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ON THE DOMAIN OF LOCAL UNIVALENCE AND STARLIKENESS IN A CERTAIN CLASS OF HOLOMORPHIC FUNCTIONS

Introduction

Let $\mathbf{C}$ be the complex plane and $E = \{z \in \mathbf{C} : |z| < 1\}$ the unit disk. Denote by H the class of functions holomorphic in the disk E, satisfying the condition

$$\operatorname{Re}\left\{(1 - z^2)\,\frac{f(z)}{z}\right\} \geq 0, \quad z \in E, \tag{1}$$

and such that $f(0) = 0$, $f'(0) = 1$.

If the coefficients of the function f satisfying condition (1) are real, then the function is typically real *).

The class H is then a proper superclass of T_r.

Some results for the class H were established by Hengartner and Schober [4]. The condition (1) can be written in the form

$$f(z) = \frac{z}{1 - z^2}\,p(z), \quad z \in E, \tag{2}$$

where $p \in P$, P being the class of functions p holomorphic in E and such that $\operatorname{Re} p(z) > 0$ for $z \in E$ and $p(0) = 1$.

*) The function f holomorphic in E and such that $f(0) = 0$, $f'(0) = 1$, is said to be typically real if it takes the real values on the segment $(-1,1)$ of the real axis and satisfies the condition $\operatorname{Im} z \cdot \operatorname{Im} f(z) > 0$ for $z \in E \setminus (-1,1)$. The class of typically real functions will be denoted by T_r.

Making use of Herglotz's formula for the function $p \in P$ we can write condition (1) in the form

$$f(z) = \int_{-\pi}^{\pi} \frac{z}{1-z^2} \frac{1+ze^{-it}}{1-ze^{-it}} d\mu(t), \tag{3}$$

where μ is a real function, non decreasing in the interval $[-\pi, \pi]$ and such that

$$\int_{-\pi}^{\pi} d\mu(t) = 1.$$

1. The domain of local univalence in the class H

We will determine the domain of local univalence in the class H. The method used in order to characterize the boundary functions and the boundary points in this problem is a modification of the method applied by Burštein [2] in the study of another problem. This modification required, however, some more precise statements.

Let z(f) denote the set of zeros of the derivative of the function f in the disk E, that is

$$z(f) = \{z \in E : f'(z) = 0\}.$$

The points of the set z(f) are said to be the critical points of the function f.

Denote by B the set $B = \bigcup_{f \in H} z(f)$. It is obvious that the set $D = E \setminus B$ does not contain critical points of the function f, that is $f'(z) \neq 0$ for any $z \in D$ and any function $f \in H$. Hence, D is the set of local univalence in the class H.

The set D is symmetrical with respect to the real axis. This follows from the fact that if the function f belongs to the class H, then the function g defined as

$$g(z) = \overline{f(\bar{z})}, \quad z \in E,$$

also belongs to the class H. Moreover, Re $f'(z) > 0$ for $z \in (-1,1)$, hence D contains the segment $(-1,1)$ of the real axis. Since the class H is compact, the set B of critical points is closed in the space E (with respect to the disk E).

Let $\Gamma = \partial B \cap E$, so $\Gamma = \partial D \cap B$.

D e f i n i t i o n 1. The point $z_0 \in \Gamma$ is said to be a regular boundary point of the set B, if there exists an $a \notin B$ such that $|z - a| \geq |z_0 - a|$ for any $z \in B$. If such an a does not exist, then z_0 is said to be a singular boundary point. This definition was introduced by M.Biernacki [1] (see also Lebedev [6]).

Denote by $\Gamma_0 \subset \Gamma$ the set of all regular boundary points of Γ. It is known (M.Biernacki [1], 13) that Γ_0 is dense in Γ, hence in order to find the boundary of Γ it suffices to determine the set Γ_0.

The function $f_0 \in H$ such that $f_0'(z_0) = 0$, where z_0 is a boundary point of the set B, is said to be a boundary function.

To determine the boundary functions and the boundary points in this problem we make use of the variational formulas of Goluzin [3] in the class of functions defined by the structural formula of the form

$$f(z) = \int_a^b s(z,t)\, d\mu(t),$$

where the function μ is non decreasing in the interval $[a,b]$ and such that $\int_a^b d\mu(t) = 1$, while s is a function (as function of the variables z, t) continuous with derivative $s_t'(z,t)$ in the set $E \times [a,b]$, and being regular in E for any $t \in [a,b]$.

