DEMONSTRATIO MATHEMATICA
Vol. XX No 3-4 1987

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ON T_i-CLIQUISH FUNCTIONS

Let X be a topological space and let $\ddot{U}$ be an open cover of X. The cover $\ddot{U}$ is called T_2-open (T_3-open) [6] if for every $U \in \ddot{U}$, the interior of $X \setminus U$ is not empty (there are open sets W and W′ such that $W \subset \overline{W} \subset W' \subset X \setminus U$). Throughout this article every open cover will be called T_1-open cover. Also in [6], K.R. Gentry and H.B.Hoyle have used the concept of T_i-open cover to define a T_i-continuous functions as follows:

Definition 1. A function $f:X \to Y$ is T_i-continuous if for every T_i-open cover $\ddot{U}$ of Y there exists an open cover $\ddot{O}$ of X such that if $O \in \ddot{O}$, then there is a $U \in \ddot{U}$ such that $f(O) \subset U$ [6].

It is easy to verify that a function $f:X \to Y$ is T_i-continuous if and only if for every $x \in X$ and for each T_i-open cover $\ddot{U}$ of Y there exist an open set $W \subset X$ containing x and a $U \in \ddot{U}$ such that $f(W) \subset U$.

Definition 2. A function $f:X \to Y$ is said to be T_i-cliquish at a point $x \in X$ if for each T_i-open cover $\ddot{U}$ of Y and for every open set $O \subset X$ containing x there exist an open non-empty set $G \subset O$ and a $U \in \ddot{U}$ such that $f(G) \subset U$. A function f is T_i-cliquish if it is T_i-cliquish at every point.

It is easy to see that such functions are natural generalizations of T_i-continuous functions. The following example shows that the class of T_i-cliquish functions is greater than the class of T_i-continuous functions.

E x a m p l e 1. Let R be the space of real numbers with the natural topology. Let $X = [0,\infty) \subset R$ have the usual subspace topology and let $Y = [0,\infty) \subset R$ have the topology generated by $\ddot{U} = \{\{0\},[1,\infty)\} \cup \{[1/(n+1),1/n): n=1,2,\dots\}$. We shall demonstrate that the function $f:X \to Y$ given by $f(x) = x$ for all $x \in X$ is T_1-cliquish but is not T_3-continuous. At first let $x = 0 \in X$. Let $\ddot{A}$ be an open cover of Y and let $W \subset X$ be an open set containing x. Then there exist a number n' and $A \in \ddot{A}$ such that $G = (1/n',1/n'-1) \subset W$ and $f(G) \subset A$; so f is T_1-cliquish at x. Secondly if $x \in \{1,1/2,1/3,\dots\}$ and $\ddot{A}$ is an open cover of Y, then there exists a number n' and $A \in \ddot{A}$ such that $f(x) = x = 1/n' \in \ddot{A}$; so $[1/n',1/(n'-1)) \subset A$ and, for every open set $W \subset X$ containing x there exists an open non-empty set $G \subset W$ $(G = W \subset [1/n',1/(n'-1))$ such that $f(G) \subset A$. Thus f is T_1-cliquish at x. We observe that f is continuous at each $x \in X \setminus \{0,1,1/2,1/3,\dots\}$. So the T_1-cliquishness of f is shown. Clearly, f is not T_3-continuous at $x = 0$. Indeed, the collection $\ddot{U}$ is a T_3-open cover of Y and, for every open set $W \subset X$ containing x we have $f(W) \not\subset U$ for any $U \in \ddot{U}$. Thus, f is not T_3-continuous. Since any T_i-continuous (T_i-cliquish) function is T_{i+1}-continuous (T_{i+1}-cliquish), $i = 1,2$, it follows that the function f is also T_2-cliquish and T_3-cliquish; and f is not T_2-continuous and f is not T_1-continuous.

D e f i n i t i o n 3. A function $f:X \to Y$ is quasi-continuous at a point $x \in X$ if for each open set $V \subset Y$ containing $f(x)$ and for every open set $O \subset X$ containing x there exists an open non-empty set $G \subset O$ such that $f(G) \subset V$ [1,7,10, 12,14].

