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THE EFFECT OF NUMERICAL INTEGRATION IN FINITE ELEMENT METHODS FOR SOLVING QUASILINEAR PARABOLIC EQUATIONS

Introduction

In finite element methods, evaluation of a solution of initial-boundary value problem necessitates the use of numerical integration of inner products, which introduced additional errors.

The main subject of this paper is to investigate the effect of numerical integration on the error estimates. It is proved that a suitable choice of the quadrature scheme leads to the optimal L^2- and H^1-convergence rates.

The types of finite element spaces used, and the assumptions on the quadrature formulae follow [7], where a corresponding problem for quasilinear elliptic equations has been analysed.

The analogous problem in the case of linear parabolic equations was considered by Raviart [9] and Fix [6]. In the case of nonlinear parabolic equations Douglas and Dupont [4] presented an alternative method of approximating the integrals by interpolating the coefficients and evaluating the integrals by exact formulae.

Finally we consider the problem of stiffness of a system of ordinary differential equations which arises in the semi-discretization with numerical integration.

1. The quasilinear parabolic problem

Let Ω be a bounded, open set in Euclidian space R^n with a sufficiently smooth boundary Γ and $J = (0,T] \subset R^1$. Consider the following nonlinear parabolic problem

$$
(1.1) \quad \begin{cases} \frac{\partial u}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(a_i(x,u)\frac{\partial u}{\partial x_i}\right) + a_0(x,u) = 0 & \text{for } (x,t) \in \Omega \times J, \\ u(x,t) = 0 & \text{for } (x,t) \in \Gamma \times J, \\ u(x,0) = u_0(x) & \text{for } x \in \Omega. \end{cases}
$$

We shall assume the following regularity of the coefficients and the ellipticity condition

(R1) $a_i(x,u)$, $i = 0,1,\dots,n$, are k-times continuously differentiable functions on $\bar{\Omega} \times R$,

(R2) for any $C > 0$ there exist constants C_0, C_1 such that

$$0 < C_0 \leq a_i(x,u) \leq C_1$$

for any $i = 1,\dots,n$ and any $(x,u) \in \Omega \times [-C,C]$ and

$$\frac{\partial}{\partial u} a_0(x,u) - \sum_{i=1}^{n} \left(\frac{\partial}{\partial u} a_i(x,u) \frac{\partial u}{\partial x_i}\right)^2 (a_i(x,u))^{-1} \geq 0$$

(here and in the following C, C_0, $C_1,\dots$ denote generic positive constants not necessarily the same in each two formulae).

For simplifying the problem, we assume all functions a_i, $i = 0,1,\dots,n$, to be autonomous.

Let $W^{m,q}(\Omega)$, for any integer $m \geq 0$ and any number $q \in [1,+\infty]$, be the Sobolev space with the norm $\|\cdot\|_{m,q,\Omega}$ and the semi-norm $|\cdot|_{m,q,\Omega}$. For $q = 2$ we abbreviate $\|\cdot\|_{m,2,\Omega}$ by $\|\cdot\|_{m,\Omega}$ and $\|\cdot\|_{0,2,\Omega}$ by $\|\cdot\|$ and we denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$. The space $W_0^{m,q}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in the norm $\|\cdot\|_{m,q,\Omega}$. The inner product on $L^2(\Omega) = H^0(\Omega)$ we denote by

$(.,.)$ and we extend $(.,.)$ to the duality pairing between $H_0^1(\Omega)$ and its dual $H^{-1}(\Omega)$.

If $(X(\Omega),\|\cdot\|_{X(\Omega)})$ is a Banach space, then, for any $p \in [1,+\infty]$, $L^p(J,X(\Omega))$ denotes the Banach space of functions $u: J \to X(\Omega)$ such that

$$\|u\|_{L^p(J,X(\Omega))} = \left(\int_J \|u(t)\|^p_{X(\Omega)}\, dt\right)^{\frac{1}{p}} < +\infty$$

with the usual modification for $p = +\infty$. Furthermore, let $\partial_i u = \frac{\partial u}{\partial x_i}$, $i = 1,\dots,n$, and $\partial_t u = \frac{\partial u}{\partial t}$.

