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Michel Hébert

MULTIREFLECTIONS AND MULTIFACTORIZATION SYSTEMS

1. Introduction

In [3], a bijection has been constructed, in sufficiently nice categories, between reflective subcategories and factorization systems for morphisms verifying a certain property of maximality. From [4], we also obtain a correspondence between epireflective subcategories and the factorization systems (E,M) for morphisms verifying a property of minimality and such that E is a class of epimorphisms (this result being equivalent to one obtained in [6] involving the so-called dispersed factorization systems for sources). It is then natural to ask if we can define a concept of "multifactorization system" which would correspond to the multireflective subcategories of Y. Diers.

In [8], W. Tholen shows how a "multiconcept" in a category C corresponds to a "concept" in a related category we will denote by Pro(D_0, $\mathcal{C}$). This translation does not preserve all the properties we need to apply the previous results to Pro(D_0, $\mathcal{C}$), but the additional conditions needed on $\mathcal{C}$ are light. The nice behaviour of factorization systems for morphisms in Pro(D_0,$\mathcal{C}$) will permit the definition of a concept of "multifactorization system" for sources in $\mathcal{C}$ which is a direct extension of the usual notion of factorization system for sources, and then to generalize the results stated above in a uniform way. Some examples and related questions will be considered in the third section.

2. The correspondances

Recall from [8] that for a class of small categories D (containing the terminal one) and a category $\mathcal{C}$, the objects of the category Pro(D,$\mathcal{C}$) are the functors $\bar{X}:\mathcal{D}^{op} \to \mathcal{C}$, where $\mathcal{D} \in D$ and its hom-sets are $\mathrm{Pro}(D,\mathcal{C})(\bar{X},\bar{Y}) = \lim_{d\in\mathcal{D}}(\mathrm{colim}_{e\in E}(\mathcal{C}(\bar{X}(d),\bar{Y}(e)))$. If D_0 is the class of all small discrete categories (=all sets), then the set of Pro(D_0,$\mathcal{C}$)-morphisms from $\bar{X}:I \to \mathcal{C}$ to $\bar{Y}:J \to \mathcal{C}$ is $\prod_{j\in J}\left(\coprod_{i\in I}(\mathcal{C}(X_i,Y_i))\right)$ (where we write X_i for $\bar{X}(i)$). In other words, an object of this category is just a set of objects of $\mathcal{C}$ and a morphism can be seen as a set (possibly empty) of disjoint sources (in this paper, source means set-indexed source unless otherwise specified). We then call a morphism in Pro(D_0,$\mathcal{C}$) a multisource in $\mathcal{C}$.

Sources and multisources in $\mathcal{C}$ will be denoted by $\bar{f},\bar{g}$, etc... If we have to be more precise, we will write $(X,f_i)_I$ or (X,f_i) for sources and $(X_j,\bar{f}^j)^J$ or $(\bar{X},\bar{f}^j)$ for multisources. If $\bar{f} = \{\bar{f}^j\} = \{f^j_{i(j)}\}$ is a multisource, $\bar{f}^j$ will be referred as a source of $\bar{f}$ and $f^j_{i(j)}$ as a ($\mathcal{C}$-)morphism of $\bar{f}^j$. Composition of multisources is made in the most naive way through composition of $\mathcal{C}$-morphisms. (X,φ) and $(\bar{X},\varphi)$ will denote the empty source and multisource respectively.

In what follows, all subcategories are assumed to be full and isomorphism-closed.

Any subcategory $\mathcal{A}$ of $\mathcal{C}$ gives rise to an obvious inclusion functor $\mathrm{Pro}(D_0,U):\mathrm{Pro}(D_0,\mathcal{A}) \hookrightarrow \mathrm{Pro}(D_0,\mathcal{C})$. Theorem 2.4 of [8] shows that $\mathcal{A}$ is multireflective in $\mathcal{C}$ if and only if Pro(D_0,U) has a left adjoint. In fact, one can prove that any reflective subcategory of Pro(D_0,$\mathcal{C}$) is of the form Pro(D_0,$\mathcal{A}$) for some reflective subcategory $\mathcal{A}$ of $\mathcal{C}$. This can be shown directly but it will follow from our results.

2.1. Definition. A multifactorization system in a category $\mathcal{C}$ consists of two classes of sources E and M such that:

(i) E and M are both closed for composition with isomorphisms (in Pro(D_0,$\mathcal{C}$)).

(ii) Each source in $\mathcal{C}$ has an <u>(E,M)-factorization:</u> that is, for each source $\bar{f} = (X, f_i)_I$ there exists a source $\bar{g} = (X, g_j)_J$ in E and a set $\left\{(Y_j, h^j_{i(j)})\right\}_{j\in J}$ of sources in M such that $\bar{h}\,\bar{g} = \bar{f}$ (in $\mathrm{Pro}(D_0, \mathcal{C})$), where $\bar{h}$ is the multisource $(Y_j, h^j_{i(j)})^J$.

(iii) $\mathcal{C}$ has the (E,M)-diagonalisation property: for each commutative square

$$\begin{array}{ccc} X & \xrightarrow{\bar{f}} & \bar{Y} \\ {\scriptstyle \bar{g}}\downarrow & \overset{\bar{\lambda}}{\swarrow} & \downarrow{\scriptstyle \bar{k}} \\ \bar{Z} & \xrightarrow[\bar{h}]{} & \bar{W} \end{array}$$

(in $\mathrm{Pro}(D_0, \mathcal{C})$) with $\bar{f} \in E$ and $\bar{h}$ a multisource with all its sources in M, there exists a unique multisource $\bar{\lambda}: \bar{Y} \to \bar{Z}$ such that $\bar{\lambda}\,\bar{f} = \bar{g}$ and $\bar{h}\,\bar{\lambda} = \bar{k}$.