(This types of functions has also been defined by N.Levine [8] who used the term semi-continuous).

The following diagram illustrates the relations between these classes of functions.

$$\begin{array}{ccc}
f \text{ is } T_3\text{-continuous} & \Longrightarrow & f \text{ is } T_3\text{-cliquish} \\
\Uparrow & & \Uparrow \\
f \text{ is } T_2\text{-continuous} & \Longrightarrow & f \text{ is } T_2\text{-cliquish} \\
\Uparrow & & \Uparrow \\
f \text{ is } T_1\text{-continuous} & \Longrightarrow & f \text{ is } T_1\text{-cliquish} \\
\Uparrow & & \Uparrow \\
f \text{ is continuous} & \Longrightarrow & f \text{ is quasi-continuous}
\end{array}$$

Definition 4. A function f from a topological space Y with a uniformity $\mathcal{U}$ is cliquish at a point $x \in X$ if for every open set $O \subset X$ containing x and for every $V \in \mathcal{U}$ there exists an open non-empty set $G \subset O$ such that $(f(x'),f(x'')) \in V$ for any $x',x'' \in G$. A function f is cliquish if it is cliquish at every point [2,3]. If a uniformity $\mathcal{U}$ is induced by a metric on Y, then the above definition coincides with the well known definition of the cliquishness [1,5,9,12,14].

Let Y be a uniform space with a uniformity $\mathcal{U}$. Simultaneously we consider Y as a topological space with the topology induced by the uniformity $\mathcal{U}$.

Proposition 1. Let Y be a uniform space with a uniformity $\mathcal{U}$. If a function $f:X \to Y$ is T_3-cliquish at a point $x \in X$, then f is cliquish at its point.

Proof. Let O be any open set containing x and let $V \in \mathcal{U}$. There exists $W \in \mathcal{U}$ such that $W = W^{-1}$ and $W^2 \subset V$. Since the collection $\{\mathrm{Int}(B(y,W)): y \in Y\}$ is a T_3-open cover of the topological space Y with the topology induced by $\mathcal{U}$, where $B(y,W) = \{s \in Y: (y,s) \in W\}$, there exist an open non-empty set $G \subset O$ and $y' \in Y$ such that $f(G) \subset \mathrm{Int}(B(y',W))$. Then $(f(x'),f(x'')) \in W^2 \subset V$ for any $x',x'' \in G$ and, consequently, f is cliquish at x.

The following example shows that a cliquish function need not be T_3-cliquish.

Example 2. Let R be the space of real numbers with the natural metric and let $Y = (0,\infty) \subset R$ have the usual subspace topology. Let us consider the set $X = [0,\infty)$ with the topology $T = \{\emptyset, X\} \cup \{(r,\infty): r > 0\}$. By Q we denote the set of rational numbers. We denote $a_n = \frac{3n+4}{3(n+1)(n+2)}$ and $b_n = \frac{3n+2}{3n(n+1)}$ for $n = 1,2,\dots$. Define $f:X \longrightarrow Y$ by

$$f(x) = \begin{cases} b_n & \text{if } x \in [n-1,n) \cap Q \\ a_n & \text{if } x \in [n-1,n) \setminus Q,\ n = 1,2,\dots \end{cases}$$

The function f is clearly cliquish. Note, however, that the collection $\ddot{U} = \{(a_n,b_n):\ n = 1,2,\dots\} \cup \{(4/6,\infty)\}$ is a T_3-open cover of Y and for every non-empty set $G \subset X$ and for every $U \in \ddot{U}$ we have $f(G) \not\subset U$. Thus, f is not T_3-cliquish.

P r o p o s i t i o n 2. Any cliquish function f from a topological space X into a uniform compact space Y is T_1-cliquish.