The weak form of (1.1) is to find $u \in L^2(J,H_0^1(\Omega))$ such that $\partial_t u \in L^2(J,H^{-1}(\Omega))$ and

$$(1.2) \qquad \begin{cases} (\partial_t u,y) + (a(u)\nabla u,\nabla y) + (a_0(u),y) = 0 \\ (u(.,0),y) = (u_0,y) \end{cases}$$

for any $y \in H_0^1(\Omega)$ and a.e. $t \in J$, where

$$(a(u)\nabla u,\nabla y) = \sum_{i=1}^{n} (a_i(.,u)\partial_i u,\partial_i y), \quad (a_0(u),y) = (a_0(.,u),y).$$

For error estimates of the numerical method we are going to present, we assume sufficient regularity of the unique solution of the problem (1.2), namely:

$$(1.3) \qquad u \in L^\infty(J,W^{k+1,\infty}(\Omega) \cap W_0^{1,\infty}(\Omega))$$

(for regularity results, see [8]).

2. <u>The semi-discrete Galerkin method</u>

To approximate (1.2), we construct for $h > 0$ a family of triangulations $\mathcal{T}_h$ of $\bar\Omega$, (see [1]), i.e., we consider a finite partition of $\bar\Omega$ into disjoint subsets $\{e: e \in \mathcal{T}_h\}$ such that $\bar\Omega = \bigcup_{e\in\mathcal{T}_h} e$ and $h = \sup_{e\in\mathcal{T}_h} \text{diam}(e)$ approaches zero. All the finite elements $e \in \mathcal{T}_h$ are affine-equivalent to a single reference finite element E through the invertible affine mapping.

With $\mathcal{T}_h$ we associate S_h, a finite-dimensional subspace of $C^0(\Omega) \cap H_0^1(\Omega)$. We make the following hypotheses about S_h:

(A1) if $y_h \in S_h$ and $e \in \mathcal{T}_h$, then $y_h|_e \in P_k(e)$ for some integer $k \geqslant 1$, where $P_k(e)$ denotes space of all polynomials in $x_1, \dots, x_n$ of degree not greater than k,

(A2) the approximation property:
for any integer $m = 2, \dots, k+1$, any real number $q \in [2, +\infty]$ and any h there exists a bounded linear operator $\Pi_h : W^{m,q}(\Omega) \cap W_0^{1,q}(\Omega) \rightarrow S_h$, such that

$$\Big(\sum_{e \in \mathcal{T}_h} \|\Pi_h y - y\|_{j,q,e}^q \Big)^{\frac{1}{q}} \leqslant C h^{m-j} \|y\|_{m,q,\Omega} \quad \text{if} \quad q < +\infty ,$$

$$\max_{e \in \mathcal{T}_h} \|\Pi_h y - y\|_{j,\infty,e} \leqslant C h^{m-j} \|y\|_{m,\infty,\Omega} \qquad \text{if} \quad q = +\infty$$

for all $y \in W^{m,q}(\Omega) \cap W_0^{1,q}(\Omega)$ and $j = 0, 1, \dots, m$, where the constant C is independent of y and h,

(A3) the inverse inequality:
for any integer $0 \leqslant j \leqslant m$ there exists a constant C independent of h such that

$$\Big(\sum_{e \in \mathcal{T}_h} |y_h|_{m,e}^2 \Big)^{\frac{1}{2}} \leqslant C h^{j-m} \Big(\sum_{e \in \mathcal{T}_h} |y_h|_{j,e}^2 \Big)^{\frac{1}{2}} ,$$

$$\max_{e \in \mathcal{T}_h} |y_h|_{m,\infty,e} \leqslant C h^{j-m} \max_{e \in \mathcal{T}_h} |y_h|_{j,\infty,e} \quad \text{for all} \quad y_h \in S_h .$$

These properties are satisfied by the finite element space introduced in [7].