A multifactorisation system in $\mathcal{C}$ for which each source in E is a $\mathcal{C}$-morphism is just a factorization system (for set-indexed sources) in the usual sense of [6]. If E and M are classes of $\mathcal{C}$-morphisms, then it is a factorization system for morphisms as defined in [1] or [3].

In a category $\mathcal{C}$ with products, a factorization system for morphisms can always be extended to a factorization system for set-indexed sources but not, in general, for class-indexed sources. This is clearly indicated in [6], but can also be seen through the fact that the letter corresponds to epireflective subcategories while the former corresponds to reflective ones. More precisely, we have the following bijections (in sufficiently nice categories):

Dispersed factorization systems for class-indexed sources	Reflective factorization systems for sources
$\wr\updownarrow$ (a)	$\wr\updownarrow$ (c)
Epireflective subcategories	Reflective subcategories
$\wr\updownarrow$ (b)	$\wr\updownarrow$ (d)
E-minimal factorization systems for morhpisms with $E \subset \mathrm{Epi}$	E-maximal (=reflective) factorization systems for morphisms

To explain the terminology involved, recall first that a factorization system $F = (E,M)$ for sources (respectively for morphisms) in a category $\mathcal{C}$ (respectively with a terminal object t) induces a reflective subcategory $\mathcal{C}_F$ in $\mathcal{C}$ having as its class of objects $\{X \in \mathcal{C} \mid (X,\varphi) \in M\}$ (respectively $\{X \in \mathcal{C} \mid (X \longrightarrow t) \in M\}$). Two distinct factorization systems can induce the same reflective subcategory, and F will be called E-minimal if for any factorization system $F' = (E',M')$ such that $\mathcal{C}_F = \mathcal{C}_{F'}$, we have $E \subset E'$. Similarly for E-minimality. Epi stands for the class of all epimorphisms.

The correspondence (a) (with the definition of dispersed factorization systems) can be found in [6]; (b) also follows from considerations in [6] (but see also Theorem 1 of [4]). F is a reflective factorization system for morphisms if $fg \in E$ and $f \in E$ imply $g \in E$. The correspondeces (c) and (d) in the diagram follow from Corollary 3.4 of [4].

Two remarks must be made here. First, the proofs of the correspondences on the left and of the ones on the right of the diagram are different in their conceptions and require distinct hypothesis on C: on the left, what is needed is cowell-poweredness and the existence of colimits (plus a terminal object for (b)); on the right, we want finite well-completeness, that is the existence of finite limits and of all (class-indexed) intersections of strong monomorphisms. Secondly, the correspondences on the right are not extensions of those on the left: there is no known bijection between E-maximal factorization systems for class-indexed sources and epireflective subcategories.

What we want to do here is to generalize these results for multireflective subcategories. We will proceed as follow. First, we find conditions on $\mathcal{C}$ sufficient to establish the above correspondences in $\mathrm{Pro}(D_0,\mathcal{C})$. Then we show how to extract a multifactorization system F' in $\mathcal{C}$ from a factorization system F for morphisms in $\mathrm{Pro}(D_0,\mathcal{C})$. Finally, we remark that if $\mathcal{C}_{F'}$ is the multireflective subcategory of $\mathcal{C}$ induced by F', then the reflective subcategory $\mathrm{Pro}(D_0,\mathcal{C}_{F'})$ of $\mathrm{Pro}(D_0,\mathcal{C})$

corresponds to F in the bijections above (this will prove in particular that any reflective subcategory of $\mathrm{Pro}(D_0,\mathcal{C})$ is of the form $\mathrm{Pro}(D_0,\mathcal{A})$ for some multireflective subcategory $\mathcal{A}$ of $\mathcal{C}$).

By Proposition 1.3 of [8], $\mathrm{Pro}(D_0,\mathcal{C})$ always has products. Bur even the existence of equalizers in $\mathcal{C}$ does not guarantee their existence in $\mathrm{Pro}(D_0,\mathcal{C})$. Similarly, the well-poweredness of $\mathcal{C}$ does not imply the well-poweredness of $\mathrm{Pro}(D_0,\mathcal{C})$. However, the existence of products in $\mathcal{C}$ will be sufficient to save the situation in both cases. No additional requirements will be necessary for the dual properties:

2.2. Lemma. Let $\mathcal{C}$ be a category.

a) If $\mathcal{C}$ is cowell-powered (respectively well-powered and has products), then the same is true for $\mathrm{Pro}(D_0,\mathcal{C})$.

b) If $\mathcal{C}$ has colimits of type T, then the same is true for $\mathrm{Pro}(D_0,\mathcal{C})$.

c) If $\mathcal{C}$ is complete, then the same is true for $\mathrm{Pro}(D_0,\mathcal{C})$.

Proof. a) The result follows from the following facts. All sources of an epimorphic (respectively monomorphic) multisource are epimorphisms (respectively monomorphisms) of $\mathrm{Pro}(D_0,\mathcal{C})$. Furthermore, all $\mathcal{C}$-morphisms of an epimorphic source are epimorphisms in $\mathcal{C}$. Finally, a source $\{f_i:X \to Y_i\}_I$ is a monomorphism in $(\mathrm{Pro}(D_0,\mathcal{C}))$ if and only if the induced $\mathcal{C}$-morphism $\langle f_i\rangle_I: X \to \prod_I Y_i$ is a monomorphism.