The first variational formula has the form

$$(4) \qquad f_*(z) = f(z) + \lambda \int_{t_1}^{t_2} s_t'(z,t)|\mu(t) - c|\, dt,$$

where $\lambda \in [-1,1]$, c is a constant with respect to t and λ (depending on the sign of λ) and t_1, t_2, $t_1 < t_2$, are arbitrary numbers from the interval $[a,b]$.

In what follows we shall make use of (4) in the following developed form

$$(5) \qquad f_1(z) = f(z) + \lambda \int_{t_1}^{t_2} s_t'(z,t)\,|\mu(t) - c_1|\,dt,$$

$$(6) \qquad f_2(z) = f(z) - \lambda \int_{t_1}^{t_2} s_t'(z,t)\,|\mu(t) - c_2|\,dt,$$

where

$$c_1 = \lim_{t \to t_1^-} \mu(t), \qquad c_2 = \lim_{t \to t_2^+} \mu(t), \qquad \lambda \in [0;\ 1]$$

and where we assume that $\lim_{t \to a^-} \mu(t) = \mu(a)$, $\lim_{t \to b^+} \mu(t) = \mu(b)$.

The second variational formula has the form

$$(7) \qquad f_{**}(z) = f(z) + \lambda\left[s(z,t_1) - s(z,t_2)\right],$$

where t_1, t_2, $a \leqslant t_1 < t_2 \leqslant b$, are discontinuity points of the function μ and λ is sufficiently small.

Making use of (3), (4), we obtain

$$(8) \qquad f_*'(z) = f_0'(z) + \lambda \int_{t_1}^{t_2} \frac{d}{dz}\,(s_t'(z,t))\,|\mu(t) - c|\,dt,$$

$$\lambda \in [-1;1].$$

Taking account of (5) and (6), we can write (8) in the form

$$(9) \qquad f_1'(z) = f_0'(z) + \lambda \int_{t_1}^{t_2} \frac{d}{dz}\,(s_t'(z,t))\,|\mu(t) - c_1|\,dt,$$

$$\lambda \in [0;1],$$

$$(10)\qquad f_2'(z) = f_0'(z) - \lambda \int_{t_1}^{t_2} \frac{d}{dz}\,(s_t'(z,t))\,|\mu(t) - c_2|\,dt, \qquad \lambda \in [0;1],$$

where

$$(11)\qquad \frac{d}{dz}\,(s_t'(z,t)) = \frac{-4ize^{-it}}{(1-z^2)^2} \cdot \frac{1-z^3e^{-it}}{(1-ze^{-it})^3}\,.$$

The second variational formula may be written as follows

$$(12)\qquad f_{**}'(z) = f_0'(z) + \lambda\left[\frac{d}{dz}\,s(z,t_1) - \frac{d}{dz}\,s(z,t_2)\right],$$

where λ is sufficiently small and

$$(13)\qquad \frac{d}{dz}\,(s(z,t)) = \frac{1+z^2}{(1-z^2)^2} \cdot \frac{1+ze^{-it}}{1-ze^{-it}} + \frac{2ze^{-it}}{(1-z^2)(1-ze^{-it})^2}\,.$$

Set $F_z'(z,\lambda) = f_*'(z)$; then formula (8) becomes

$$(14)\qquad F_z'(z,\lambda) = f_0'(z) + \lambda A(z,c),$$

where

$$(15)\qquad A(z,c) = \int_{t_1}^{t_2} \frac{d}{dz}\,(s_t'(z,t))\,\left|\mu_{f_0}(t) - c\right|dt$$

and $\frac{d}{dz}\,(s_t'(z,t))$ is given by formula (11).