P r o o f . Let $x \in X$ and let $W \subset X$ be any open set containing x. If $\ddot{U}$ is an open cover of Y, then there exists a $V \in \mathcal{U}$ such that if $A \in \ddot{O} = \{B(y,V):\ y \in Y\}$ then there is a $U \in \ddot{U}$ such that $A \subset U$ since Y is compact. By the cliquishness of f there exists an open non-empty set $G \subset W$ such that $(f(x'),f(x'')) \in V$ for every $x',x'' \in G$. Hence there exists set $A' \in \ddot{O}$ and exists set $U \in \ddot{U}$ such that $f(G) \subset A' \subset U'$ and the proof is complete.

The following two results give simple characterization of T_i-continuous and T_i-cliquish functions.

P r o p o s i t i o n 3. A function $f:X \longrightarrow Y$ is T_i-continuous if and only if for every T_i-open cover U of Y the collection $\{\mathrm{Int}\ f^{-1}(U):\ U \in \ddot{U}\}$ covers X.

P r o o f . Assume that f is T_i-continuous. Let $\ddot{U}$ be an T_i-open cover of Y and let $x \in X$. Then there exist an open set $W \subset X$ containing x and a $U \in \ddot{U}$ such that $f(W) \subset U$. Thus $x \in W \subset \mathrm{Int}\ f^{-1}(U) \subset \bigcup \{\mathrm{Int}\ f^{-1}(U):\ U \in \ddot{U}\}$.

Conversely, let $x \in X$ and let $\ddot{U}$ be an T_i-open cover of Y. Since the collection $\{\mathrm{Int}\ f^{-1}(U):\ U \in \ddot{U}\}$ covers X there exists a $U \in \ddot{U}$ such that $x \in \mathrm{Int}\ f^{-1}(U)$. Hence $f(\mathrm{Int}\ f^{-1}(U)) \subset U$ and f is T_i-continuous at x.

P r o p o s i t i o n 4. A function $f:X \to Y$ is T_i-cliquish if and only if for every T_i-open cover U of Y we have $\overline{\bigcup\{\mathrm{Int}\, f^{-1}(U):\ U \in \ddot{U}\}} = X$.

P r o o f . Assume that f is T_i-cliquish. Let $\ddot{U}$ be an T_i-open cover of Y and let $x \in X$. For every open set $W \subset X$ containing x there exist an open non-empty set $G \subset W$ and a $U \in \ddot{U}$ such that $f(G) \subset U$. Thus $G \subset \mathrm{Int}\, f^{-1}(U) \subset \bigcup\{\mathrm{Int}\, f^{-1}(U):\ U \in \ddot{U}\}$, which implies that $x \in \overline{\bigcup\{\mathrm{Int}\, f^{-1}(U):\ U \in \ddot{U}\}}$.

Let us assume now that $\overline{\bigcup\{\mathrm{Int}\, f^{-1}(U):\ U \in \ddot{U}\}} = X$ for every T_i-open cover $\ddot{U}$ of Y. If $W \subset X$ is an open set containing $x \in X$, and $\ddot{U}$ is an T_i-open cover of Y, then $W \cap (\bigcup\{\mathrm{Int}\, f^{-1}(U): U \in \ddot{U}\}) \neq \emptyset$. So $\emptyset \neq W \cap \mathrm{Int}\, f^{-1}(U') \subset W$ for some $U' \in \ddot{U}$ and, $f(W \cap \mathrm{Int}\, f^{-1}(U')) \subset U'$. Hence f is T_i-cliquish at x and the proof is complete.

For any function $f:X \to Y$ we will denote by $A_i(f)$ and $C_i(f)$ the set of all T_i-cliquishness and T_i-continuity points of f respectively.

P r o p o s i t i o n 5. Let $f:X \to Y$ be an arbitrary function. There exists a family $\{G_s: s \in S\}$ of open subsets of X such that $C_i(f) = \bigcap\{G_s:\ s \in S\}$ and $A_i(f) = \bigcap\{\bar{G}_s:\ s \in S\}$.

