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Wiesław Sasin, Zbigniew Żekanowski

# ON LOCALLY FINITELY GENERATED DIFFERENTIAL SPACES

In this paper we consider Sikorski differential space [1], [2] whose a differential structure is generated by a finite set of real-valued functions. In Corollary 2.3 it is given a characterisation such a differential structure.

Any Hausdorff differential space which is locally finitely generated is a differential space of class $D_0$ [4].

## 1. Preliminaries

Let M be an arbitrary non empty set and C be a family of real functions on M. By scC we denote the set of all real functions on M of the form $\omega \circ (\alpha_1, \ldots, \alpha_n)$, where $\omega$ is an arbitrary smooth real function on $R^n$, $\alpha_1, \ldots, \alpha_n \in C$ and $n=1,2,\ldots$ . A set C is said to be closed with respect to composition with the smooth function on $R^n$ iff $C = scC$ [2].

By $\tau_C$ we denote the weakest topology on M in which all functions of C are continuous. Now let A be an arbitrary subset of M. By $C_A$ we denote the set of all functions $g: A \to R$ such that for each point $p \in A$ there exists an open neighbourhood U of p and a function $f \in C$ such that $f|U \cap A = g|U \cap A$.

A set C is said to be closed with respect to localization iff $C_M = C$. If $C = (scC)_M$ then the set C is said to be a differential structure on M and the couple (M,C) is called a differential space [1], [2].

Let now (M,C) and (N,D) be differential spaces. A mapping $F: M \to N$ is said to be a smooth mapping of the differential space $(M,C)$ into the differential space $(N,D)$ iff for each $f \in D$ $f \circ F \in C$. If $F$ is a smooth mapping of the differential space $(M,C)$ into the differential space $(N,D)$ we write $F\colon (M,C) \to (N,D)$ [2]. Moreover for an arbitrary mapping $F\colon M \to N$ by $F^*\colon D \to C$ we define the mapping defined by the formula $F^*(\alpha) = \alpha \circ F$ for any $\alpha \in D$.

The notion of tangent vector to $(M,C)$ at the point $p \in M$ we define as a linear mapping $v\colon C \to R$ satisfying the following condition

$$v(\alpha \cdot \beta) = \alpha(p)v(\beta) + \beta(p)v(\alpha) \quad \text{for any } \alpha, \beta \in C.$$

The set of all tangent vectors to $(M,C)$ at the point $p \in M$ we denote by $M_p$ and call the tangent space to $(M,C)$ at the point $p$.

If $F\colon (M,C) \to (N,D)$ is a smooth mapping between differential spaces then for each point $p \in M$ the mapping $F_{*p}\colon M_p \to N_{F(p)}$ defined by the formula $(F_{*p}v)(\beta) = v(F^*\beta)$ for any $\beta \in D$ and $v \in M_p$ is a linear mapping.

By a smooth vector field tangent to $(M,C)$ we mean each mapping $X\colon C \to C$ such that $X(\alpha \cdot \beta) = \alpha X(\beta) + \beta X(\alpha)$ for any $\alpha, \beta \in C$. The set of all smooth tangent vector field to $(M,C)$ we denote by $\mathfrak{X}(M)$.

Now we prove some lemmas which will be usefull in the sequel.

L e m m a 1.1. Let $F\colon (M,C) \to (N,D)$ be a smooth mapping of a differential space $(M,C)$ onto a differential space $(N,D)$ such that $F^*\colon D \to C$ is an isomorphism between linear rings. Then

(i) $F$ is an open mapping

(ii) $F$ is a diffeomorphism iff $F$ is one to one

(iii) the topologies $\tau_C$ and $\tau_D$ are equipollent and the mapping $F_\tau\colon \tau_C \to \tau_D$ defined by the formula $F_\tau(U) = F(U)$ for $U \in \tau_C$ establishes the equipollence of $\tau_C$ and $\tau_D$.

(iv) if $(M,C)$ is a Hausdorff differential space then $F$ is a diffeomorphism.

