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AN EXISTENCE THEOREM FOR THE HYPERBOLIC EQUATION $z_{xy}=f(x,y,z)$ IN BANACH SPACES

We are interested in the existence of solutions of the Darboux problem for the hyperbolic equation $z_{xy} = f(x,y,z)$ $(x,y \geq 0)$ when f with values in a Banach space satisfies some regularity condition expressed in terms of Kuratowski's measure of noncompactness α. Our result will be proved via the fixed point theorem of Sadovskii given in [5] as Theorem 3.4.3.

Let $J = \langle 0, \infty)$. Throughout this paper $Q = J \times J$ and E will denote a Banach space with norm $\|\cdot\|$. The measure of noncompactness $\alpha(A)$ of nonempty bounded subset A of E is defined as the infimum of all $\varepsilon > 0$ such that there exists a finite covering of A by sets of diameter $\leq \varepsilon$.

Denote by $C(Q,E)$ the set of all continuous functions from Q to E. The set $C(Q,E)$ will be considered as a vector space endowed with the topology of almost uniform convergence. For $V \subset C(Q,E)$ we denote by $V(x,y)$ the set of all $z(x,y)$ with $z \in V$. Further, we will use standard notation. The closure of a set A and its closed convex hull will be denoted, respectively, by $\bar{A}$ and $\overline{co}\, A$. For the properties of α we refer to [2].

The Lemma below is an adaptation of the corresponding result of Ambrosetti ([1], Lemma 2.2). It is special result of Heinz lemma (see [3]).

L e m m a . If P is a compact subset of Q and V is a bounded equicontinuous subset of the usual Banach space of continuous E-valued functions on P, then

$$\alpha(\bigcup\{V(x,y):\ (x,y)\in P\}) = \sup\{\alpha(V(x,y)):\ (x,y)\in P\}.$$

Denote by S_∞ the set of all nonnegative real sequences. For $\xi = (\xi_n)$, $\eta = (\eta_n)\in S_\infty$ we write $\xi < \eta$ if $\xi \leqslant \eta$ (i.e. $\xi_n \leqslant \eta_n$ for $n = 1,2,\ldots$) and $\xi \neq \eta$. Let $\mathfrak{X}$ be a closed convex subset of $C(Q,E)$ and Φ be a function which assigns to each nonempty subset Z of $\mathfrak{X}$ a sequence $\Phi(Z)\in S_\infty$ such that

(1) $\Phi(\{z\}\cup Z) = \Phi(Z)$ for $z\in\mathfrak{X}$,

(2) $\Phi(\overline{co}\ Z) = \Phi(Z)$,

(3) if $\Phi(Z) = \theta$ (the zero sequence) then $\bar{Z}$ is compact.

Here we use the Sadovskii fixed point theorem in the following form (cf. [4]):

If $F:\mathfrak{X}\to\mathfrak{X}$ is a continuous mapping satisfying $\Phi(F[Z]) < \Phi(Z)$ for arbitrary nonempty subset Z of $\mathfrak{X}$ with $\Phi(Z) > \theta$, then F has a fixed point in $\mathfrak{X}$.

Let f be an E-valued function defined on $Q\times E$. By (+) we shall denote the problem of finding a continuous solution of

$$\begin{cases} z_{xy} = f(x,y,z) \\ z(x,0) = 0, \quad z(0,y) = 0 \end{cases}$$

for $x\geqslant 0$, $y\geqslant 0$.

T h e o r e m . Let $G: J\times J\times J\to J$ be a continuous function nondecreasing in the last variable, and let $L: J\times J\times J\to J$ be a function such that for each $u\in J$ the mapping $(x,y)\mapsto L(x,y,u)$ is continuous and $L(x,y,0)\equiv 0$ on Q. If

1^0 f is continuous,

2^0 $\|f(x,y,u)\|\leqslant G(x,y,\|u\|)$ for $(x,y)\in Q$ and $u\in E$,

3^0 $\alpha(f[P\times W])\leqslant \sup\{L(x,y,\alpha(W)):\ (x,y)\in P\}$ for any compact subset P of Q and each nonempty bounded subset W of E,

4^0 the scalar integral inequality

$$g(x,y)\geqslant \int_0^x\int_0^y G(u,v,g(u,v))\,du\,dv$$

has a locally bounded solution g_0 existing on Q, and

$$\int_0^{+\infty}\int_0^{+\infty} L(u,v,r)\,du\,dv < r \quad \text{for all } r > 0,$$

then there exists a solution z of (+) such that $\|z(x,y)\| \leq g_0(x,y)$ for $(x,y) \in Q$.