P r o o f . For any T_i-open cover $\ddot{U}$ of Y, let $G(\ddot{U})$ be the set of all points $x \in X$ at which the following condition is satisfied: there exist an open set $W \subset X$ containing x and a $U \in \ddot{U}$ such that $f(W) \subset U$. It is obvious that the set $G(\ddot{U})$ is open. We shall prove that $C_i(f) = \bigcap\{G(\ddot{U}):\ \ddot{U}$ is an T_i-open cover of $Y\}$.

If f is T_i-continuous at a point x, then for every T_i-open cover $\ddot{U}$ of Y there exist an open set $W \subset X$ containing x and a $U \in \ddot{U}$ such that $f(W) \subset U$; so $x \in G(\ddot{U})$ for every T_i-open cover $\ddot{U}$ of Y.

Conversely, if for each T_i-open cover $\ddot{U}$ of Y $x \in G(\ddot{U})$, then evidently f is T_i-continuous at x.

Now we shall prove that

$$A_i(f) = \bigcap\{\overline{G(\ddot{U})}:\ \ddot{U} \text{ is an } T_i\text{-open cover of } Y\}.$$

Assume that f is T_i-cliquish at a point x. Let $W \subset X$ be an open set containing x. If $\ddot{U}$ is an T_i-open cover of Y, then there exist an open non-empty set $G \subset W$ and a $U \in \ddot{U}$ such that $f(G) \subset U$. It is obvious that $G \subset G(U)$. Thus $x \in \overline{G(\ddot{U})}$ for every T_i-open cover $\ddot{U}$ of Y.

Conversely, let $x \in \overline{G(\ddot{U})}$ for every T_i-open cover $\ddot{U}$ of Y. We observe that for every T_i-open cover $\ddot{U}$ of Y the set $G(\ddot{U})$ is open, then for every open set $W \subset X$ containing x and for every T_i-open cover $\ddot{U}$ of Y there exists an open non-empty set $G \subset W$ such that $G \subset W \cap G(\ddot{U})$ and, by the definition of $G(\ddot{U})$, there exist an open non-empty set $G' \subset G$ and a $U \in \ddot{U}$ such that $f(G') \subset U$. Hence f is T_i-cliquish at x and the proof is complete.

C o r o l l a r y 1. For each function f the set $A_i(f)$ is closed.

C o r o l l a r y 2. If for a function $f:X \longrightarrow Y$ the set $C_i(f)$ is dense in X, then f is T_i-cliquish.

P r o o f . It follows by: $X = \overline{C_i(f)} =$

$$= \bigcap\left\{\overline{G(\ddot{U}): \ddot{U} \text{ is an } T_i\text{-open cover of } Y}\right\} \subset$$

$$\subset \bigcap\left\{\overline{G(\ddot{U})}: \ddot{U} \text{ is an } T_i\text{-open cover of } Y\right\} = A_i(f).$$

C o r o l l a r y 3. If f is a function of a Baire space X with values in a topological space Y and the set $X \setminus C_i(f)$ is of the first category, then f is T_i-cliquish.

Since any function f of a Baire space X into a metric space is cliquish if and only if the set $X \setminus C(f)$ is of the first category [5,12], where C(f) is the set of all continuity points of f, we have the following corollary:

C o r o l l a r y 4. Any cliquish function f from a Baire space X into a metric space Y is T_1-cliquish.

P r o o f . If f is cliquish, then the set $X \setminus C(f)$ is of the first category and since $C(f) \subset C_1(f)$, the set $X \setminus C_1(f)$ is of the first category. Since X is a Baire space, the set $C_1(f)$ is dense in X. Thus by Corollary 2, f is T_1-cliquish.

We say that a topological space X has property P_i if, there exists a sequence $\{U_n: n = 1,2,\dots\}$ of T_i-open covers of X such that:

if $\ddot{O}$ is an T_i-open cover of X, then there exists a number n' such that if $U \in \ddot{U}_n$, then there is a $O \in \ddot{O}$ such that $U \subset O$.