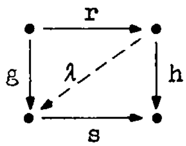
b) We construct the coequalizer of $\bar{f},\bar{g}:\bar{X} \rightrightarrows \bar{Y}$, where $\bar{x}:X \to \mathcal{C}$ and $\bar{Y}:J \to \mathcal{C}$, as follows. For each $j \in J$ there is exactly one $i = i(j)$ and one $i' = i'(j)$ in I such that there exist $\mathcal{C}$-morphisms $f_j^i: X_i \to Y_j$ and $g_j^{i'}: X_{i'} \to Y_j$ in $\bar{f}$ and $\bar{g}$ respectively. Define $\hat{J} = \{j \in J \mid i(j) = i'(j)\}$ and $\bar{Z} : \hat{J} \to \mathcal{C}$ by Z_j = codomain of $\mathrm{coeq}(f_j^i,g_j^{i'})$. Then one proves easily that $\bar{h}:\bar{Y} \to \bar{Z}$, defined by $\bar{h}^j = \mathrm{coeq}(f_j^i,g_j^{i'})$, is the coequalizer of $\bar{f}$ and $\bar{g}$ (if $\hat{J}$ is empty, set $\bar{h} = (\bar{Y},\varphi)$). The coproduct of a set $\{\bar{X}^j \mid j \in J\}$ of $\mathrm{Pro}(D_0,\mathcal{C})$-objects, where $\bar{X}^j = \{X_i^j \mid i \in I_j\}$, is the object $\bar{Y}$ consisting of all possible coproducts $\coprod_{j\in J} X_i^j$

in $\mathcal{C}$ (with the obvious morphisms) in which exactly one i from each I_j appears.

c) By a remark above, it suffices to provide a construction for the equalizers of couples $\bar{f},\bar{g}:\bar{X} \rightrightarrows \bar{Y}$ in $\mathrm{Pro}(D_0,\mathcal{C})$. Consider the diagram in $\mathcal{C}$ formed by the morphisms in $\bar{f}$ and $\bar{g}$. Then it is readily verified that the set of all limit objects (in $\mathcal{C}$) of the connected components of this diagram, together with the limit morphisms to the $\mathcal{C}$-objects in $\bar{X}$, is the equalizer of $\bar{f}$ and $\bar{g}$. □

We will need some additional notations:

If S is a class of morphisms in a category $\mathcal{C}$, then $S^{\uparrow}$ is the class of all morphisms r such that for any commutative square

(1) 

in $\mathcal{C}$ with $s \in S$, there exists a unique λ such that $\lambda r = g$ and $s\lambda = h$. If S is a class of sources in $\mathcal{C}$, then, depending of the context, $S^{\uparrow}$ is either the class of all morphisms or the class of all sources r having the above property (with (1) being now in $\mathrm{Pro}(D_0,\mathcal{C})$).

If R is a class of morphisms, then, depending of the context, $R^{\downarrow}$ is either the class of all morphisms or the class of all sources s such that for any commutative square (1) with $r \in R$, there exists a unique λ such that $\lambda r = g$ and $s\lambda = h$. If R is a class of sources, $R^{\downarrow}$ is the class of all sources s having the similar property with respect to R.

Recall that in any factorization system (E,M) for morphisms or for sources, $E^{\downarrow} = M$ and $M^{\uparrow} = E$. It is easy to see that the same is true for multifactorization systems.

2.3. Lemma. Let $\mathcal{C}$ be a category.

a) Let $\mathcal{C}$ be cocomplete and cowell-powered, and let B be an epireflective subcategory of $\mathrm{Pro}(D_0,\mathcal{C})$. Then $([\bar{\eta}]^{\downarrow\uparrow}, [\bar{\eta}]^{\downarrow})$ is a factorization system for morphisms in $\mathrm{Pro}(D_0,\mathcal{C})$, where $\bar{\eta}$

is the unit of the adjunction and $[\bar{\eta}] = \{\bar{\eta}_{\bar{X}} \mid \bar{X} \in \mathrm{Pro}(D_0,\mathfrak{C})\}$. Furthermore, $[\bar{\eta}]^{\downarrow\uparrow} \subset \mathrm{Epi}$.

b) Let C be complete and well-powered, and let $\mathfrak{B}$ be a reflective subcategory of $\mathrm{Pro}(D_0,\mathfrak{C})$. Then $(\Sigma_R, \Sigma_R^{\downarrow})$ is a reflective factorization system for morphisms in $\mathrm{Pro}(D_0,\mathfrak{C})$, where R is the reflection functor and $\Sigma_R = \{\bar{f} \mid R(\bar{f})$ is an isomorphism$\}$.

P r o o f . a) By Lemma 2.2, $\mathrm{Pro}(D_0,\mathfrak{C})$ is also cocomplete and cowell-powered. By [5], Section 34, it is then an (Epi,Extremal mono) category. This implies that $\mathrm{Epi}^{\downarrow\uparrow} = \mathrm{Epi}$. $[\bar{\eta}]$ being a class of epimorphisms, we have $[\bar{\eta}]^{\downarrow\uparrow} \subset \mathrm{Epi}^{\downarrow\uparrow} = \mathrm{Epi}$. The cowell-poweredness of $\mathrm{Pro}(D_0,\mathfrak{C})$ then allows the application of Theorem 3.1 of [1] to conclude that $([\bar{\eta}]^{\downarrow\uparrow}, [\bar{\eta}]^{\downarrow})$ is a factorization system for morphisms.

b) By Lemma 2.2, $\mathrm{Pro}(D_0,\mathfrak{C})$ is also complete and well-powered. This implies in particular that it is finitely well-complete, and we can apply Corollary 3.4 of [3].

2.4. L e m m a . Let (E,M) be a factorization system for morphisms in $\mathrm{Pro}(D_0,\mathfrak{C})$. Let E′ (respectively M′) be the class of all sources of multisources in E (respectively M). Then (E′,M′) is a multifactorization system in $\mathfrak{C}$.