P r o o f . (i). Let $U = (\beta_1,\ldots,\beta_n)^{-1}[Q]$ be an arbitrary open set of the base of the topology $\tau_C$. Because by assumption F is an isomorphism therefore there exists exactly one sequence of functions $\alpha_1,\ldots,\alpha_n \in D$ such that $\beta_i = F^*\alpha_i = \alpha_i \circ F$ for $i = 1,2,\ldots,n$. Hence $U = (\beta_1,\ldots,\beta_n)^{-1}[Q] = (F^*\alpha_1,\ldots,F^*\alpha_n)^{-1}[Q] = F^{-1}(\alpha_1,\ldots,\alpha_n)^{-1}[Q]$ and consequently $F(U) = (\alpha_1,\ldots,\alpha_n)^{-1}[Q]_D$. Hence F is an open mapping.

(ii) Observe that $(F \circ G)^* = G^* \circ F^*$. Since F is a bijection then $(F^{-1} \circ F)^* = F^* \circ F^{-1*} = id_C$. On the other hand because $F^*$ is an isomorphism so $F^* \circ F^{*-1} = id_C$. From the least two identities we get $F^* \circ F^{-1*} = F^* \circ F^{*-1}$ or equivalently $F^{-1*} = F^{*-1}$. Hence for each $\alpha \in C$ we have $F^{*-1}(\alpha) = F^{-1*}(\alpha) = \alpha \circ F^{-1} \in D$. So $F^{-1}$ by definition is a smooth mapping and consequently F is a diffeomorphism.

(iii) Since for any $V \in \tau_D$ $F^{-1}(V) \in \tau_C$ and $F_\tau(F^{-1}(V)) = F(F^{-1}(V)) = V$ therefore F is a mapping "onto". It is easy to prove also that $F(U) = F(V)$ iff $U = V$ for any $V,U \in \tau_C$. Hence F is one to one and consequently $F_\tau$ is a bijection.

(iv) Let (M,C) be a Hausdorff differential space that means $\tau_C$ is a Hausdorff topology. Now let $p_1,p_2 \in M$ be such points that $F(p_1) = F(p_2)$ and V be an arbitrary neighbourhood of the point $F(p_1)$. Hence for the set $U = F^{-1}(V)$ we get $p_1 \in U$ and $p_2 \in U$. Therefore $p_1 = p_2$ because (M,C) is a Hausdorff differential space and consequently F is one to one. By (iii) F is a diffeomorphism.

Lemma 1.2. Let $f\colon (M,C) \to (N,D)$ be a smooth mapping of a differential space (M,C) onto a differential space (N,D) such that $f^*\colon D \to C$ is an isomorphism between linear rings. Then

(i) for each point $p \in M$ the mapping $f_{*p}\colon M_p \to N_{f(p)}$ is an isomorphism between the linear spaces

(ii) the mapping $f_\#\colon \mathfrak{X}(M) \to \mathfrak{X}(N)$ defined by the formula $(f_\# X)(\beta) = f^{*-1}(X(f^*\beta))$ for any $X \in \mathfrak{X}(M)$ and $\beta \in D$ is an isomorphism between modules

(iii) for each point $p \in M$ and for any vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ there are identities:

$$(1) \qquad (f_{\#}X)(f(p)) = f_{*p}\, X(p),$$

$$(2) \qquad f_{*p}((f_{\#}^{-1}Y)(p)) = Y(f(p)).$$

P r o o f . (i) Let $v \in M_p$ be a tangent vector such that $f_{*p}\, v = 0$. For an arbitrary $\alpha \in C$ $(f_{*p}v)(f^{*-1}\alpha) = v(f^{*} \circ f^{*-1}\alpha) = v(\alpha) = 0$. Therefore $v = 0$ and consequently $f_{*p}$ is a one to one mapping. We will show now that $f_{*p}$ is "onto". Let $w \in N_{f(p)}$ be an arbitrary tangent vector to $(N,D)$ at the point $f(p)$. It is easy to observe that the mapping $v: C \to R$ defined by the formula $v(\alpha) = w(f^{*-1}\alpha)$ for any $\alpha \in C$ is a tangent vector to $(M,C)$ at the point $p$ and moreover $f_{*p}v = w$. So $f_{*p}$ is onto. In consequence $f_{*p}: M_p \to N_{f(p)}$ is an isomorphism.