Let z_0 be a single zero of the function $f_0'(z)$, that is $f_0'(z_0) = 0$ and $f_0''(z_0) \neq 0$. Denote by $z(\lambda)$ the zero of the function $f_z'(z,\lambda)$ which is nearest to z_0. Then we have

$$\lim_{\lambda \to 0} z(\lambda) = z_0, \qquad z(0) = z_0.$$

The function $z(\lambda)$ has a derivative for $\lambda = 0$ and

$$z'(0) = -\frac{1}{f_0''(z_0)}\,A(z_0,c) = h(z_0,c),$$

where A(z,c) is given by formula (15); hence

$$z^* := z(\lambda) = z_0 + \lambda h(z_0,c) + o(\lambda), \text{ where } \lim_{\lambda \to 0} \frac{o(\lambda)}{\lambda} = 0.$$

In view of (15), we have

$$(16)\quad z^* = z_0 + \lambda \frac{-1}{f_0''(z_0)} \int_{t_1}^{t_2} \frac{-4iz_0 e^{-it}}{(1-z_0^2)^2} \cdot \frac{1-z_0^3 e^{-it}}{(1-z_0 e^{-it})^3} \left| \mu_{f_0}(t) - c \right| dt + o(\lambda).$$

Now, if z^{**} is a zero of the function f'_{**} given by (12), that is, if $f'_{**}(z^{**}) = 0$, then applying formula (7) and proceeding as before, we obtain

$$(17)\quad z^{**} = z_0 + \lambda \frac{1}{f_0''(z_0)} \left[\frac{d}{dz}(s(z_0,t_1)) - \frac{d}{dz}(s(z_0,t_2)) \right] + o(\lambda),$$

where λ is sufficiently small, t_1 and t_2 are discontinuity points of the function μ_{f_0} and $\frac{d}{dz}(s(z,t))$ is given by (13).

Assume now that z_0 is a regular boundary point. Then there exists an $a \notin B$ such that

$$(18)\quad |z^* - a|^2 \geqslant |z_0 - a|^2.$$

Further, we have

$$|z^* - a|^2 = |z_0 - a|^2 + 2\lambda \operatorname{Re}\left[(\bar{z}_0 - \bar{a}) h(z_0,c) \right] + o(\lambda),$$

whence, on account of formulas (9) and (10), we obtain

$$|z_1^* - a|^2 = |z_0 - a|^2 + 2\lambda \operatorname{Re}\left[(\bar{z}_0 - \bar{a}) h_1(z_0,c_1) \right] + o(\lambda),$$

$$|z_2^* - a|^2 = |z_0 - a|^2 - 2\lambda \operatorname{Re}\left[(\bar{z}_0 - \bar{a}) h_2(z_0,c_2) \right] + o(\lambda),$$

where $0 \leqslant \lambda < 1$ and

$$h_1(z_0,c_1) = -\frac{1}{f_0''(z_0)} A(z_0,c_1), \quad h_2(z_0,c_2) = -\frac{1}{f_0''(z_0)} A(z_0,c_2).$$

The inequality (18) implies the following ones

$$\operatorname{Re}\left[(\bar{z}_0-a)h_1(\bar{z}_0,c_1)\right] \leqslant 0, \qquad \operatorname{Re}\left[(\bar{z}_0-a)h_2(\bar{z}_0,c_2)\right] \geqslant 0,$$

which can be written in the form

$$(19) \qquad \int_{t_1}^{t_2} \Phi(t)\left|\mu_{f_0}(t) - c_1\right| dt \leqslant 0,$$

$$(20) \qquad \int_{t_1}^{t_2} \Phi(t)\left|\mu_{f_0}(t) - c_2\right| dt \geqslant 0,$$

where

$$(21) \quad \Phi(t) = \operatorname{Re}\left[\frac{-4iz_0e^{-it}}{(z_0-a)f_0''(z_0)(1-z_0^2)^2} \cdot \frac{1-z_0^3e^{-it}}{(1-z_0e^{-it})^3}\right].$$

If the equation $\Phi(t) = 0$ has no roots in the interval (t_1,t_2), then in inequality (19) we have $\Phi(t) < 0$ and the sign in inequality (20) must be inverted. Hence, it follows that

$$(22) \qquad \int_{t_1}^{t_2} \Phi(t)\left|\mu_{f_0}(t) - c_2\right| dt = 0.$$

By a similar argument, we find that also

$$(23) \qquad \int_{t_1}^{t_2} \Phi(t)\left|\mu_{f_0}(t) - c_1\right| dt = 0.$$

Therefore, $\mu_{f_0}(t)$ is constant in any interval (t_1,t_2) which does not contain zeros of $\Phi(t)$.