P r o o f . $E = M^{\uparrow}$ and $M = E^{\downarrow}$ imply that a multisource is in E (respectively M) if and only if each one of its is; for example, if $\bar{f}:\bar{Z} \to \bar{U} = \{\bar{f}^i : Z_i \to \bar{U}^i\}^I$ is a multisource in M and

$$\begin{array}{ccc} \bar{X} & \xrightarrow{\bar{g}} & \bar{Y} \\ {\scriptstyle \bar{h}}\downarrow & & \downarrow{\scriptstyle \bar{k}} \\ Z_i & \xrightarrow[\bar{f}^i]{} & \bar{U}^i \end{array}$$

is a commutative square with $\bar{g} \in E$, consider then the square

$$\begin{array}{ccc} \bar{X}' & \xrightarrow{\bar{g}'} & \bar{Y}' \\ {\scriptstyle \bar{h}'}\downarrow & & \downarrow{\scriptstyle \bar{k}'} \\ \bar{Z} & \xrightarrow[\bar{f}]{} & \bar{U} \end{array}$$

where $\bar{X}' = \bar{X} \cup \{Z_j\}_{j\neq i}$, $\bar{Y}' = \bar{Y} \cup \{Z_j\}_{j\neq i}$, $\bar{g}' = \bar{g} \cup \{1_{Z_j}\}_{j\neq i}$, $\bar{h}' = \bar{h} \cup \{1_{Z_j}\}_{j\neq i}$ and $\bar{k}' = \bar{k} \cup \{\bar{f}^j\}_{j\neq i}$. This diagram clearly commutes and it is easy to check that $\bar{g} \in E$ and $E = M^{\dagger}$ imply that $\bar{g}' \in E$. Then there must exist $\bar{\lambda}' : \bar{Y}' \to \bar{Z}$ such that $\bar{k}' = \bar{f}\bar{\lambda}'$ and $\bar{\lambda}'\bar{g}' = \bar{h}'$. But any source of $\bar{\lambda}'$ having one of the Z_j with $j \neq i$ in its codomain must be 1_{Z_j}, and then there exists $\bar{\lambda} : \bar{Y} \to Z_i$ such that $\bar{\lambda}\bar{g} = \bar{h}$ and $\bar{f}^i\bar{\lambda} = \bar{k}$. The unicity is clear.

Hence E' (respectively M') is the class of all sources in E (respectively M). (E', M') is then a multifactorization system in $\mathcal{C}$. Q.E.D.

A multireflective subcategory of $\mathcal{C}$ such that for each $X \in \mathcal{C}$ the multireflection $\bar{\eta}_X : X \to \bar{Y}$ is an episource, that is an epimorphism in $Pro(D_0, \mathcal{C})$, will be called an epimultireflective subcategory. This must not be confused with the weaker notion of multiepireflective subcategory of [7] (see 3.5).

A multifactorization system (E,M) will be called reflective if for any source $\bar{f}$ and any multisource $\bar{g}$ such that $\bar{g}\,\bar{f}$ is in E and all sources of $\bar{g}$ are in E, $\bar{f}$ is in E.

2.5. Theorem. For a multifactorization system $F = (E,M)$ in a category $\mathcal{C}$, consider the subcategory $\mathcal{C}_F$ of $\mathcal{C}$ having as objects the class of all X in $\mathcal{C}$ such that $(X, \varphi) \in M$. Then,

a) if $\mathcal{C}$ is cocomplete and cowell-powered, the correspondence $F \mapsto \mathcal{C}_F$ restricts to a bijection from the class of E-minimal multifactorization systems with $E \subset$ Episource into the class of epimultireflective subcategories;

b) if $\mathcal{C}$ is complete and well-powered, then the correspondence $F \mapsto \mathcal{C}_F$ restricts to a bijection from the class of E-maximal (=reflective) multifactorization systems into the class of multireflective subcategories.

Proof. Let $X \in \mathcal{C}$ and $X \xrightarrow{\bar{\eta}_X} \bar{Y} \to \varphi$ be the F-multifactorization of (X, φ). It is straightforward to verify that $\bar{\eta}$ is the "multiunit" defining the multireflective subcategory $\mathcal{C}_F$.

a) If $\mathcal{A}$ is epimultireflective in $\mathcal{C}$, then $Pro(D_0, \mathcal{A})$ is epireflective in $Pro(D_0, \mathcal{C})$, and if $\bar{\eta}$ is the unit of this adjunction, then $([\bar{\eta}]^{\downarrow\uparrow},[\bar{\eta}]^{\downarrow})$ is a factorization system for morphisms in $\mathrm{Pro}(D_0,\mathcal{C})$ with $[\bar{\eta}]^{\downarrow\uparrow}\subset$ Epi by Lemma 2.3a). This induces a multifactorization system $F'=(E',M')$ in $\mathcal{C}$, by Lemma 2.4. It is easily seen that $E'=[\bar{\eta}']^{\downarrow\uparrow}$, where $\bar{\eta}'$ is the multiunit of the multireflection, and that E' is a class of episources.

If $F''=(E'',M'')$ is a multifactorization system such that $\mathcal{C}_{F''}=\mathcal{C}_{F'}$, then $[\bar{\eta}']\subset E''$, so that $[\bar{\eta}']^{\downarrow\uparrow}\subset(E'')^{\downarrow\uparrow}=E''$. Hence F' is E-minimal.