(ii) Observe that for any $\alpha, \beta \in C$ and $X,Y \in \mathfrak{X}(M)$ there is the identity

$$f_{\#}(\alpha X + \beta Y) = f^{*-1}\alpha \cdot f_{\#}X + f^{*-1}\beta \cdot f_{\#}Y.$$

It is easy to verify also that the mapping $f_{\#}^{-1}: \mathfrak{X}(N) \to \mathfrak{X}(M)$ defined by the formula: $(f_{\#}^{-1}Y)(\alpha) := f^{*}(Y(f^{*-1}\alpha))$, for each $Y \in \mathfrak{X}(N)$ and $\alpha \in C$, is linear and inverse of $f_{\#}$. Hence $f_{\#}$ is an isomorphism between modules.

(iii) Since $f^{*}: D \to C$ is an isomorphism we have the identity $\alpha = f^{*-1}\alpha \circ f$ for any $\alpha \in C$. Using the last identity we verify the following equality

$$(1.1) \qquad (f_{\#}X)(f(p))(\beta) = (f_{\#}X)(\beta)(f(p)) = f^{*-1}(X(f^{*}\beta))(f(p)) =$$
$$= (f^{*-1}(X(f^{*}\beta)) \circ f)(p) = X(f^{*}\beta)(p) = X(p)(f^{*}\beta) =$$
$$= (f_{*p}\, X(p))(\beta)$$

for each $p \in M$, $\beta \in C$ and $X \in \mathfrak{X}(M)$. From (1.1) it follows

$$(f_{\#}X)(f(p)) = f_{*p}\, X(p)$$

for each $p \in M$ and $X \in \mathfrak{X}(M)$.

Let now $Y \in \mathfrak{X}(N)$ and p be an arbitrary point of M. By an immediate calculus we verify the following identity:

$$(1.2)\quad f_{*p}((f_{\#}^{-1}Y)(p))(\beta) = (f_{\#}^{-1}Y)(p)(f^{*}\beta) = (f_{\#}^{-1}Y)(f^{*}\beta)(p) =$$
$$= f^{*}(Y(f^{*-1}f^{*}\beta))(p) = f^{*}(Y(\beta))(p) = Y(\beta)(f(p)) =$$
$$= Y(f(p))(\beta)$$

for an arbitrary $\beta \in D$. Hence from (1.2) we get

$$f_{*p}((f_{\#}^{-1}Y)(p)) = Y(f(p))$$

for each point $p \in M$ and $Y \in \mathfrak{X}(N)$.

From Lemma 1.2 it follows that

C o r o l l a r y 1.3. Let $f: (M,C) \to (N,D)$ be a smooth mapping of a differential space (M,C) onto a differential space (N,D) such that $f^{*}: D \to C$ is an isomorphism. Then (M,C) is a differential space of constant differential dimension n iff (N,D) is a differential space of constant differential dimension n.

P r o o f . Let (M,C) be a differential space of constant differential dimension n. Then, by definition, for an arbitrary point $p \in M$ there exists a local vector base of C-module $\mathfrak{X}(M)$. We will show that (N,D) is a differential space of constant differential dimension n too. In fact, let q be an arbitrary point of N. Since f is a mapping "onto" so there exists point $p \in M$ such that $f(p) = q$. Let now $W_1,\dots,W_n$ be a local vector base of C-module $\mathfrak{X}(M)$ at the point p. From Lemma 1.2 it follows that $f_{\#}W_1,\dots,f_{\#}W_n$ create a local vector base of D-module $\mathfrak{X}(N)$ at the point q. Indeed, if an open set U is the domain of the local vector base $W_1,\dots,W_n$ then f(U) is the domain of the vector fields $f_{\#}W_1,\dots,f_{\#}W_n$. From the lemma 1.2 - (ii) it follows that the vector fields $(f|U)_{\#}W_1,\dots,(f|U)_{\#}W_n$ make a module base of $D_{f(U)}$-module $\mathfrak{X}(f(U))$. Let us observe yet that the vectors $((f|U)_{\#}W_1)(q),\dots,((f|U)_{\#}W_n)(q)$ are linearly independent. In fact, from (iii) of Lemma 1.2 there follow equalities

