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FIXED POINTS FOR CONTRACTIVE CORRESPONDENCES

In the present paper we prove some fixed and common fixed point theorems for contractive type mappings and multivalued mappings in the metric space. These theorems extend theorems of Achari [1], Ray [6], Reich [7], Singh and Whitfield [8].

Let (X,d) be a metric space and let S, T be two correspondences (i.e. mappings from points to sets) from X to CB(X), which are neither necessarily continuous nor commuting. We shall denote by CB(X) the set of all non-empty closed and bounded subsets of X, by CL(X) the set of all closed subsets of X, by H(A,B) the Hausdorff distance of $A,B \in CB(X)$. We shall also write $D(A,B) = \inf\{d(a,b) : a \in A, b \in B\}$, $\delta(A,B) = \sup\{d(a,b) : a \in A, b \in B\}$. A point $x \in X$ will be called a fixed point of S or T is $x \in Sx$ or $x \in Tx$ respectively.

The theorems proved in this paper generalize some results from [1-10] for mappings and correspondences.

Theorem 1. Let (X,d) be a compact metric space and $S,T : X \to CL(X)$ and let S or T be continuous and satisfy condition : there exists a real valued function $\varphi : R_+^5 \to R_+$ such that

$$(1) \quad \delta(Sx,Ty) < \varphi(d(x,y),\delta(x,Sx),\delta(y,Ty),\delta(x,Ty),\delta(y,Sx))$$

for any distinct x,y in X, where φ is non-decreasing with respect to the each variable and

$$(2) \qquad \varphi(t,t,t,2t,2t) \leqslant t.$$

Then S or T has a fixed point u satisfying $\{u\} = Su$ or $\{v\} = Tv$. If both S and T have such fixed points then $u = v$.

P r o o f . Let S be continuous and put

$$f(x) = H(x,Sx) \quad \text{for all} \quad x \in X.$$

Then f is continuous on X. Hence f takes its minimum value at some x_0. We prove that x_0 is a fixed point of S or some $x_1 \in Sx_0$ is a fixed point of T.

Let

$x_1 \in Sx_0$ be such that $d(x_0,x_1) = H(x_0,Sx_0) = b_0$,

$x_2 \in Tx_1$ be such that $d(x_1,x_2) = H(x_1,Tx_1) = b_1$ and

$x_3 \in Sx_2$ be such that $d(x_2,x_3) = H(x_2,Sx_2) = b_2$.

If $b_0 > 0$ and $b_1 > 0$, then from (1) it follows that

$$b_1 = H(x_1,Tx_1) \leqslant \delta(Sx_0,Tx_1) <$$

$$< \varphi(d(x_0,x_1),\delta(x_0,Sx_0),\delta(x_1,Tx_1),\delta(x_0,Tx_1),\delta(x_1,Sx_0)) \leqslant$$

$$\leqslant \varphi(d(x_0,x_1),d(x_0,x_1),d(x_1,x_2),d(x_0,x_1) + \delta(x_1,Tx_1),$$

$$d(x_1,x_0) + \delta(x_0,Sx_0)) \leqslant \varphi(b_0,b_0,b_1,b_0+b_1,2b_0).$$

If $b_0 < b_1$ then

$$b_1 < \varphi(b_1,b_1,b_1,2b_1,2b_1) \leqslant b_1$$

and we obtain a contradiction. So $b_0 > b_1$.

Now

$$b_2 = H(x_2,Sx_2) \leqslant \delta(Tx_1,Sx_2) <$$

$$< \varphi(d(x_1,x_2),\delta(x_1,Tx_1),\delta(x_2,Sx_2),\delta(x_1,Sx_2),\delta(x_2,Tx_1)) \leqslant$$

$$\leqslant \varphi(b_1,b_1,b_2,b_1+b_2,b_1+b_1).$$

If $b_1 > b_2$ then

$$b_2 < \varphi(b_1, b_1, b_1, 2b_1, 2b_1) \leq b_1 < b_0,$$

which contradicts the minimality of b_0.

If $b_1 < b_2$ then,

$$b_2 < \varphi(b_2, b_2, b_2, 2b_2, 2b_2) \leq b_2$$

and we obtain a contradiction. Therefore $b_0 = 0$ or $b_1 = 0$.

Now let $\{u\} = Su$ and $\{v\} = Tv$ then $u = v$, because otherwise from (1)

$$d(u,v) = \delta(Su,Tv) < \varphi(d(u,v), \delta(u,Su), \delta(v,Tv), \delta(u,Tv), \delta(v,Tu)) \leq$$

$$\leq d(u,v)$$

we obtain a contradiction. This proves our theorem.