Also, $\mathcal{C}_{F'}=\mathcal{A}$ because an object X of $\mathcal{C}$ is in $\mathcal{A}$ if and only if $(X,\varphi)\in[\bar{\eta}]^{\downarrow}$. Conversely, if $F=(E,M)$ is an E-minimal multifactorization system with $E\subset$ Episource, then E must be $[\bar{\eta}']^{\downarrow\uparrow}$, where $\bar{\eta}'$ is the multiunit for $\mathcal{C}_F$, because $[\bar{\eta}']\subset E$ and $E^{\downarrow\uparrow}=E$ imply $[\bar{\eta}']^{\downarrow\uparrow}\subset E$. This insures that the stated correspondence is bijective.

b) The proof is analogous to the one of part a), using this time part b) of Lemma 2.3. If F' is the multifactorization system $(\Sigma'_R,(\Sigma_R^{\downarrow})')$ obtained from a factorization system $(\Sigma_R,\Sigma_R^{\downarrow})$ for morphisms in $\mathrm{Pro}(D_0,\mathcal{C})$, as in Lemma 2.4, for some reflection functor R to a reflective subcategory $\mathrm{Pro}(D_0,\mathcal{A})$ then for any multifactorization system $F=(E,M)$ such that $\mathcal{C}_F=\mathcal{C}_{F'}$, $R(\bar{f})$ must be an isomorphism (in $\mathrm{Pro}(D_0,\mathcal{C})$) for any $\bar{f}\in E$. This shows that $(\Sigma'_R,(\Sigma_R^{\downarrow})')$ is E-maximal.

We finally show that a multifactorization system $F=(E,M)$ is reflective if and only if it is of the type $(\Sigma'_R,(\Sigma_R^{\downarrow})')$. If $\bar{g}\,\bar{f}:X\to\bar{Y}\to\bar{Z}$ is in Σ'_R and $\bar{g}$ has all its sources in Σ'_R, then $R(\bar{f})=R(\bar{g})^{-1}\cdot R(\bar{g}\,\bar{f})$ and is then also an isomoprhism, so that $\bar{f}\in\Sigma'_R$. Conversely, if $F=(E,M)$ is reflective and R is the reflective functor for $\mathrm{Pro}(D_0,\mathcal{C}_F)$, then, for $\bar{f}:X\to\bar{Y}$ in Σ'_R, $\bar{\eta}_{\bar{Y}}\cdot\bar{f}=R(\bar{f})\cdot\bar{\eta}'_X$ is in E because $\bar{\eta}'_X$ is in E and $R(\bar{f})$ is an isomorphism; $\bar{\eta}_{\bar{Y}}$ having each of its sources in E, $\bar{f}$ must be in E; hence $\Sigma'_R\subset E$. The other inclusion holding in any case, we have $\Sigma'_R=E$. Q.E.D.

2.6. R e m a r k s . a) As in [6], one can show that in a cowell-powered category $\mathcal{C}$ multifactorization systems

(E,M) with $E \subset$ Episource extend to multifactorization systems for class-indexed sources. Part a) of Theorem 2.5 can be reformulated accordingly.

b) Let U be the inclusion functor for a multireflective subcategory $\mathcal{A}$ of $\mathcal{C}$ and let R be the reflection functor at the Pro-D_0-level. If S is the class of all $\mathcal{C}$-morphisms which are diagonally universal for U in the sense of 2.0 of [2], then a morphism of $\mathcal{C}$ is in S if and only if it is in some source in Σ'_R. One can also show that if $I \neq \varphi$, then a source $\{f_i\}_I : X \to \{Y_i\}_I$ is in $(\Sigma'_R)^{\downarrow} (= \Sigma^{\downarrow}_R)')$ if and only if each f_i and the induced morphism $\langle f_i \rangle_I : X \to \prod_I Y_i$ are in $S^{\downarrow}$. It follows that any morphism in $\mathcal{C}$ has a unique $(S, S^{\downarrow})$-factorization. What prevents $(S, S^{\downarrow})$ to be a factorization system in $\mathcal{C}$ (for morphisms) is that $S^{\downarrow\uparrow} \not\subset S$ in general. In fact, it is a factorization system if and only if $\mathcal{A}$ is reflexive. From what follows, it is readily seen that this will be the case precisely when $(S, S^{\downarrow}) = (\Sigma'_R, (\Sigma^{\downarrow}_R)')$.

We can express Theorem 2.5 as a generalization of the correspondances a) and c) above: we show that any multifactorization system $F = (E,M)$ in $\mathcal{C}$ such that $\mathcal{C}_F$ is reflective in $\mathcal{C}$ is such that all sources in E are $\mathcal{C}$-morphisms.

If $\mathcal{C}$ has a terminal object t, then we can see this as a consequence of the easily verified fact that a multireflective subcategory of $\mathcal{C}$ is reflective if and only if it contains t (compare this to Theorem 2.3 of [8]): $\mathcal{C}_F$ is then reflective if and only if $(t,\varphi) \in M$, and for $(X,\bar{f}) \in E$, the fact that

$$\begin{array}{ccc} X & \xrightarrow{\bar{f}} & \bar{Y} \\ \downarrow & & \downarrow \\ t & \longrightarrow & \varphi \end{array}$$

is always commutative implies, if $(t,\varphi) \in M$, that there is only one Pro$(D_0,\mathcal{C})$-morphism $\bar{\lambda} : \bar{Y} \to t$; this insures that $\bar{Y} \in \mathcal{C}$, and then that $\bar{f}$ is a morphism in $\mathcal{C}$.

In the general case, one remarks first that $E \subset \Sigma_R$, where R is the reflection functor to $\mathrm{Pro}(D_0,\mathcal{C}_F)$, and then that for any source $\bar{f}:X \to \bar{Y}$, the fact that $R(X) \in \mathcal{C}$ when $\mathcal{C}_F$ is reflective implies that $R(\bar{f})$ can be an isomorphism only if $\bar{Y} \in \mathcal{C}$.

We have then proved the following:

2.7. Theorem. Any multifactorization system F in a category C such that $\mathcal{C}_F$ is reflective in $\mathcal{C}$ is a factorization system for sources. Hence the correspondences of Theorem 2.5 reduce to ones between factorization systems for sources (with the properties mentioned) and (epi)reflective subcategories. Q.E.D.