$$((f|U)_{\#}W_i)(q) = ((f|U)_{\#}W_i)(f(p)) = f_{*p}W_i(p)$$

for $i=1,2,\dots,n$. Hence because $W_1(p),\dots,W_n(p)$ are R-linear independent and $f_{*p}$ is an isomorphism, the vectors $f_{*p}W_1(p),\dots,f_{*p}W_n(p)$ are linearly independent too.

Let now $(N,D)$ be a differential space of constant differential dimension n and p be an arbitrary point of M. There exists an open neighbourhood V of the point f(p) and a local vector base $\widetilde{W}_1,\dots,\widetilde{W}_n$ of D-module $\mathfrak{X}(N)$ on the set V. We will show now that the vector fields $f_{\#}^{-1}\widetilde{W}_1,\dots,f_{\#}^{-1}\widetilde{W}_n$ make a local vector base of C-module $\mathfrak{X}(M)$ on the set $f^{-1}(V)$.

Evidently those vector fields make a module base of $C_{f^{-1}(V)}$-module $\mathfrak{X}(f^{-1}(V))$. Next, since $f_{*p}$ is an isomorphism and the vectors $\widetilde{W}_1(f(p)),\dots,\widetilde{W}_n(f(p))$ are linear independent for each point $p\in f^{-1}(V)$ therefore from the equality

$$f_{*p}(f_{\#}^{-1}(\widetilde{W}_k)(p)) = \widetilde{W}_k(f(p))$$

for $k=1,2,\dots,n$ there follows linear independence of the vectors $f_{\#}^{-1}(\widetilde{W}_1)(p),\dots,f_{\#}^{-1}(\widetilde{W}_n)(p)$.

Now let $(M,C)$ be an arbitrary differential space. A set $C_0$ of real functions defined on M is said to be a set of generators for the differential structure C on M iff $(scC_0)_M = C$.

It is easy to prove the following lemma.

L e m m a 1.3. Let $f: (M,C) \to (N,D)$ be a smooth mapping of a differential space $(M,C)$ onto a differential space $(N,D)$ such that $f^*: D \to C$ is an isomorphism. Then if $D_0$ is a set of generators for the differential structure D then $C_0 = f^*D_0$ is a set of generators for C.

## 2. Finitely generated differential spaces

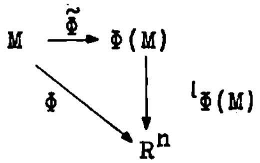
Let $(M,C)$ be a differential space. A differential space $(M,C)$ is said to be finitely generated by a set $C_0 = \{\alpha_1,\dots,\alpha_n\}$ if $(scC_0)_M = C$.

Now, by the symbol $\Phi: M \to R^n$ we denote the mapping defined by the formula

$$\Phi(p) = (\alpha_1(p), \dots, \alpha_n(p))$$

for any point $p \in M$. Evidently $\Phi$ is a smooth mapping.

Let $\tilde{\Phi}: (M,C) \longrightarrow (\Phi(M), \varepsilon_{n\Phi(M)})$ be the mapping $\Phi$ onto the image $\Phi(M)$ with the natural differential structure induced from $(R^n, \varepsilon_n)$ on $\Phi(M)$. The following diagram




is commutative, where $\iota_{\Phi(M)}$ is a natural imbedding.

Let $\pi_k: R^n \longrightarrow R$ be the projection map onto the k-axis, for $k=1,\dots,n$. Of course the differential structure $\varepsilon_{n\Phi(M)}$ is finitely generated by the n-element set $\{\pi_1|\Phi(M), \dots, \pi_n|\Phi(M)\}$ of functions. The functions $\pi_1|\Phi(M), \dots, \pi_n|\Phi(M)$ are the successive components of the imbedding map $\iota_{\Phi(M)}$. A base of the topology $\tau_{\varepsilon_{n\Phi(M)}}$ is composed of sets of the form $\iota^{-1}_{\Phi(M)}(P) = P \cap \Phi(M)$, where P is an arbitrary open interval in $R^n$ [2].