Let now t_1, t_2 be discontinuity points of the function μ_{f_0}. Calculating the expression $|z^*-a|^2$, $a \notin B$, applying formula (17) and taking into account the fact that $|z^*-a| \geq |z-a|$, we obtain the condition

$$\text{(24)} \qquad \operatorname{Re} \frac{1}{(z-a)f_0''(z_0)} \frac{d}{dz} \left[s(z_0,t_1) - s(z_0,t_2) \right] = 0.$$

This means that the function

$$\text{(25)} \qquad \psi(t) = \operatorname{Re} \frac{1}{(z_0-a)f_0''(z_0)} \frac{d}{dz} (s(z_0,t))$$

takes on equal values at the discontinuity points t_1, t_2 of the function μ_{f_0}. Hence, the function μ_{f_0} can be discontinuous only at points which are roots of the equation $\Phi(t) = 0$, $t \in [-\pi,\pi)$, and for which the function $\psi(t)$ takes on equal values. The equation $\Phi(t) = 0$ has the form

$$\text{(26)} \qquad \operatorname{Re} \left[\frac{-4iz_0}{(z_0-a)f_0''(z_0)(1-z_0^2)^2} \cdot \frac{e^{-it}(1-z_0^3e^{-it})}{(1-z_0e^{-it})^3} \right] = 0.$$

Setting $e^{it} = u$ and $\dfrac{iz_0}{(z_0-a)f_0''(z_0)(1-z_0^2)^2} = b$ we can write (26) in the equivalent form

$$b(u-z_0^3)(1-\bar{z}_0u)^3 + \bar{b}(1-\bar{z}_0^3u)(u-z_0)^3 = 0.$$

It is easily seen that this equation has at most 4 roots with respect to $u = e^{it}$, $-\pi \leq t < \pi$. Hence, the function μ_{f_0} can have at most 4 discontinuity points in the interval $[-\pi,\pi)$.

We shall prove that μ_{f_0} can have at most 2 discontinuity points. For the indirect proof assume that t_1, t_2, t_3, $-\pi \leq t_1 < t_2 < t_3 < \pi$, are discontinuity points of the function μ_{f_0}. Since $\psi(t_1) = \psi(t_2) = \psi(t_3)$, $\Phi(t_k) = 0$, $k=1,2,3$, and $\Phi(t) = \psi'(t)$, there exist points $t_4 \in (t_1,t_2)$ and $t_5 \in (t_2,t_3)$ such that $\Phi(t_4) = \psi'(t_4) = 0$ and $\Phi(t_5) = \psi'(t_5) = 0$. This would mean that the equation $\Phi(t) = 0$ has 5 roots in the interval $[-\pi,\pi)$ which is impossible.

Let now t_1, t_2, t_3, t_4, $-\pi \leqslant t_1 < t_2 < t_3 < t_4 < \pi$, be roots of the equation $\Phi(t) = 0$. It should be noted that the points t_1 and t_2 cannot be at the same time discontinuity points of μ_{f_0}. Otherwise the function $\psi(t)$ would take on equal values at these points; by $\psi'(t) = \Phi(t)$, this would lead to the conclusion that in the interval (t_1, t_2) there exists a root t_5 of the equation $\Phi(t) = 0$.

A similar argument shows that the points t_2 and t_3, resp. t_3 and t_4, cannot be at the same time discontinuity points of the function μ_{f_0}.

Assume now that the points t_1 and t_3 are discontinuity points of μ_{f_0}. In each of the intervals (t_1, t_2) and (t_2, t_3), the equation $\Phi(t) = 0$ has no roots. Hence, by virtue of formulas (22) and (23), we have

$$\mu_{f_0}(t) = c_1 = \lim_{t \to t_1^-} \mu(t) \quad \text{for} \quad t \in (t_1, t_2),$$

$$\mu_{f_0}(t) = c_2 = \lim_{t \to t_3^+} \mu(t) \quad \text{for} \quad t \in (t_2, t_3).$$

Moreover, since the points t_1 and t_3 are discontinuity points, we have $c_1 \neq c_2$. The point t_2 would then be a discontinuity point which is impossible, inasmuch as the function μ_{f_0} can have at most 2 discontinuity points. An analogous proof shows that the points t_2 and t_4, resp. t_1 and t_4, cannot be at the same time discontinuity points of μ_{f_0}.

From the preceding considerations it follows that if z_0 is a regular boundary point such that $f_0''(z_0) \neq 0$, then the corresponding boundary function has the form

$$f_0(z) = \frac{z}{1-z^2} \cdot \frac{1+ze^{-it}}{1-ze^{-it}}, \quad z \in E, \quad t \in [-\pi, \pi). \tag{27}$$

Let us note that for $t = -\pi$ and $t = \pi$ the boundary function is ~~the~~ ~~same~~ and has the form $f_0(z) = \dfrac{z}{(1+z)^2}$.