3. Examples and problems

3.1. Example. Partially ordered sets.

Let $\mathcal{C}$ be a partially ordered set (considered as a category in the usual way) and let $\mathcal{A}$ be a full subcategory of $\mathcal{C}$ verifying the following property: for each $X \in \mathcal{C}$ and each $A \in \mathcal{A}$ such that $X \leq A$, the set $\{A' \in \mathcal{A} \mid X \leq A' \leq A\}$ has a smallest element. As remarked by H. Herrlich (see [7]), this property characterizes the multireflective subcategories of $\mathcal{C}$ (among the full ones). If R is the reflection functor from $\mathrm{Pro}(D_0,\mathcal{C})$ to $\mathrm{Pro}(D_0,\mathcal{A})$, then one verifies that

$$\Sigma'_R = \{X \to \{Y_i\}_I \mid \text{for all } A \in \mathcal{A} \text{ with } X \leq A, \text{ there exists exactly one } i \in I \text{ such that } A \leq Y_i\}.$$

One can find examples where $(\Sigma'_R,(\Sigma^{\downarrow}_R)')$ is not a multifactorization system: suppose there exist X,Y_1,Y_2 in $C \setminus \mathcal{A}$ and $A \in \mathcal{A}$ such that $X < Y_i < A$, $i = 1,2$, $Y_1 \neq Y_2$ and there is no Y in $\mathcal{C}$ such that $X < Y < Y_i$, $i = 1,2$; then $X \to \{Y_1,Y_2\}$ does not necessarily have a $(\Sigma'_R,(\Sigma^{\downarrow}_R)')$-multifactorization. On the other hand, if $\mathcal{C}$ is complete, then $X = Y_1 \times Y_2$ and $X \to \{Y_1,Y_2\}$ is itself in $(\Sigma^{\downarrow}_R)'$. In fact $(\Sigma'_R,(\Sigma^{\downarrow}_R)')$ is a multifactorization system in this case by Theorem 2.5. Remark that $\mathcal{A}$ is multiepireflective but not epimultireflective in $\mathcal{C}$.

3.2. Example. Fields in commutative rings.

The full subcategory $\mathcal{F}$ of (commutative) fields is multireflective in the category $\mathcal{CRing}$ of commutative rings with unity (see for example [2]). Let R be the reflecion functor from $\mathrm{Pro}(D_0,\mathcal{CRing})$ to $\mathrm{Pro}(D_0,\mathcal{F})$. By Theorem 2.5, $(\Sigma'_R,(\Sigma^{\downarrow}_R)')$ is a multifactorization system in $\mathcal{CRing}$. We will construct in an effective way the multifactorization of sources having an Artin ring as domain.

Consider any source $\bar{f} = \{f_i\}_I : X \to \bar{Y} = \{Y_i\}_I$ in $\mathcal{CRing}$. Denote by $\bar{\eta}_X : X \to \hat{X}$ and $\bar{\eta}_{\bar{Y}} : \bar{Y} \to \hat{\bar{Y}}$ the reflections; then

$$\bar{\eta}_X = \left\{\eta^X_\alpha : X \to X(\alpha) \mid \alpha \in \mathrm{Spec}(X)\right\}$$

and

$$\bar{\eta}_{\bar{Y}} = \left\{\left\{\eta^{Y_i}_\beta : Y_i \to Y_i(\beta) \mid \beta \in \mathrm{Spec}(Y_i)\right\} \mid i \in I\right\}$$

where Spec(X), the <u>spectrum of</u> X, is the set of all prime ideals of X and η^X_α is the canonical homomorphism into the residual field of the ring of fractions X_α (similarly for $\mathrm{Spec}(Y_i)$ and $\eta^{Y_i}_\beta$). Also $R(\bar{f}) : \hat{X} \to \hat{\bar{Y}}$ is the multisource

$$\left\{\left\{f_\beta : X(f_i^{-1}(\beta)) \to Y_i(\beta) \mid \beta \in \mathrm{Spec}(Y_i)\right\} \mid i \in I\right\}.$$

Denote by $\mathrm{Spec}(\bar{Y})$ the disjoint union $\bigcup_I \mathrm{Spec}(Y_i)$ and by $\mathrm{Spec}(\bar{f})$ the function from $\mathrm{Spec}(\bar{Y})$ to $\mathrm{Spec}(X)$ defined by $\mathrm{Spec}(\bar{f})(\beta) = f_i^{-1}(\beta)$ (where $\beta \in \mathrm{Spec}(Y_i)$). Then it is easily verified that $\bar{f}$ is in Σ'_R if and only if the two following properties are satisfied:

i) For each $i \in I$, $\beta \in \mathrm{Spec}(Y_i)$ and y,y' in $Y_i \setminus \beta$, there exist x,x' in $X \setminus f_i^{-1}(\beta)$ such that $y \cdot f_i(x) - y' \cdot f_i(x')$ is in β.

ii) $\mathrm{Spec}(\bar{f})$ is bijective.

Remark also that a particular $f_i : X \to Y_i$ in $\bar{f}$ is diagonally universal (see 2.6) for the inclusion functor if and only if:

i)′ For each $\beta \in \mathrm{Spec}(Y_i)$ and y, y' in $Y_i \setminus \beta$, there exist x, x' in $X \setminus f_i^{-1}(\beta)$ such that $y \cdot f_i(x) - y' \cdot f_i(x')$ is in β and

ii)′ $\mathrm{Spec}(f_i)$ is injective.