The Galerkin problem takes now the form:
let $\{\xi_s(x)\}_{s=1}^M$ be a basis of S_h, we seek $u_h \in L^2(J, S_h)$ of the form

$$u_h(x,t) = \sum_{s=1}^{M} u_s(t)\xi_s(x)$$

such that

$$(2.1) \quad \begin{cases} (\partial_t u_h, y_h) + (a(u_h)\nabla u_h, \nabla y_h) + (a_0(u_h), y_h) = 0, \\ u_h(0) \in S_h \end{cases}$$

for any $y_h \in S_h$ and a.e. $t \in J$. Problem (2.1) is an initial problem for a nonlinear system of ordinary differential equations of first order. It has been extensively studied; e.g., see [3] and [10], where, under suitable assumptions on the smoothness of u, optimal estimates of the error $u-u_h$ in the spaces $L^2(J,H^1(\Omega))$ and $L^\infty(J,L^2(\Omega))$ are derived, namely

$$\|u-u_h\|_{L^\infty(J,L^2(\Omega))} + h\|u-u_h\|_{L^2(J,H^1(\Omega))} \leq C_1(u)h^{k+1},$$

$$\|\partial_t(u-u_h)\|_{L^2(J,L^2(\Omega))} \leq C_1(u)h^{k+1},$$

where

$$C_1(u) = C(\|u\|_{L^\infty(J,H^{k+1}(\Omega))} + \|\partial_t u\|_{L^\infty(J,H^{k+1}(\Omega))}).$$

In [2] the uniform error estimate is derived

$$\|u-u_h\|_{L^\infty(J,L^\infty(\Omega))} \leq C_2(u)h^{k+1}|\ln h|^{c(n)},$$

where

$$C_2(u) = C(\|u\|_{L^\infty(J,W^{k+1,\infty}(\Omega))} + \|\partial_t u\|_{L^\infty(J,W^{k+1,\infty}(\Omega))}),$$

and $c(n) = (n/2)+2$.

The regularity of Galerkin solution follows from the above estimates and properties (A2) and (A3):

L e m m a 2.1. Let $u \in L^{\infty}(J, W^{k+1,\infty}(\Omega) \cap W_0^{1,\infty}(\Omega))$ be the solution of (1.2)' such that $\partial_t u \in L^{\infty}(J, W^{k+1,\infty}(\Omega))$. Then for a solution u_h of (2.1) the following estimates hold

$$\max_{e \in \mathcal{T}_h} \|u_h\|_{k,\infty,e} \leq \bar{C}_2(u),$$

$$\Big(\sum_{e \in \mathcal{T}_h} \|u_h\|^2_{k,e}\Big)^{\frac{1}{2}} \leq \bar{C}_1(u),$$

$$\Big(\sum_{e \in \mathcal{T}_h} \|\partial_t u_h\|^2_{k,e}\Big)^{\frac{1}{2}} \leq \bar{C}_1(u) \quad \text{for a.e.} \quad t \in J.$$

P r o o f . We shall prove the first estimate. From the triangle inequality we get

$$\max_{e \in \mathcal{T}_h} \|u_h\|_{k,\infty,e} \leq \|u\|_{k,\infty,\Omega} + \max_{e \in \mathcal{T}_h} \|u-u_h\|_{k,\infty,e} \leq \tag{2.2}$$

$$\leq \|u\|_{k,\infty,\Omega} + \max_{e \in \mathcal{T}_h} \|u-\Pi_h u\|_{k,\infty,e} + \max_{e \in \mathcal{T}_h} \|\Pi_h u-u_h\|_{k,\infty,e}.$$

Using the property (A2), we have for the second term on the right-hand side of (2.2)

$$\max_{e \in \mathcal{T}_h} \|u-\Pi_h u\|_{k,\infty,e} \leq Ch\|u\|_{k+1,\infty,\Omega}.$$