Let now z_0 be a zero of multiplicity m of the function f_0', that is, $(f_0'(z_0) = f_0''(z_0) = \dots = f_0^{(m)}(z_0) = 0$, $f_0^{(m+1)}(z_0) \neq 0$, $m \geq 2$, $|z_0| < 1$. Denote by z^* the zero of the function f_*' which is nearest to z_0 and belongs to an arbitrary small neighbourhood of z_0. Proceeding in the same way as in [2],we obtain

$$f_0'(z^*) = (z^*-z_0)^m \frac{f_0^{(m+1)}(z_0)}{m!} + (z^*-z_0)^{m+1} \frac{f_0^{(m+2)}(z_0)}{(m+1)!} + \dots .$$

Making use of formulas (9) and (10), we find

$$f_*'(z^*) = (z^*-z_0)^m \frac{f_0^{(m+1)}(z_0)}{m!} + \dots +$$

$$+ (-1)^{k+1} \lambda \int_{t_1}^{t_2} \frac{d}{dz} (s_t'(z^*,t)) \left| \mu_{f_0}(t)-c_k \right| dt,$$

$$k=1,2,\ \lambda \in [0,1].$$

Passing to the limit as $\lambda \to 0$ and in view of $\lim_{\lambda \to 0} z^* = z_0$, we obtain

$$\lim_{\lambda \to 0} \frac{(z^*-z_0)^m}{\lambda} = (-1)^k \frac{m!}{f_0^{(m+1)}(z_0)} \int_{t_1}^{t_2} \frac{d}{dz} (s_t'(z_0,t)) \left| \mu_{f_0}(t)-c_k \right| dt,$$

whence

$$z^* = z_0 + \sqrt[m]{(-1)^k \frac{m!\lambda}{f_0^{(m+1)}(z_0)} \int_{t_1}^{t_2} \frac{d}{dz} (s_t'(z_0,t)) \left| \mu_{f_0}(t)-c_k \right| dt} + o\left(\lambda^{\frac{1}{m}}\right),$$

where it is assumed that $\lambda^{\frac{1}{m}} > 0$. Hence

$$|z^*-a|^2 = |z_0-a|^2 + 2\mathrm{Re}\left\{(\bar{z}_0-\bar{a})(-1)^{\frac{k}{m}} \sqrt[m]{\lambda}\, \varphi(z_0,t_1,t_2,m)\right\} + o\left(\lambda^{\frac{1}{m}}\right),$$

where

$$\varphi(z_0,t_1,t_2,m) = \sqrt[m]{\frac{m!}{f_0^{(m+1)}(z_0)} \int_{t_1}^{t_2} \frac{d}{dz}\,(s_t'(z_0,t))\,\left|\mu_{f_0}(t)-c_k\right|dt}.$$

Since $|z^*-a|^2 \geqslant |z_0-a|^2$, we have

$$\operatorname{Re}\left\{(-1)^{\frac{k}{m}}\,(\bar{z}_0-\bar{a})\sqrt[m]{\lambda}\;\varphi(z_0,t_1,t_2,m)\right\} + o\left(\lambda^{\frac{1}{m}}\right) \geqslant 0.$$

Dividing the latter inequality by $\sqrt[m]{\lambda}$ and passing to the limit as $\lambda \to 0$, we obtain

$$\operatorname{Re}\left\{(-1)^{\frac{k}{m}}\,(\bar{z}_0-\bar{a})\varphi(z_0,t_1,t_2,m)\right\} \geqslant 0. \tag{28}$$

The inequality (28) implies

$$(\bar{z}-\bar{a})\;\varphi(z_0,t_1,t_2,m) = 0.$$

Since $(\bar{z}_0-\bar{a}) \neq 0$ and $f_0^{(m+1)}(z_0) \neq 0$, this gives

$$\int_{t_1}^{t_2} \frac{d}{dz}\,(s_t'(z_0,t))\,\left|\mu_{f_0}(t) - c_k\right|dt = 0. \tag{29}$$