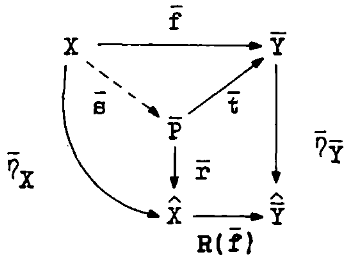
We want to show that in some cases, the multifactorization $(\Sigma'_R, (\Sigma_R^{\downarrow})')$ can be obtained through the pullback (in $\mathrm{Pro}(D_0, \mathcal{CRing})$) of $R(\bar{f})$ along $\bar{\eta}_{\bar{Y}}$. For this we have to look more closely at its construction.

Consider the smallest equivalence relation on the set $K = I \cup \mathrm{Spec}(X)$ such that i in I is equivalent to α in $\mathrm{Spec}(X)$ if $f_i^{-1}(\beta) = \alpha$ for some $\beta \in \mathrm{Spec}(Y_i)$. For each equivalence class $[k]$, denote by $P[k]$ the subring

$$P[k] = \left\{ \{x_\alpha, y_i \mid x_\alpha \in X(\alpha),\ y_i \in Y_i,\ \alpha, i \in [k]\} \;\middle|\; f_\beta(x_\alpha) = \eta_\beta^{Y_i}(y_i) \text{ when } \alpha = f_i^{-1}(\beta) \right\}$$

of $\prod_{\alpha \in [k]} X(\alpha) \times \prod_{i \in [k]} Y_i$.

Then $\bar{P} = \{P[k] \mid k \in K\}$, with the obvious projections, is the pullback of $R(\bar{f})$ along $\bar{\eta}_{\bar{Y}}$. We then obtain a factorization of $\bar{f}$ in $\mathrm{Pro}(D_0, \mathcal{CRing})$ through the induced source from X to $\bar{P}$:



It follows from the standard properties of factorization systems (in $\mathrm{Pro}(D_0, \mathcal{CRing})$) that $\bar{t}$ is in $\Sigma_R^{\downarrow}$ (see for example [1]), and then its sources are in $(\Sigma_R^{\downarrow})'$. We show first that if each $[k]$ is finite and each prime ideal of X is maximal (in particular if X is an Artin ring and I is finite), then $\bar{s}$ is in Σ'_R (and then $\bar{t}\,\bar{s}$ is the $(\Sigma'_R, (\Sigma_R^{\downarrow})')$-multifactorization of $\bar{f}$).

Denote by $s[k] : X \to P[k]$ the $[k]$-component of $\bar{s}$. Then $s[k](x) = \left\{ \frac{x}{1} + M_\alpha, f_i(x) \mid \alpha, i \in [k] \right\}$, where M_α is the maximal ideal of X_α.

For $\alpha_0 \in [k] \cap \mathrm{Spec}(X)$, it is easily verified that the subset $\alpha_0' = \left\{ \{x_\alpha, y_i\} \mid x_{\alpha_0} = 0 \right\}$ of $P[k]$ is a prime ideal. Obviously $s[k]^{-1}(\alpha_0') = \alpha_0$, and then the induced homomorphism $s[k]_{\alpha_0} : X(\alpha_0) \to P[k](\alpha_0')$ (component of $R(s[k])$) is an isomorphism because it has an injective left inverse, namely the (unique) homomorphism λ such that

$$\begin{array}{ccc} P[k] & \longrightarrow & P[k](\alpha_0') \\ {\scriptstyle r_{\alpha_0}} \downarrow & \swarrow {\scriptstyle \lambda} & \\ X(\alpha_0) & & \end{array}$$

commute (where r_{α_0} is the relevant component of $\bar{r}$). We show next that $\mathrm{Spec}(P[k]) = \{\alpha' \mid \alpha \in [k] \cap \mathrm{Spec}(X)\}$. Let $\rho \in \mathrm{Spec}(P[k])$. If $[k] \cap \mathrm{Spec}(X) = \{\alpha_1, \ldots, \alpha_m\}$, consider an element $\{x_\alpha, y_i\}$ of $\alpha_1' \cap \ldots \cap \alpha_m'$. The fact that $x_\alpha = 0$ for each α in $[k]$ implies that each y_i belongs to each prime ideal of Y_i, and is then nilpotent. $I \cap [k]$ being finite, $\{x_\alpha, y_i\}$ is also nilpotent and then is an element of ρ. Hence $\alpha_1' \cap \ldots \cap \alpha_m' \subseteq \rho$, and this implies that $\alpha_j' \subseteq \rho$ for some j. Furthermore, $\rho \subseteq \alpha_1' \cup \ldots \cup \alpha_m'$ because if $\{x_\alpha, y_i\} \in P[k]$ is such that $x_\alpha \neq 0$ for each α, then y_i is invertible for each $i \in [k] \cap I$ and then $\{x_\alpha, y_i\}$ is invertible. Hence $\rho \subseteq \alpha_n'$ for some n. But the inclusions $\alpha_j \subseteq s[k]^{-1}(\rho) \subseteq \alpha_n$ imply that $j = n$ because all prime ideals of X are maximal, and then $\rho = \alpha_j'$. This shows that each $s[k]$ is diagonally universal for the inclusion functor. By the very definition of the equivalence relation, we conclude that $\bar{s}$ is in Σ_R'.

We mention that in the absence of any restriction on $\bar{f}$, the source $\bar{s}$ obtained above is not necessarily in Σ_R' (and hence R is not a <u>simple</u> reflexion, in the terminology of [3]):

for example, one can easily find $\bar{f}:X \to \bar{Y}$ such that $[k] \cap I$ is infinite for some $k \in K$ and where there exists an element $\{x_\alpha, y_i\}$ of $P[k]$ with each y_i nilpotent but not itself nilpotent. Such an element is necessarily in each α'_j but there must exist a prime ideal ρ of $P[k]$ such that $\{x_\alpha, y_i\} \notin \rho$. Hence $\rho \neq \alpha'_j$ for all $\alpha_j \in [k] \cap \mathrm{Spec}(X)$, and $\mathrm{Spec}(\bar{s})$ is not bijective.