Lemma 1. If f is a function into a topological space Y with the property P_i, then there exists a sequence $\{\ddot{U}_n: n = 1,2,\dots\}$ of T_i-open covers of Y such that $C_i(f) = \bigcap\{G(\ddot{U}_n): n = 1,2,\dots\}$ and $A_i(f) = \bigcap\{\overline{G(\ddot{U}_n)}: n = 1,2,\dots\}$, where $G(\ddot{U}_n)$ is as in Proposition 5.

Proof. The proof is simple and is thus omitted.

Corollary 5. If f is a function into a topological space with the property P_i, then the set $C_i(f)$ is G_δ.

Proposition 6. A function f of a Baire space X into a topological space Y with the property P_i is T_i-cliquish if and only if the set $X \setminus C_i(f)$ is of the first category.

Proof. If f is T_i-cliquish, then by Lemma 1, $X \setminus C_i(f) = \bigcap\{\overline{G(\ddot{U}_n)}: n = 1,2,\dots\} \setminus \bigcap\{G(\ddot{U}_n): n = 1,2,\dots\} \subset \bigcup\{\overline{G(\ddot{U}_n)} \setminus G(\ddot{U}_n): n = 1,2,\dots\}$ and since for every number n the set $G(\ddot{U}_n)$ is open, the set $\overline{G(\ddot{U}_n)} \setminus G(\ddot{U}_n)$ is nowhere dense and hence the set $X \setminus C_i(f)$ is of the first category.

Conversely, if the set $X \setminus C_i(f)$ is of the first category, then by Corollary 3, f is T_i-cliquish.

A topological space X is called nearly T_i-space, i=1,2,3, if for each point $x \in X$ and for every open set $W \subset X$ containing x there exists a T_i-open cover $\ddot{U}$ of X, i = 1,2,3, respectively, such that $St(x,\ddot{U}) \subset W$ [13], where $St(x,\ddot{U})$ denotes the star of the point x with respect to $\ddot{U}$. Clearly, nearly T_i, i = 1,2,3, is strictly weaker than T_i, i = 1,2,3, respectively.

From the fact that any T_i-continuous function, i = 1,2,3, into a nearly T_i-space, i = 1,2,3, respectively, is continuous [13], we obtain the following result:

Corollary 6. A function f of a Baire space X **into a nearly T_i-space Y with the property P_i is T_i-cliquish if and only if the set $X \setminus C(f)$ is of the first category.**

P r o o f . It follows immediately from Proposition 6.

A function $f:X \to Y$ is barely continuous, if for every non-empty closed set $M \subset X$ the restriction f/M has at least one point of the continuity [11].

The following result improves Theorem 1.1 of [3].

P r o p o s i t i o n 7. Any barely continuous function $f:X \to Y$ is T_1-cliquish.

P r o o f . Let $x \in X$ and let $\ddot{U}$ be an open cover of Y. If $O \subset X$ is an open set containing x, then by the barely continuity of f there exists a point $x' \in \bar{O}$ of the continuity of $f/\bar{O}$ and, consequently, $f/\bar{O}$ is T_1-continuous at x'. Then there exists an open set $W \subset X$ containing x' such that $f(W \cap O) \subset U'$ for some $U' \in \ddot{U}$. Thus $G = W \cap O \subset O$ is an open non-empty set such that $f(G) \subset U'$. This means that f is T_1-cliquish at x and the proof is complete.

A topological space X is said to be a Baire space in the narrow sense, if every closed subspace of X is a Baire space [4].

P r o p o s i t i o n 8. A function f of a Baire space in the narrow sense X into a nearly T_1-space Y with the property P_1 is barely continuous if and only if for every non-empty closed set $M \subset X$ a function f/m is T_1-cliquish.

P r o o f . If f is barely continuous and $M \subset X$ is a closed non-empty set, then f/M is clearly barely continuous; so by Proposition 7 the function f/M is T_1-cliquish.

Conversely, if the restriction f/M is T_1-cliquish for any closed non-empty set $M \subset X$, then by Corollary 6, the set $C(f/M)$ is dense in M. It follows that $C(f/M) \neq \emptyset$ so that f is barely continuous.

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Received January 15, 1986.