By an argument analogous to the one applied in the case, where z_0 is a simple zero of $f_0'(z)$, we conclude that the boundary functions have the form (27). Hence, the regular boundary points $z_0 \in \Gamma_0$ are the solution of the equation $f_0'(z_0) = 0$ which belong to the disk E, the function f_0 being given by formula (27). The equation $f_0'(z) = 0$ has the form

$$\frac{1+z^2}{(1-z^2)^2}\cdot\frac{1+ze^{-it}}{1-ze^{-it}} + \frac{2ze^{-it}}{(1-z^2)(1-ze^{-it})^2} = 0, \tag{30}$$
$$z=z(t),\quad t \in [-\pi,\pi].$$

After some transformations we get

$$e^{-2it}z^4 + 2e^{-it}z^3 + (e^{-2it}-1)z^2 - 2e^{-it}z-1 = 0. \tag{31}$$

Let us now set $z = re^{i\theta}$, $r = r(\theta) \geqslant 0$, $\theta = \theta(t)$, $\theta \in [0,2\pi]$, $t \in [-\pi,\pi]$ in equation (31) and separate the real and imaginary parts; we obtain the system of equations

$$(32)\quad \begin{cases} \left(r - \frac{1}{r}\right)\left[\left(r + \frac{1}{r}\right)\cos(2\theta - t) + 2\cos\theta\right] = 0, \\ \left(r^2 + \frac{1}{r^2}\right)\sin(2\theta - t) + 2\left(r + \frac{1}{r}\right)\sin\theta - 2\sin t = 0. \end{cases}$$

From the first equation it follows that either

$$\left(r + \frac{1}{r}\right)\cos(2\theta - t) + 2\cos\theta = 0$$

or $r = 1$ for any $t \in [-\pi,\pi]$.

Consider the first case. Then, setting $R = r + \frac{1}{r}$ and taking into account the fact that $r \leqslant 1$, we can write the system (32) in the form

$$\begin{cases} R\cos 2\theta\cos t + R\sin 2\theta\sin t = -2\cos\theta, \\ (R^2-2)\sin 2\theta\cos t - \left[(R^2-2)\cos 2\theta + 2\right]\sin t = -2R\sin\theta, \end{cases}$$

where $R \geqslant 2$.

Eliminating t from this system we get the following equation

$$(33)\quad (16R^2+64)\sin^4\theta - (8R^4+64)\sin^2\theta + R^6 - 4R^4 = 0,$$

whence

$$(34)\quad \sin^2\theta = \frac{R^4 + 8 - 4\sqrt{2R^4 + 4}}{4R^2 + 16}$$

or

$$(35)\quad \sin^2\theta = \frac{R^4 + 8 + 4\sqrt{2R^4 + 4}}{4R^2 + 16}.$$

Since the right-hand side of (35) is greater than 1 for $R \geqslant 2$, it suffices to consider the equation (34).

Let

$$M(R) = \frac{R^4 + 8 - 4\sqrt{2R^4 + 4}}{4R^2 + 16}.$$

A short calculation gives

$$M'(R) = \frac{1}{(4R^2+16)^2}\left[\left(4R^3 - \frac{16R^3}{\sqrt{2R^4+4}}\right)(4R^2+16) - 8R\left(R^4+8-4\sqrt{2R^2+4}\right)\right] \geqslant 0$$

for $R \geqslant 2$, whence it follows that $M(R)$ is an increasing function for $R \geqslant 2$. Moreover, we have $M(2) = 0$ and $M(1+\sqrt{5}) = 1$. Hence, the formula (34) holds for $R \in [2, 1+\sqrt{5}]$. Since the domain D and its boundary Γ_0 are symmetrical with respect to the real axis, we have merely to restrict ourselves to the case of $0 \leqslant \theta \leqslant \pi$.

For $\theta \in \left[0, \frac{\pi}{2}\right]$ and $\theta \in \left[\frac{\pi}{2}, \pi\right]$ there exists functions $R = R_1(\theta)$ and $R = R_2(\theta)$, respectively, inverse to the function $M(R)$, where $R_1(\theta)$ is increasing for $\theta \in \left[0, \frac{\pi}{2}\right]$ and $R_2(\theta)$ is decreasing for $\theta \in \left[\frac{\pi}{2}, \pi\right]$. This proves, in view of the relation $R = r + \frac{1}{r}$, $0 < r \leqslant 1$, that the function $r = r(\theta)$, which represents in polar coordinates the boundary Γ_0, is decreasing from 1 to $r_0 = \frac{1+\sqrt{5}}{2} - \sqrt{\frac{1+\sqrt{5}}{2}}$ for $\theta \in \left[0, \frac{\pi}{2}\right]$ and increasing from $r_0 = \frac{1+\sqrt{5}}{2} - \sqrt{\frac{1+\sqrt{5}}{2}}$ to 1 for $\theta \in \left[\frac{\pi}{2}, \pi\right]$.

