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THE NOTION OF DIRECTED FIELD AS AN ALGEBRAIC COUNTERPART OF ORDERED EUCLIDEAN PLANE

Introduction

One can investigate analytical plane Euclidean geometry "without coordinates". An elegant way here is to use complex field ([3], comp. [2]). In a sufficiently strong system of the Euclidean geometry the betweenness relation is definable. But usualy we assume there is an order given in the coordinate field, which defines the betweenness. We wanted to investigate also analytical ordered Euclidean planes without coordinates. Therefore we must consider our (complex) field together with a relation which will correspond to betweenness (it cannot be simply an order in the field). Below we propose an axiom system which will describe such fields; we call them "directed fields".

An axiom system for directed fields. The structures we shall consider are of the type $\langle F;+,\cdot,0,1,\uparrow\rangle$, where $0,1 \in F$; $+,\cdot$ are two binary operations and $\uparrow$ is a binary relation in F, called directedness. Any such structure will be called a directed field provided it satisfies

DF0: $\langle F;+,\cdot,0,1\rangle$ is a field

DF1 $a \uparrow a$

DF2 $a \uparrow b \rightarrow b \uparrow a$

DF3 $a \uparrow -a \rightarrow a = 0$

DF4 $a \neq 0 \wedge a \uparrow b,c \rightarrow b \uparrow c$

DF5 $a \uparrow b,c \rightarrow a \uparrow b+c$

DF5′ $a \uparrow b,c \rightarrow a \uparrow b-c \vee a \uparrow c-b$

DF6 $a \uparrow c \rightarrow ba \uparrow bc$

Let F be a directed field. We define over F two structures A(F) and R(F). Given $a,b,c,d \in F$ we define $ab \upuparrows_F cd :\Leftrightarrow b-a \uparrow d-c$ and put $A(F) := \langle F, \upuparrows_F \rangle$. Let $R(F) := \{x \in F: x \uparrow 1 \vee x \uparrow -1\}$, $P(F) := \{x \in F: x \uparrow 1\}$. By DF5′ we easily obtain $a \uparrow 0$.

L e m m a 1. R(F) is an ordered field, with P(F) being a positive cone in R(F).

P r o o f . We simultaneously prove that R(F) is a subfield of F (i.e. R(F) is closed under operations of F) and that P(F) satisfies axioms of positive cone in a field (c.f. [6]). We note at first that $0,1 \in P(F) \subseteq R(F)$.

Now let $a,b \in R(F)$. We are going to show that $a-b \in R(F)$ and $ab^{-1} \in R(F)$, provided $b \neq 0$. Assume $a,b \uparrow 1$. Then by DF5′ $1 \uparrow a-b$, or $1 \uparrow b-a$ and $-1 \uparrow a-b$. If $a,b \uparrow -1$, then we proceed analogously. Let $a \uparrow 1$, $b \uparrow -1$; then DF5 yields $a-b \uparrow 1$. Notice that DF5 gives also $a,b \uparrow 1 \Rightarrow a+b \uparrow 1$. Thus we have proved $a,b \in R(F) \Rightarrow a-b \in R(F)$, and $a,b \in P(F) \Rightarrow a+b \in P(F)$.

Let $b \neq 0$, $e = b^{-1}$. If $b \uparrow 1$ then we have $1 \cdot e \uparrow b \cdot e$, and thus $b^{-1} \uparrow 1$. Analogously $b \uparrow -1 \Rightarrow b^{-1} \uparrow 1$. Let $a,b \uparrow 1$. Then $ab^{-1} \uparrow a \cdot 1 \uparrow 1$. Other cases are considered analogously. Thus we have proved $a,b \in R(F)$, $b \neq 0 \Rightarrow ab^{-1} \in R(F)$, and $a,b \in P(F) \Rightarrow a \cdot b \in P(F)$. Finally by DF3 we obtain $a \in P(F) \wedge -a \in P(F) \Rightarrow a = 0$, and clearly $a \in R(F) \Rightarrow a \in P(F) \vee -a \in P(F)$ (by DF6). Therefore P(F) is a positive cone in R(F) and thus R(F) is ordered by P(F).

C o r o l l a r y 2. F can be considered as a linear space over R(F).

L e m m a 3. $a,b \neq 0 \Rightarrow (a \uparrow b \Leftrightarrow (\exists\, \lambda \in P(F) \setminus \{0\})$ $[a = \lambda b])$.

P r o o f . Let $a,b \neq 0$, $a \uparrow b$. Consider $\lambda = ab^{-1}$. We have $\lambda = ab^{-1} \uparrow bb^{-1} = 1$. Now if $\lambda \uparrow 1$, $a = \lambda b$, then we have $b \cdot 1 \uparrow b\lambda = a$.

C o r o l l a r y 4. A(F) is an ordered affine space (with directed parallelity of segments) (see [1]).

An axiom system for directed complex fields, and their geometry. Given any two fields $F_1 \subseteq F_2$ such that F_1 is ordered one can define relation $\uparrow$ (as in Lemma 4). The resulting structure is a directed field then. When we want to use our system in geometry (especially in Euclidean geometry) we must assume that F forms a complex field and R(F) consists of fix points of conjugacy. A directed complex field will be any system

$\mathbf{F} = \langle F; +, \cdot, {}^{-}, 0, 1, \uparrow \rangle$ such that

$\langle F; +, \cdot, {}^{-}, 0, 1 \rangle$ is a complex field (see [3]),

$\langle F; +, \cdot, 0, 1, \uparrow \rangle$ is a directed field, and

DCF: $a \uparrow b \to \bar{a} \uparrow \bar{b}$

$a+\bar{a} \uparrow 1 \vee a+\bar{a} \uparrow -1$.

