1,2,\ldots, \tag{12}$$

where $c_n(t+\pi) = c_n(t)$ and $c_n(t) \in L^2(0,\pi)$. Put

$$C_n = \int_0^{\pi} |c_n(t)|\, dt.$$

T h e o r e m 2. If the functions a, b satisfy the hypotheses of Theorem 1, then there exists a constant $K > 0$ independent of n such that the solutions s_{no} satisfying

$$s_{no}(0) = s'_{no}(0) = 0 \tag{13}$$

of the equations (12) can be estimated as follows

$$|s_{no}(t)| \leq KC_n\lambda_n^{-1}, \quad |s'_{no}(t)| \leq KC_n. \tag{14}$$

P r o o f . Since the hypotheses of Theorem 1 are satisfied, the corresponding homogeneous equations (1) have no periodic solutions with period π. Therefore, for each n (12) has the unique solution s_n periodic with period π (see [1], p.251). Denote by r_{n1}, r_{n2} the solutions of (1) satisfying (7) and by y_{n1}, y_{n2} the solutions of (2) satisfying (9). Put

$$R_n(t) = \begin{bmatrix} r_{n1}(t) & r_{n2}(t) \\ r'_{n1}(t). & r'_{n2}(t) \end{bmatrix}, \quad c_n^*(t) = \begin{bmatrix} 0 \\ c_n(t) \end{bmatrix}.$$

Then we have

$$(15)\qquad \begin{bmatrix} s_n(t) \\ s_n'(t) \end{bmatrix} = R_n(t) \begin{bmatrix} s_n(0) \\ s_n'(0) \end{bmatrix} + \int_0^t R_n(t)\, R_n^{-1}(s)\, c_n^*(s)\, ds,$$

where

$$(16)\qquad \begin{bmatrix} s_n(0) \\ s_n'(0) \end{bmatrix} = \left[E - R_n(\pi)\right]^{-1} R_n(\pi) \int_0^\pi R_n^{-1}(t)\, c_n^*(t)dt.$$

Choose n_1 so that $\exp N \lambda_{n_1}^{-1}\pi < \operatorname{ch}\alpha\pi$. Then for $n \geq n_1$ by (10), (11) we get

$$|y_{n1}(\pi) + y_{n2}'(\pi)| \leq 2 \exp(N \lambda_{n_1}^{-1}\pi),$$

$$|r_{n1}(\pi) + r_{n2}'(\pi)| \leq 2 \exp\left[(N \lambda_{n_1}^{-1} - \alpha)\pi\right],$$

$$\det\left[E - R_n(\pi)\right] = 1 - \left[r_{n1}(\pi) + r_{n2}'(\pi)\right] + \det R_n(\pi) \geq$$

$$\geq 2 \exp(-\alpha\pi)\left[\operatorname{ch}\alpha\pi - \exp(N \lambda_{n_1}^{-1}\pi)\right] = d > 0.$$

By (16) we obtain

$$s_n(0) = \frac{1}{\det\left[E-R_n(\pi)\right]}\left[\left(r_{n1}(\pi) - e^{-2\alpha\pi}\right)\int_0^\pi \left(-\exp\int_0^t a(s)ds\right) r_{n2}(t)c_n(t)\,dt +\right.$$

$$\left. + r_{n2}(\pi)\int_0^\pi \left(\exp\int_0^t a(s)ds\right) r_{n1}(t)\; c_n(t)dt.\right.$$

Putting $H = \frac{1}{d}\max\left\{\exp\int_0^t a(s)ds \;:\; t \in [0,\pi]\right\}$ together with (8) give

$$(17)\qquad |s_n(0)| \leq H\, D\, (2D+1)\, C_n \lambda_n^{-1}.$$

Analogously, we have

$$(18)\qquad |s_n'(0)| \leq H\, D\, (2D+1)\, C_n.$$

The periodicity of s_n and (8), (15)-(18) assure

$$|s_n(t)| \leqslant 2HD^2(2D+1+d)\ C_L\lambda_n^{-1},$$

$$|s_n'(t)| \leqslant 2HD^2(2D+1+d)\ C_n.$$

On the other hand, for each n there exists K_n such that

$$|s_n(t)| \leqslant K_n C_n \lambda_n^{-1}, \quad |s_n'(t)| \leqslant K_n C_n.$$

Hence, with $K_o = \max\{2HD^2(2D+1+d),\ K_n\ :\ n \leqslant n_1\}$ we have for all $n \in N$

$$(19) \qquad |s_n(t)| \leqslant K_o C_n \lambda_n^{-1}, \quad |s_n'(t)| \leqslant K_o C_n.$$

It is easy to see that

$$s_{no}(t) = s_n(t) - s_n(0) r_{n1}(t) - s_n'(0) r_{n2}(t)$$

is the solution of (12) satisfying (13). By (8), (17)-(19) we get (14) with $K = K_o(2D+1)$.

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