From the first equation of the system (32) it follows that $r = 1$ for any $t \in [-\pi, \pi]$. In this case the second equation of the system (32) takes on the form

$$\sin(\theta - t) = -\operatorname{tg}\theta, \quad t \in [-\pi, \pi].$$

Hence, it follows that the solutions of equation (30) are also the points $z = e^{i\theta(t)}$ of the unit disk such that $-\frac{\pi}{4} \leqslant \arg z \leqslant \frac{\pi}{4}$ and $\frac{3}{4}\pi \leqslant \arg z \leqslant \frac{5}{4}\pi$. This, however, is not essential for our considerations, since our task was to find solutions of the equation (30) in the unit disk E.

From the preceding considerations it follows that the boundary Γ_0 of the domain D satisfies the equation (34) for $R \in [2,\ 1+\sqrt{5}]$, where $R = R(r) = r + \frac{1}{r} = R\left(\frac{1}{r}\right)$.

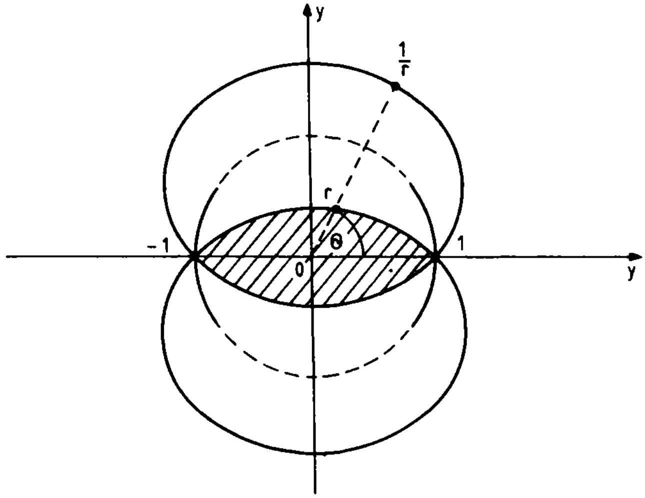

Fig.1

This equation represents a system of curves, as is shown in fig.1; each of them has the polar equation $r = r(\theta)$. We are concerned with the boundary Γ_0, composed of curve segments contained in E (in fig.1 the domain D is hatched). Thus we obtain the following theorem.

T h e o r e m 1. The set of local univalence in the class H is a domain starlike with respect to the origin; the boundary of this domain is a curve whose polar equation is $r = r(\theta)$, $\theta \in [0,\ 2\pi]$, where $r \in (0,1]$ satisfies the equation

$$\sin^2\theta = \frac{R^4 + 8 - 4\sqrt{2R^4 + 4}}{4R^2 + 16} \quad \text{with} \quad R = r + \frac{1}{r}.$$

R e m a r k . For $f \in H$ we have

$$f'(z) = \int_{-\pi}^{\pi} \frac{d}{dz}\left(\frac{z}{1 - z^2} \cdot \frac{1 + ze^{-it}}{1 - ze^{-it}}\right) d\mu(t).$$

From a known result of Ašnevič-Ulina it follows that the domain D of local univalence in the set of the points $z \in E$ such that the convex hull of the curve

$$(35') \qquad w = w(t) = \frac{d}{dz}\left(\frac{z}{1-z^2}\cdot\frac{1+ze^{-it}}{1-ze^{-it}}\right), \quad t \in \langle -\pi, \pi\rangle ,$$

does not contain the origin.

The determination of the domain D by investigating the convex hull of the curve (35′) turns to be more complicated than the presented proof.