P r o o f . Denote by $\mathfrak{X}$ the set of all $z \in C(Q,E)$ with $\|z(x,y)\| \leq g_0(x,y)$ on Q and

$$\|z(x_1,y_1) - z(x_2,y_2)\| \leq \left|\int_0^{x_2}\int_{y_1}^{y_2} G(u,v,g_0(u,v))\,du\,dv\right| +$$

$$+ \left|\int_{x_1}^{x_2}\int_0^{y_1} G(u,v,g_0(u,v,g_0(u,v))\,du\,dv\right| \quad \text{for} \quad (x_1,y_1),(x_2,y_2) \in Q.$$

The set $\mathfrak{X}$ is a closed convex and almost equicontinuous subset of $C(Q,E)$. We define a continuous mapping F of $\mathfrak{X}$ into itself as follows

$$(Fz)(x,y) = \int_0^x\int_0^y f(u,v,z(u,v))\,du\,dv \quad \text{for} \quad z \in \mathfrak{X}.$$

Let n be a positive integer and $P_n = [0,n] \times [0,n]$. Let Z be a nonempty subset of $\mathfrak{X}$ and $W = \bigcup\{Z(x,y): (x,y) \in P_n\}$. Fix (x,y) in P_n. For any given $\varepsilon > 0$ there exists $\delta > 0$ such that $u',u'' \in [0,x]$, $v',v'' \in [0,y]$ with $|u'-u''| < \delta$ and $|v'-v''| < \delta$ imply $|L(u',v',\alpha(W)) - L(u'',v'',\alpha(W)| < \varepsilon$. We divide the intervals $[0,x]$, $[0,y]$ into m parts $x_0 = 0 < x_1 < \ldots < x_m = x$, $y_0 = 0 < y_1 < \ldots < y_m = y$ in such a way that $|x_i - x_{i-1}| < \delta$, $|y_i - y_{i-1}| < \delta$ for $i = 1,2,\ldots,m$.

Put $P_{ij} = [x_{i-1},x_i] \times [y_{j-1},y_j]$, $i,j = 1,2,\ldots,m$, and let (σ_i,τ_j) be a point in P_{ij} such that

$$L(x,y,\alpha(W)) \leq L(\sigma_i,\tau_j,\alpha(W)) \quad \text{for} \quad (x,y) \in P_{ij}.$$

Now, by the integral mean value theorem, our condition 3° and Lemma, we obtain

$$\alpha(F[Z](x,y)) \leqslant \alpha\Big(\sum_{i,j=1}^{m} \operatorname{mes}(P_{ij})\ \overline{\operatorname{co}}(f[P_{ij}\times W])\Big) \leqslant$$

$$\leqslant \sum_{i,j=1}^{m} \operatorname{mes}(P_{ij})\ L(\sigma_i,\tau_j,\alpha(W)) \leqslant \sum_{i,j=1}^{m} \iint_{P_{ij}} L(u,v,\alpha(W))\,du\,dv +$$

$$+ \sum_{i,j=1}^{m} \iint_{P_{ij}} |L(u,v,\alpha(W)) - L(\sigma_i,\tau_j,\alpha(W))|\,du\,dv <$$

$$< \int_0^x\int_0^y L(u,v,\alpha(W))\,du\,dv + \varepsilon xy =$$

$$= \varepsilon xy + \int_0^x\int_0^y L(u,v,\sup\{\alpha(Z(x,y)) : (x,y)\in P_n\})\,du\,dv.$$

As $\varepsilon > 0$ is arbitrary, this implies

$$\sup\{\alpha(F[Z](x,y)) : (x,y)\in P_n\} \leqslant \tag{4}$$

$$\leqslant \sup_{(x,y)\in Q} \int_0^x\int_0^y L(u,v,\sup\{\alpha(Z(x,y)) : (x,y)\in P_n\})\,du\,dv.$$

Define

$$\Phi(Z) = \Big(\sup_{(x,y)\in P_1} \alpha(Z(x,y)),\ \sup_{(x,y)\in P_2} \alpha(Z(x,y)),\dots\Big)$$

for any nonempty subset Z of $\mathfrak{X}$. Evidently $\Phi(Z)\in S_{\infty}$. By the corresponding properties of α, the function Φ satisfies conditions (1)-(3) listed above. From our assumption on L and inequality (4) it follows that $\Phi(F[Z]) < \Phi(Z)$ whenever $\Phi(Z) > \theta$. Thus all assumptions of Sadovskiĭ's fixed point theorem being satisfied, F has a fixed point in $\mathfrak{X}$ and the proof is complete.

R e m a r k . In particular, the function $L(u,v,r) = L(u,v)\cdot\varphi(r)$, where

$$\int_0^{+\infty}\int_0^{+\infty} L(u,v)\,du\,dv \leqslant 1 \quad \text{and} \quad 0 < \varphi(r) < r \quad \text{for} \quad r > 0$$

satisfies conditions from the theorem.

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