The third term on the right-hand side is estimated by

$$\max_{e \in \mathcal{T}_h} \|\Pi_h u-u_h\|_{k,\infty,e} \leq Ch^{-k}\|\Pi_h u-u_h\|_{0,\infty,\Omega} \leq$$

$$\leq Ch^{-k}(\|\Pi_h u-u\|_{0,\infty,\Omega} + \|u-u_h\|_{0,\infty,\Omega}) \leq$$

$$\leq Ch^{-k}(h^{k+1}\|u\|_{k+1,\infty,\Omega} + \bar{C}_2(u)h^{k+1}\,|\ln h|^{c(n)}) \leq$$

$$\leq Ch\|u\|_{k+1,\infty,\Omega} + \bar{C}_2(u)h\,|\ln h|^{c(n)},$$

where we have first used the inverse inequality (A3) and then the approximation property (A2) and the uniform error estimate of the Galerkin method. Combining the previous inequalities we obtain

$$\max_{e\in \mathcal{J}_h} \|u_h\|_{k,\infty,e} \leq \bar{C}_2(u),$$

where

$$\bar{C}_2(u) = C(\|u\|_{k+1,\infty,\Omega} + \|\partial_t u\|_{k+1,\infty,\Omega}).$$

Analogously, we can prove the second and third estimates of Lemma 2.1.

3. The effect of quadrature errors

Since it is either too costly or impossible to calculate exactly the values of integrals over Ω which appear in (2.1), we must take into account the fact that numerical integration is used for evaluating these integrals. Thus, for any finite element $e \in \mathcal{J}_h$, we introduce a quadrature formula over e:

$$\int_e f(x)dx \quad \text{is approximated by} \quad \sum_{l=1}^{L} w_{l,e} f(b_{l,e})$$

for some specified points $b_{l,e} \in e$ and weights $w_{l,e} > 0$, $l = 1,\ldots,L$.

In view of affine-equivalence, we see that the quadrature scheme over the set E automatically induces a quadrature scheme over the set e, namely let

(3.1) $$F_e : E \ni \hat{x} \longrightarrow x = F_e(\hat{x}) = G_e\hat{x} + g_e \in e$$

be the invertible affine mapping which maps E onto e. Then

$$\int_E \hat{f}(\hat{x})d\hat{x} \sim \sum_{l=1}^{L} \hat{w}_l \hat{f}(\hat{b}_l),$$

where $f(x) = \hat{f}(\hat{x})$ for all $x = F_e(\hat{x})$, $\hat{x} \in \hat{E}$, and

$$(3.2) \qquad w_{l,e} = |\det G_e|\hat{w}_l, \quad b_{l,e} = F_e(\hat{b}_l), \quad l=1,\dots,L.$$

By using the quadrature formulae, we replace the semi-discrete problem (2.1) by the following one:
find a function $U_h \in L^2(J,S_h)$ of the form

$$U_h(x,t) = \sum_{s=1}^{M} U_s(t)\xi_s(x)$$

such that

$$(3.3) \quad \begin{cases} (\partial_t U_h, y_h)_h^* + (a(U_h)\nabla U_h, \nabla y_h)_h + (a_0(U_h), y_h)_h = 0, \\ U_h(0) \in S_h \text{ will be defined later on,} \end{cases}$$

for any $y_h \in S_h$ and a.e. $t \in J$, where for $z_h, y_h \in S_h$

$$(3.4) \quad \begin{cases} (\partial_t z_h, y_h)_h^* = \sum_{e\in J_h} \sum_{l=1}^{L^*} w_{l,e}^* \partial_t z_h(b_{l,e}^*) y_h(b_{l,e}^*), \\ (a(z_h)\nabla z_h, \nabla y_h)_h = \\ = \sum_{e\in J_h} \sum_{l=1}^{L} w_{l,e} \left[\sum_{i=1}^{n} a_i(b_{l,e}, z_h(b_{l,e})) \times \right. \\ \left. \times\, \partial_i z_h(b_{l,e}) \partial_i y_h(b_{l,e}) \right], \\ (a_0(z_h), y_h)_h = \sum_{e\in J_h} \sum_{l=1}^{L} w_{l,e}\, a_0(b_{l,e}, z_h(b_{l,e})) y_h(b_{l,e}). \end{cases}$$