2. The set of starlikeness in the class H

The set

$$(36) \qquad D^* = \left\{z \in E:\ \mathrm{Re}\,\frac{zf'(z)}{f(z)} \geqslant 0 \quad \text{for any function} \quad f \in H\right\},$$

is said to be the set of starlikeness in the class H. From (2) it follows that

$$\frac{zf'(z)}{f(z)} - \frac{1+z^2}{1-z^2} = \frac{zp'(z)}{p(z)} ,$$

where $p \in P$. Making use of the exact estimate

$$\left|\frac{zp'(z)}{p(z)}\right| \leqslant \frac{2|z|}{1-|z|^2} \quad \text{for} \quad p \in P,$$

we obtain

$$(37) \qquad \left|\frac{zf'(z)}{f(z)} - \frac{1+z^2}{1-z^2}\right| \leqslant \frac{2|z|}{1-|z|^2} .$$

From (37) it follows that the inequality

$$\mathrm{Re}\,\frac{zf'(z)}{f(z)} \geqslant 0$$

will be satisfied provided that

$$\mathrm{Re}\,\frac{1+z^2}{1-z^2} \geqslant \frac{2|z|}{1-|z|^2} .$$

Denote

$$\tilde{D} = \left\{ z \in E:\ \mathrm{Re}\ \frac{1+|z|^2}{1-|z|^2} \geqslant \frac{2|z|}{1-|z|^2} \right\}.$$

From the preceding argument it follows that $\tilde{D} \subset D^*$. Let us observe that if $z \in \tilde{D}$, then also $(-z) \in \tilde{D}$, $\bar{z} \in \tilde{D}$ and the segment $(-1,1)$ of the real axis belongs to $\tilde{D}$. Therefore, the set $\tilde{D}$ is symmetrical with respect to the real axis and to the point $z = 0$.

The boundary of $\tilde{D}$ is given by the equation

$$\mathrm{Re}\ \frac{1+z^2}{1-z^2} = \frac{2|z|}{1-|z|^2}, \quad z \in E. \tag{38}$$

Setting $z = re^{i\theta}$ in equation (38) and performing some transformations we obtain

$$\frac{1-r^4}{1-2r^2\cos 2\theta + r^4} = \frac{2r}{1-r^2}. \tag{39}$$

Setting $r + \frac{1}{r} = R$, $R \geqslant 2$, we can write equation (39) in the form

$$\sin^2\theta = \frac{1}{8}(R-2)^2(R+2). \tag{40}$$

Since the domain $\tilde{D}$, as well as its boundary are symmetrical with respect to the real axis, we may restrict the argument to the case of $0 \leqslant \theta \leqslant \pi$. Hence, the formula (40) holds for $R \in [2;\ 1+\sqrt{5}]$. The function $\varphi(R) = \frac{1}{8}(R-2)^2(R+2)$ is increasing in this interval. This leads, in view of the relation $R = r + \frac{1}{r}$, to the conclusion that for $\theta \in \left[0, \frac{\pi}{2}\right]$ and $\theta \in \left[\frac{\pi}{2}, \pi\right]$ there exist functions $r = r_1(\theta)$ and $r = r_2(\theta)$, respectively, inverse to the function $\varphi(R) = \sin^2\theta$. The function $r = r_1(\theta)$ is decreasing for $\theta \in \left[0, \frac{\pi}{2}\right]$ and we have $r_1(0) = 1$, $r_1\left(\frac{\pi}{2}\right) = \frac{1+\sqrt{5}}{2} - \sqrt{\frac{1+\sqrt{5}}{2}}$, while $r = r_2(\theta)$ is increasing for $\theta \in \left[\frac{\pi}{2}, \pi\right]$ and we have $r_2\left(\frac{\pi}{2}\right) = \frac{1+\sqrt{5}}{2} - \sqrt{\frac{1+\sqrt{5}}{2}}$, $r_2(\pi) = 1$. The set $\tilde{D}$ thus determined is not the full set of starlikeness of the class H.

The preceding considerations enable us to state the following theorem.

T h e o r e m 2. The set $\tilde{D}$ is a set starlike with respect to the origin; its boundary is given by the polar equation $r = r(\theta)$, $\theta \in [0, 2\pi]$, where $r \in (0;1]$ satisfies the equation

$$\sin^2\theta = \frac{1}{8}(R-2)^2(R+2) \quad \text{with} \quad R = r + \frac{1}{r}.$$

R e m a r k . The above determined domains D and $\tilde{D}$ contain the disk of univalence $K_{r_H} = \{z \in \mathbb{C} : |z| < r_H\}$ in the class H; the number $r_H = \frac{1+\sqrt{5}}{2} - \sqrt{\frac{1+\sqrt{5}}{2}}$ has been determined by L.Koczan in paper [5].

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