R e m a r k 1. If

$$\delta(Sx,Ty) < \varphi(d(x,y), \delta(x,Sx), \delta(y,Ty), D(x,Ty), D(y,Sx))$$

then (2) can be replaced by

$$\varphi(t,t,t,2t,0) \leq t, \quad t \geq 0.$$

R e m a r k 2. Condition (2) cannot be weakened to the condition $\varphi(t,t,t,t,t) \leq t$.

E x a m p l e . Let $X = \{1,2,3\}$, $d(x,y) = |x-y|$, $F(1) = 2$, $F(2) = \{1,2,3\}$, $F(3) = 2$.

Let

$$\varphi(t_1,t_2,t_3,t_4,t_5) = \frac{2}{3} \max \{t_1,t_2,t_3,t_4,t_4\}.$$

Then $F = S = T$ satisfies condition (1) but F has not a fixed point u satisfying $\{u\} = Fu$.

R e m a r k 3. If the condition

$$(1') \quad H(Sx,Ty) < \varphi(d(x,y), D(x,Sx), D(y,Sy), D(x,Ty), D(y,Sx))$$

is satisfied then S or T has a fixed point.

In the proof let $f(x) = D(x,Sx)$ and

$x_1 \in Sx_0$ be such that $d(x_0,x_1) = D(x_0,Sx_0) = b_0$,

$x_2 \in Tx_1$ be such that $d(x_1,x_2) = D(x_1,Tx_1) = b_1$,

$x_3 \in Sx_2$ be such that $d(x_2,x_3) = D(x_2,Sx_2) = b_2$.

(see [3]).

If S and T are mappings we can prove similarly as in [4].

Theorem 2. Let (X,d) be a metric space, $S,T : X \to X$ be continuous mappings and suppose that there exists $x_0 \in X$ such that the sequence (x_n) defined by $x_{2n+1} = Sx_{2n}$, $x_{2n+2} = Tx_{2n+1}$ for $n = 0,1,2,\dots$ satisfies $x_n \neq x_{n+1}$ and contains a subsequence (x_j) convergent, to, say, u.

Suppose that there exists a real valued function $\varphi : R_+^5 \to R_+$ non-decreasing with respect to each variable and

$$\text{(a)} \qquad \begin{cases} \varphi(t,t,t,2t,0) \leq t, \\ \varphi(t,t,t,0,2t) \leq t \end{cases}$$

and such that

$$\text{(b)} \quad d(Sx,Ty) < \varphi(d(x,y),d(x,Sx),d(y,Ty),d(x,Ty),d(y,Sx))$$

for any distinct x,y in X. Then u is the unique common fixed point of S and T.

Proof. By (b) there exists $\lim d(x_n,x_{n+1})$. Let j be even and $u \neq Su$. Because

$$\lim d(x_{j+1},Su) = 0 \quad \text{and} \quad \lim d(x_{j+2},TSu) = 0$$

we have

$$d(u,Su) = \lim d(x_j,x_{j+1}) = \lim d(x_{j+1},x_{j+2}) \leq$$

$$< \lim d(x_{j+1},Su) + d(Su,TSu) + \lim d(TSu,x_{j+2}) <$$

$$< d(u,Su) \quad \text{a contradiction. Hence } u = Su.$$

Since $\lim d(x_{j+2},Tu) = 0$ and $\lim d(x_{j+3},STu) = 0$ it follows that

$$d(u,Tu) = d(Su,Tu) = \lim d(x_{j+1},x_{j+2}) =$$
$$= \lim d(x_{j+2},x_{j+3}) = d(Tu,STu).$$

If $u \neq Tu$ then

$$d(u,Tu) = d(Tu,STu) <$$
$$<\varphi(d(u,Tu),d(u,Tu),d(Tu,STu),d(u,STu),d(Tu,Tu)) \leqslant d(u,Tu)$$

a contradiction. Therefore $u = Tu$.

If v is another fixed point of S and T such that $u \neq v$, then by (b)

$$d(u,v) = d(Su,Tv) < d(u,v). \quad \text{Hence} \quad u = v.$$

This completes the proof.

Remark 4. If X is compact metric space $S = T$ then there exists a unique fixed point of T.

If

$$\varphi(t_1,t_2,t_3,t_4,t_5) = \max\left\{t_1, \tfrac{1}{2}(t_2+t_3), \tfrac{1}{2}(t_4+t_5)\right\}$$

we obtain theorem of Achari [1]. This result extends earlier ones of Wong [10], Leader and Hoyle [5], Su and Seghal [9], Singh and Whitfield [8].

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