Here we use two kinds of quadrature formulae:

(Q1) $\hat{w}_l^* > 0$, $\hat{b}_l \in E$, $l = 1,\dots,L^*$, are the weights and the nodes, respectively, of such quadrature formula of degree $2k-2$ that the union $\bigcup_{l=1}^{L^*} \{\hat{b}_l^*\}$ contains a $P_k(E)$-unisolvent subset,

(Q2) $\hat{w}_l > 0$, $\hat{b}_l \in E$, $l = 1,\dots,L$, are the weights and the nodes, respectively, of a quadrature formula of degree $2k-2$.

In [7] the following properties of the forms (3.4) were derived:

(P1) the consistency properties on the Galerkin solution u_h, i.e. there exists C depending only on the coefficients and solution of parabolic problem such that

$$|(a(u_h)\nabla u_h, \nabla y_h) - (a(u_h)\nabla u_h, \nabla y_h)_h| \leq Ch^k |y_h|_{1,\Omega},$$

$$|(a_0(u_h), y_h) - (a_0(u_h), y_h)_h| \leq Ch^k |y_h|_{1,\Omega},$$

$$|(\partial_t u_h, y_h) - (\partial_t u_h, y_h)_h^*| \leq Ch^k |y_h|_{1,\Omega}$$

for any $y_h \in S_h$,

(P2) **the uniform** S_h-coercivity, i.e. there exists a constant $\alpha > 0$ independent of S_h such that

$$(a(y_h)\nabla y_h, \nabla(y_h - z_h))_h - (a(z_h)\nabla z_h, \nabla(y_h - z_h))_h +$$

$$+ (a_0(y_h), y_h - z_h)_h - (a_0(z_h), y_h - z_h)_h \geq \alpha |y_h - z_h|_{1,\Omega}^2$$

for all $y_h, z_h \in S_h$.

Now we prove the next property of form $(\cdot,\cdot)_h^*$.

L e m m a 3.1. Let the union $\bigcup_{l=1}^{L^*} \{\hat{b}_l^*\}$ contain a $P_k(E)$-unisolvent subset and let $\hat{w}_l^* > 0$, $l = 1,\dots,L^*$. Then the mapping

$$y_h \to |y_h|_h = ((y_h, y_h)_h^*)^{\frac{1}{2}} = \Big(\sum_{e \in \mathcal{T}_h} \sum_{l=1}^{L^*} w_{l,e}^* \, y_h^2(b_{l,e}^*)\Big)^{\frac{1}{2}}$$

is a norm over S_h and there exists a constant $D > 0$ independent of h such that

(3.5) $$D^{-1}\|y_h\| \leq |y_h|_h \leq D\|y_h\|.$$

P r o o f . Using the strict positivity of the weights, we find that

if $\hat{p} \in P_k(E)$ and $\sum_{l=1}^{L^*} \hat{w}_l^* \hat{p}^2(\hat{b}_l^*) = 0$, then $\hat{p}(\hat{b}_l^*) = 0$ for $l=1,\dots,L^*$.

Thus, by the unisolvency assumption, we have $\hat{p} \equiv 0$. As the consequence, the mapping

$$\hat{p} \rightarrow \Big(\sum_{l=1}^{L^*} \hat{w}_l^* \hat{p}^2(\hat{b}_l^*)\Big)^{\frac{1}{2}}$$

defines a norm over $P_k(E)$, since $\Big(\sum_{l=1}^{L^*} \hat{w}_l^* x_l^2\Big)^{\frac{1}{2}}$ defines a norm over R^{L^*}. Since the space $P_k(E)$ is finite dimensional, there exists constants $\hat{D}_1, \hat{D}_2 > 0$ such that