Now we will prove

L e m m a 2.1. Let $(M,C)$ be a differential space finitely generated by the set $C_0 = \{\alpha_1, \dots, \alpha_n\}$. Then:

(i) the empty set and the sets of the form $\Phi^{-1}(P)$ make a base of the topology $\tau_C$, where P is an arbitrary open interval in $R^n$. The bases $\tau_C$ and $\tau_{\varepsilon_{n\Phi(M)}}$ are equipollent,

(ii) the mapping $\tilde{\Phi}: (M,C) \longrightarrow (\Phi(M), \varepsilon_{n\Phi(M)})$ is open,

(iii) $\tau_C$ is the Hausdorff topology iff $\tilde{\Phi}: M \longrightarrow \Phi(M)$ is a homeomorphism,

(iv) if $p_1$ and $p_2$ are such points of M that $\Phi(p_1) = \Phi(p_2)$ then for each function $\alpha \in C$ $\alpha(p_1) = \alpha(p_2)$.

P r o o f. Let p be an arbitrary point of M and $U \in \tau_C$ be an arbitrary open neighbourhood of p. We will show that there exists a set $\Phi^{-1}(P)$ such that $p \in \Phi^{-1}(P) \subset U$. Indeed, since the sets of the form $(\alpha_{i_1}, \dots, \alpha_{i_k})^{-1}(P)$ make a base of the topology $\tau_C$, then there exists a set $(\alpha_{i_1},\dots,\alpha_{i_k})^{-1}(P_1)$ such that $p \in (\alpha_{i_1},\dots,\alpha_{i_k})^{-1}(P_1) \subset U$.

Of course it suffices to show that there exists an open interval P in $R^n$ such that $p \in \Phi^{-1}(P) \subset (\alpha_{i_1},\dots,\alpha_{i_k})^{-1}(P_1)$.

Observe that $(\alpha_{i_1},\dots,\alpha_{i_k}) = (\pi_{i_1},\dots,\pi_{i_k}) \circ \Phi$. Hence it results that $(\alpha_{i_1},\dots,\alpha_{i_k})^{-1}(P_1) = \Phi^{-1}((\pi_{i_1},\dots,\pi_{i_k})^{-1}(P_1))$. Evidently the set $(\pi_{i_1},\dots,\pi_{i_k})^{-1}(P_1)$ is open in $R^n$ and contains the point $\Phi(p)$. So there exists an open interval P in $R^n$ such that $\Phi(p) \in P \subset (\pi_{i_1},\dots,\pi_{i_k})^{-1}(P_1)$. In consequence

$$p \in \Phi^{-1}(P) \subset \Phi^{-1}((\pi_{i_1},\dots,\pi_{i_k})^{-1}(P_1)) = (\alpha_{i_1},\dots,\alpha_{i_k})^{-1}(P_1)$$

which proves that the sets $\Phi^{-1}(P)$, where P is an arbitrary open interval in $R^n$, make a base of the topology $\tau_C$.

To each set $\Phi^{-1}(P)$ we may assign the set $\iota^{-1}_{\Phi(M)}(P) = \Phi(M) \cap P$. Evidently this assignement is one to one. Hence the base of the topology $\tau_C$ composed of the set of the form $\Phi^{-1}(P)$ is equipollent with the base of the topology $\tau_{\varepsilon_n \Phi(M)}$ composed of the sets $\Phi(M) \cap P$. Moreover $\tilde{\Phi}(\Phi^{-1}(P)) = \Phi(M) \cap P$.

(ii) Let $U \in \tau_C$. There exists an open covering $\{\Phi^{-1}(P_i)\}_{i\in I}$ of U. It is easy to observe that the sets $\iota^{-1}_{\Phi(M)}(P_i)$ make an open covering of the set $\tilde{\Phi}(U)$. So $\tilde{\Phi}(U)$ is an open set and consequently $\tilde{\Phi}$ is an open map.