Theorem 6. A system $\mathbf{F} = \langle F; +, \cdot, {}^{-}, 0, 1, \uparrow \rangle$ is a directed complex field iff $\langle F; +, \cdot, {}^{-}, 0, 1 \rangle$ is a complex field, $\langle F; +, \cdot, 0, 1, \uparrow \rangle$ is a directed field and $\mathrm{Fix}(^{-}) = R(F)$.

Proof. Let F be a complex directed field. Assume $x \in R(F)$, $x \uparrow 1$. Then $\bar{x} \uparrow \bar{1}$ and $x-\bar{x} \uparrow 1$ or $\bar{x}-x \uparrow 1$. In both cases we get $x-\bar{x} \uparrow 1$, $\bar{x}-x \uparrow 1$. Therefore $x = \bar{x}$. Similarly $x \uparrow -1$ yields $x = \bar{x}$. Let $x = \bar{x}$. Then $x+\bar{x} = 2x \uparrow 1$ and $x \uparrow 1$ or $x+\bar{x} \uparrow -1$, $x \uparrow -1$.

Conversely, if F is a complex field with directedness and $\mathrm{Fix}(^{-}) = R(F)$ then F can be considered as an Euclidean plane over R(F) (see [3]) and also as an ordered affine plane over R(F) (see Theorem 5). Operation $^{-}$ is an axial symmetry and thus this function preserves directed parallelity of vectors. On the second hand $a+\bar{a} \in \mathrm{Fix}(^{-})$, thus $a+\bar{a} \in R(F)$. This proves DCF.

Given a directed complex field $\mathbf{F} = \langle F; +, \cdot, {}^{-}, 0, 1, \uparrow \rangle$ we define the ordered Euclidean plane $\mathbf{E}(\mathbf{F})$ to be the structure $\langle F; \equiv_F, B_F \rangle$ where for $a,b,c,d \in F$ we have $ab \equiv_F cd \Leftrightarrow (a-b)(\overline{a-b}) = (c-d)(\overline{c-d})$ and $B_F(a,b,c) \Leftrightarrow a-b \uparrow b-c$. The structures $\langle F; \equiv_F \rangle$ i.e. unordered Euclidean planes were broadly investigated in [4]. The directedness $\uparrow$ allows us only to introduce a geometrical order to this plane. We shall not develop the analytical theory of ordered Euclidean planes in more details. Dealing with notions specific for ordered geometrical structures we shall only remark that positive dilatations of $\mathbf{E}(\mathbf{F})$ are transformations $f(x) = ax+b$ with $a \uparrow 1$, $a \neq 0$.

As we have noted above directed complex field can be considered as an algebraic counterpart of ordered Euclidean planes, or as a system for developing analytical geometry of such planes, without coordinates (comp. [3]). Let us recall that any such plane can be obtained as a Gauss-plane (L:K) with L being a quadratic extension $L = K(\sqrt{k}\,)$, k not a square in K (see [2,5]). The field L can be considered as a complex field with $R(L) = K$, and conversely every complex field F is a quadratic extension of R(F). Notice that if K is ordered (e.g. if $K = Q$) then we have two possibilities: $k > 0$ or $k < 0$. The second one is more "classical" ($k = -1$) and the geometry we obtain is more "Euclidean". For example the following is true: $B(apb) \wedge pa \equiv pb \equiv pc \wedge cx \perp ab \wedge a \neq b \wedge \wedge L(abx) \Longrightarrow B(abx)$. This is not the case if we can choose $k > 0$. The resulting geometrical theory is a mixture of Euclidean and Minkowskian geometry. And it seems of some interest to investigate such planes for their own.

In terms of directed complex fields the two cases can be distinguished as follows:

Euclidean: $(\forall x)[x\bar{x} \uparrow 1]$

Eucl/Mink: $(\exists x)[x\bar{x} \uparrow -1 \wedge x \neq 0]$

R e m a r k 7. Other plane geometries can also be developed in suitable algebras L over K, for $L = K(\sqrt{k})$. If $k = 0$, then we obtain Galileian geometry, and if k is a square in K then we obtain Minkowskian geometry (c.f. [7]). When we look for a natural notion which would suffice to introduce an order to these planes we see that the directedness (defined as in Lemma 4) is a suitable relation. In particular a convenient algebraic counterpart of ordered Minkowskian (Galileian) plane geometry is the notion of directed pseudo-complex algebra (dual numbers) and of directed double numbers resp. These algebraic systems can be characterized as follows

$$H = \langle H;+,\cdot,\bar{\ },0,1,\uparrow\rangle$$

$\langle H;+,\cdot,^{-},0,1\rangle$ is a suitable algebra (c.f. [7]);
DCF holds;
DF1 - DF6 are true.

R e m a r k 8. One can try to construct axiomatic systems to describe directed complex fields in more algebraic or more geometrical way. It's worthwhile to mention one such system. The structures under consideration are $\langle F;+,0,\omega,\dagger\rangle$ where $\langle F;+,0\rangle$ is an abelian group, ω is an antiinvolutory automorphism of this group and $\dagger$ is a suitable directedness. From the results of [4] it follows that the Euclidean plane can be constructed now. Other systems of this kind has been recently developed by M.Muzalewski.

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