(3.6) $$\hat{D}_1 |\hat{p}|^2_{0,E} \leq \sum_{l=1}^{L^*} \hat{w}_l^* \hat{p}^2(\hat{b}_l^*) \leq \hat{D}_2 |\hat{p}|^2_{0,E}.$$

Let $y_h|_e = p_e$ and let $\hat{p}_e \in P_k(E)$ be the polynomial associated with p_e through the usual correspondence (3.1), i.e. $p_e = \hat{p}_e \cdot F_e^{-1}$. We can write, using (3.2) and (3.6),

$$\sum_{l=1}^{L^*} w_{l,e}^* y_h^2(b_{l,e}^*) = \sum_{l=1}^{L^*} w_{l,e}^* p_e^2(b_{l,e}^*) = \sum_{l=1}^{L^*} |\det G_e| \hat{w}_l^* \hat{p}_e^2(\hat{b}_l^*) \geq$$

$$\geq \hat{D}_1 |\det G_e| \, |\hat{p}_e|^2_{0,E} = \hat{D}_1 |\det G_e| \, |\det G_e|^{-1} \, |p_e|^2_{0,e},$$

where we have also used the formula of change of variables in multiple integrals. Hence,

$$|y_h|_h^2 = \sum_{e\in\mathcal{T}_h} \sum_{l=1}^{L^*} w_{l,e}^* y_h^2(b_{l,e}^*) \geq D^{-2}\|y_h\|^2$$

which implies the left part of (3.5), the right one can be derived analogously.

We can now turn to the error estimates for the quadrature-Galerkin method (3.3). Let us write

$$u - U_h = u - u_h + u_h - U_h = u - u_h + Z_h,$$

where u is the solution of the differential problem (1.2), u_h is the solution of the Galerkin approximation (2.1), and U_h is the solution of the quadrature-Galerkin method (3.3). The estimates of $u-u_h$ are known, so we have to estimate $Z_h \in S_h$.

Let us begin defining the initial function $U_h(0) \in S_h$. We assume that $U_h(0)$ is the projection of the given function u_0 on the subspace S_h with regard to the discrete inner product $(\cdot,\cdot)_h$, i.e.

$$(3.7) \qquad \sum_{e\in\mathcal{T}_h} \sum_{l=1}^{L^*} w_{l,e}^* (u_0-U_h(0))(b_{l,e}^*) y_h(b_{l,e}^*) = 0$$

for any $y_h \in S_h$.

Lemma 3.2. Let $U_h(0)$ be the projection of $u_0 \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$ on S_h defined in (3.7) and let $u_h(0)$ be the projection of u_0 on S_h with regard to the inner product on $L^2(\Omega)$. Then

$$\|Z_h(0)\| = \|u_h(0)-U_h(0)\| \leq Ch^k\|u_0\|_{k+1,\Omega}.$$

Proof. By the assumption, $(u_0-U_h(0),y_h)_h^* = (u_0-u_h(0),y_h)$ for any $y_h \in S_h$. Thus

$$(u_h(0)-U_h(0),y_h)_h^* \leq |(u_0-u_h(0),y_h) - (u_0-u_h(0),y_h)_h^*|$$

for any $y_h \in S_h$. From Lemma 3.1 and the "local" quadrature error estimate (see [7]) we get, with $y_h = u_h(0)-U_h(0) \in S_h$, the estimate

$$D^{-1}\|u_h(0)-U_h(0)\|^2 \leq Ch^k\Big(\sum_{e\in\mathcal{T}_h}\|u_0-U_h(0)\|^2_{k,e}\Big)^{\frac{1}{2}}|u_h(0)-U_h(0)|_{1,\Omega}.$$

In [5] is shown that

$$\|u_0-u_h(0)\| \leq Ch^{k+1}\|u_0\|_{k+1,\Omega}$$

and thus, in the way analogous to the proof of Lemma 2.1, we obtain

$$\Big(\sum_{e\in\mathcal{T}_h}\|u_0-U_h(0)\|^2_{k,e}\Big)^{\frac{1}{2}} \leq Ch\|u_0\|_{k+1,\Omega}.$$

From (A3)

$$|u_h(0)-U_h(0)|_{1,\Omega} \leq Ch^{-1}\|u_h(0)-U_h(0)\|.$$

Combining all previous inequalities, we get the proposition of Lemma 3.2.