If X is an Artin ring, then, whatever is $\bar{f}$, there is only a finite number of equivalence classes in K, and two applications of the above procedure leads to the $(\Sigma'_R, (\Sigma^{\downarrow}_R)')$-multifactorization of $\bar{f}$: indeed, $\bar{s}:X \to \bar{P}$ verifies the hypothesis mentioned above for $\bar{f}$ and then, if $\bar{u}\,\bar{v}$ is its factorization through the pullback of $R(\bar{s})$ along $\bar{\eta}_{\bar{P}}$, $\bar{v}$ is in Σ_R and $\bar{t}\,\bar{u}$ in $\Sigma^{\downarrow}_R$.

In the general case, the $(\Sigma'_R, (\Sigma^{\downarrow}_R)')$-multifactorization of $\bar{f}$ can be obtained through the $(M^{\downarrow}, M)$-factorization of $\bar{s}$ in $\mathrm{Pro}(D_0, \mathcal{C})$, where $M = (\Sigma^{\downarrow}_R)' \cap \mathrm{SMon}$, SMon the class of strong monomorphisms in $\mathrm{Pro}(D_0, \mathcal{C})$ (see the proof of Theorem 3.3 in [3]); no effective way to obtain this factorization is known to us.

3.3. D-pro-factorizations

What we did for $D = D_0$ can be done for any class of categories D, leading to a concept of "D-pro-factorization system" and to the analogous correspondences involving the D-pro-(epi)reflective subcategories. However, it seems to us that in general the hypothesis needed on $\mathrm{Pro}(D, \mathcal{C})$ to apply the method of section 2 do not translate nicely enough into conditions on $\mathcal{C}$ to make such a result very interesting. It remains to see to what extent these hypothesis are necessary.

3.4. Multireflective hulls

If $\{(E_\alpha, M_\alpha)\}_A$ is a class of multifactorization systems in a category $\mathcal{C}$ which is cocomplete and cowell-powered and $E_\alpha \subseteq \mathrm{Episource}$ for each $\alpha \in A$, then one can prove that $((\bigcap_{\alpha \in A} M_\alpha)^{\downarrow}, \bigcap_{\alpha \in A} M_\alpha)$ is also a multifactorization system, by applying the analogous result for factorization systesm for morphisms (see Theorem 2 of [4]) to $Pro(D_0,\mathcal{C})$. From this and Theorem 2.5a), one concludes that the intersection of epimultireflective subcategories is an epimultireflective subcategory. The problem of the existence of a (general) multireflective hull remains the same than in the reflective case: we do not know if $((\bigcap_{\alpha\in A} M_\alpha)^{\uparrow}, \bigcap_{\alpha\in A} M_\alpha)$ is always a multifactorization system when one of the E_α contains non-epi sources.(*)

3.5. Multiepireflective subcategories

Recall from [7] that if $\mathcal{A}$ is a multireflective subcategory of $\mathcal{C}$ with $\bar\eta$ the unit of the adjunction at the $Pro(D_0,\mathcal{C})$-level, we call $\mathcal{A}$ multiepireflective in $\mathcal{C}$ if each $\bar\eta_X$ is what we will call a weak episource, that is, each of its $\mathcal{C}$-morphisms is an epimorphism. This concept might be more interesting than epimultireflectivity which is abusively strong.

If we replace, in part a) of Theorem 2.5, "epimultireflective" by "multiepireflective" and "Episource" by "Weak episource", then the result is still true. The proof follows the same lines, except that we cannot use Lemma 2.3. We have here to prove directly that $([\bar\eta]^{\downarrow\uparrow}, [\bar\eta]^{\downarrow})$ is a factorization system for morphisms in $Pro(D_0,\mathcal{C})$. But it is not difficult to verify that $[\bar\eta]^{\downarrow\uparrow} \subseteq$ Weak episource, from which it follows that for each $\bar X \in Pro(D_0,\mathcal{C})$, the class of all morphisms in $[\bar\eta]^{\downarrow\uparrow}$ with domain $\bar X$ has a representative set (by the cowell-poweredness of $\mathcal{C}$). This permits again the utilization of Theorem 3.1 of [1].

(*) J.Adamek and J.Rosicky recently showed by a counterexample that even in categories which are well- and cowell-powered, complete and cocomplete, the subcategories do not always have reflective hulls: see [Kelly, G.M., On the ordered set of reflective subcategories, Sydney Cat. Sem. Rep., August 1986].

REFERENCES

[1] A.K. B o u s f i e l d : Constructive of factorization systems in categories, J. Pure Appl. Algebra 9 (1977), 207-220.

[2] Y. D i e r s : Une construction universelle des spectres, topologies spectrales et faisceaux structuraux, Comm. Algebra, 12 (1984), 2143-2183.

[3] C. C a s s i d y , M. H é b e r t , G.M. K e l l y: Reflective subcategories, localizations, and factorization systems, J. Austral. Math. Soc. A, 38 (1985), 287--329.

[4] M. H é b e r t : Factorization systems and the reflective hull problem, to appear in SAMSA Journal.

[5] H. H e r r l i c h , G.E. S t r e c k e r : Category theory, Haldermann Verlag, Berlin 1979.

[6] H. H e r r l i c h , G. S a l i c r u p , R. V a s q u e z : Dispersed factorization structures, Canad, J. Math., 30 (1979), 1059-1071.

[7] G. S a l i c r u p : Multiepireflective subcategories, General Topology and Appl. 13 (1982), 191-200.

[8] W. T h o l e n : Pro-categories and multiadjoint functors, Canad. J. Math. 36 (1984), 144-155.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ZAMBIA,
P.O.Box 32379, LUSAKA, ZAMBIA
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