We are now in a position to estimate the error of the quadrature-Galerkin method (3.3).

T h e o r e m 3.1. Let the unique solution u of the problem (1.2) satisfy the condition (1.3). Then under the assumption (R1), (R2), (A1), (A2), (A3), and (Q1), (Q2) the following error estimates hold

$$\|u-U_h\|_{L^\infty(J,L^2(\Omega))} + \|u-U_h\|_{L^2(J,H^1(\Omega))} \leq C(u)h^k,$$

$$\|u-U_h\|_{L^2(J,H^r(\Omega))} \leq C(u)h^{k+1-r} \quad \text{with } r=0 \text{ or } r=1.$$

P r o o f . Since u_h and U_h are the solutions of the Galerkin and the quadrature-Galerkin problem, respectively, we have

$$(\partial_t u_h, y_h) + (a(u_h)\nabla u_h, \nabla y_h) + (a_0(u_h), y_h) =$$

$$= (\partial_t U_h, y_h)_h^* + (a(U_h)\nabla U_h, \nabla y_h)_h + (a_0(U_h), y_h)_h,$$

for any $y_h \in S_h$, which implies

$$(\partial_t Z_h, y_h)_h^* + (a(u_h)\nabla u_h, \nabla y_h)_h - (a(U_h)\nabla U_h, \nabla y_h)_h +$$

$$+ (a_0(u_h), y_h)_h - (a_0(U_h), y_h)_h = (\partial_t u_h, y_h)_h^* - (\partial_t u_h, y_h) +$$

$$+ (a(u_h)\nabla u_h, \nabla y_h)_h - (a(u_h)\nabla u_h, \nabla y_h) + (a_0(u_h), y_h)_h - (a_0(u_h), y_h).$$

Using the properties (P1), (P2) and Lemma 3.1, with $y_h = Z_h$, we get the estimate

$$\frac{d}{dt}\|Z_h\|^2 + 2D\alpha|Z_h|^2_{1,\Omega} \leq 2\bar{C}(u)Dh^k|Z_h|_{1,\Omega}.$$

Applying the ε-inequality (with $\varepsilon = \alpha D$), we find

$$\|Z_h(s)\|^2 + \alpha D\int_0^s |Z_h(t)|^2_{1,\Omega}\,dt \leq \frac{2D}{\alpha}\bar{C}^2(u)h^{2k} + \|Z_h(0)\|^2,$$

for $0 \leq s \leq T$. This, together with Lemma 3.2 and error estimates for the Galerkin solution (2.1), implies the proposition of the theorem.

R e m a r k . Let us consider the following problem: whether numerical integration has a negative influence on stiffness of a system of ordinary differential equations

$$M_h \frac{du_h}{dt} + K_h u_h = f_h$$

which arises in the semi-discretization (2.1) of a linear parabolic equation?

Both matrices M_h and K_h are symmetric and it is well-known that the stiffness ratio

$$s_h = \left(\max_{i=1,..,M} \lambda_i\right) : \left(\min_{i=1,..,M} \lambda_i\right) = \left(\max_u \frac{u^T K_h u}{u^T M_h u}\right) : \left(\min_u \frac{u^T K_h u}{u^T M_h u}\right)$$

is of the order h^{-2}, where λ_i, $i=1,\dots,M$, are the eigenvalues of $-M_h^{-1} K_h$. If

$$\tilde{M}_h \frac{dU_h}{dt} + \tilde{K}_h U_h = \tilde{f}_h$$

is the perturbated by numerical integration system (3.3), then the stiffness ratio is

$$\tilde{s}_h = \left(\max_u \frac{u^T \tilde{K}_h u}{u^T \tilde{M}_h u}\right) : \left(\min_u \frac{u^T \tilde{K}_h u}{u^T \tilde{M}_h u}\right),$$

since both matrices $\tilde{M}_h$ and $\tilde{K}_h$ are also symmetric.

Let us first observe that

$$r_h^1(u) = \frac{\dfrac{u^T K_h u}{u^T u} - C_1 h^{k+1}}{\dfrac{u^T M_h u}{u^T u} + C_2 h^{k+1}} \leqslant \frac{u^T \tilde{K}_h u}{u^T \tilde{M}_h u} \leqslant \frac{\dfrac{u^T K_h u}{u^T u} + C_1 h^{k+1}}{\dfrac{u^T M_h u}{u^T u} - C_2 h^{k+1}} = r_h^2(u),$$

since in linear case the consistency properties (P1) are valid with (k+1)-order of accuracy (see Fix [6]).

Hence

$$\tilde{s}_h = \left(\max_u \frac{u^T \tilde{K}_h u}{u^T \tilde{M}_h u}\right) : \left(\min_u \frac{u^T \tilde{K}_h u}{u^T \tilde{M}_h u}\right) \leqslant (\max_u r_h^2(u)) : (\min_u r_h^1(u)).$$

Putting

$$\mu_h = \max_u \frac{u^T K_h u}{u^T M_h u}, \quad \gamma_h = \min_u \frac{u^T K_h u}{u^T M_h u};$$

we get

$$\tilde{s}_h \leqslant \frac{(1+C_2h^{k+1}\mu_h)(s_h+C_1h^{k+1}\mu_h\gamma_h^{-1})}{(1-C_2h^{k+1}\mu_h)(1-C_1h^{k+1}\mu_h\gamma_h^{-1})} .$$

Let us now assume that

$$\gamma_h = \min_u \frac{u^T K_h u}{u^T M_h u}$$

is constant (the typical case in finite element methods). Then, $\tilde{s}_h$ does not exceed s_h, if $k \geqslant n$, since μ_h is of the order h^{-n}.

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REFERENCES

[1] P.G. Ciarlet: The finite element method for elliptic problems. North-Holland, 1978.

[2] M. Dorowolski: L-convergence of finite element approximation to quasilinear initial boundary value problems, RAIRO Anal. Num. 12 (1978) 247-266.

[3] J. Douglas Jr., T. Dupont: Galerkin methods for parabolic equations, SIAM J.Num.Anal. 7 (1970) 576-626.

[4] J. Douglas Jr., T. Dupont: The effect of interpolating the coefficients in nonlinear parabolic Galerkin procedures, Math. Comput. 29 (1975) 360-389.

[5] J. Douglas Jr., T. Dupont, L. Wahlbin: The stability in L^q of the L^2-projection into finite element function space, Numer. Math. 23 (1975) 193-197.

[6] G.J. F i x : Effects of quadrature errors in finite element approximation of steady state, eigenvalue and parabolic problems. The mathematical foundation of the finite element method with applications to partial differential equations, A.K.Aziz, ed., Ac. Press (1972) 525-556.

[7] T.J. J a n i k : The effect of quadrature errors in the finite element solution of some nonlinear elliptic problems, Demonstratio Math. 19 (1986) 1023-1039.

[8] О.А. Л а д ы ж е н с к а я, В.А. С о л о н н и к о в, Н.Н. У р а л ь ц е в а : Линейные и квазилинейные уравнения параболического типа. Москва 1967.

[9] P.A. R a v i a r t : The use of numerical integration in finite element methods for parabolic equations, Topics in numerical analysis, Proc. of the Royal Irish Academy Conf. in Num. Anal., J.J.Miller, ed. Ac. Press (1973) 233-264.

[10] M.F. W h e e l e r : A priori L^2 error estimates for Galerkin approximations to parabolic partial differential equations, SIAM J.Num.Anal. 10 (1973) 723-759.

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