(iii) Assume that $\tau_C$ is a Hausdorff topology. Then for any points $p_1, p_2 \in M$ there exists a function $\alpha_k \in C_0$ such that $\alpha_k(p_1) \neq \alpha_k(p_2)$. Hence $\tilde{\Phi}$ is an injective mapping. In consequence $\tilde{\Phi}$ as an open mapping is a homomorphism.

(iv) Let now $p_1, p_2 \in M$ be such points that $\tilde{\Phi}(p_1) = \tilde{\Phi}(p_2)$. We will show that each open set V which contains $p_1$ contains $p_2$ too. Indeed, let $V \in \tau_C$ be an arbitrary open set containing the point $p_1$. Of course $\tilde{\Phi}(V)$ is an open set containing the point $\tilde{\Phi}(p_1) = \tilde{\Phi}(p_2)$. Moreover $p_2 \in \tilde{\Phi}^{-1}(\tilde{\Phi}(V))$.

Let us take an arbitrary function $\alpha \in C$ and let $U \ni p_1$ be an arbitrary neighbourhood of $p_1$ such that there exists a smooth function $\omega \in \varepsilon_n$ and the following condition is fulfilled

$$\alpha(q) = (\omega \circ \Phi)(q)$$

for each point $q \in U$. Since $p_1, p_2 \in U$ hence $\alpha(p_2) = (\omega \circ \Phi)(p_2)$ and $\alpha(p_1) = (\omega \circ \Phi)(p_1)$. In consequence from the equality $\Phi(p_1) = \Phi(p_2)$ we get

$$\alpha(p_1) = \omega(\Phi(p_1)) = \omega(\Phi(p_2)) = \alpha(p_2).$$

Now we prove

Theorem 2.2. If $(M,C)$ is a finitely generated differential space by the set $C_0 = \{\alpha_1, \ldots, \alpha_n\}$ of real functions, then the mapping $\tilde{\Phi}^*: \varepsilon_{n\Phi(M)} \longrightarrow C$ is an isomorphism between linear rings.

Proof. Since $\tilde{\Phi}: (M,C) \longrightarrow (\Phi(M), \varepsilon_{n\Phi(M)})$ is a smooth surjection then $\tilde{\Phi}^*$ is a one to one homomorphism. Indeed, if $\tilde{\Phi}^*\alpha = 0$ for $\alpha \in \varepsilon_{n\Phi(M)}$ then $\alpha \circ \tilde{\Phi} = 0$. Hence $\alpha(q) = (\alpha \circ \tilde{\Phi})(p) = \alpha(\tilde{\Phi}(p)) = 0$ for $q \in \Phi(M)$. In consequence $\alpha = 0$.

Now we will prove that $\tilde{\Phi}^*$ is "onto". So let $\alpha \in C$ be an arbitrary real function on $M$. Let $\omega_\alpha: \Phi(M) \longrightarrow R$ be the function defined by the formula

$$(2.1) \qquad \omega_\alpha(q) = \alpha(p)$$

for any $q \in \Phi(M)$, where $p \in M$ is such a point of $M$ that $q = \Phi(p)$.

From the lemma (2.1) - (iv) there follows the correctness of the definition (2.1). Moreover the equality

$$(2.2) \qquad \omega_\alpha \circ \tilde{\Phi} = \alpha$$

holds. Now we will show that $\omega_\alpha$ is a smooth function on $(\Phi(M), \varepsilon_{n\Phi(M)})$. Indeed, for an arbitrary point $q \in \Phi(M)$ let us choose a point $p \in M$ such that $\Phi(p) = q$. There exists an open neighbourhood $V$ of $p$ as well as a function $\omega \in \varepsilon_n$ such that $\alpha|V = \omega \circ \Phi|V$.

Evidently from the lemma 2.1 it follows that $\tilde{\Phi}(V)$ is an open set containing the point $q$ and moreover the equality

$$\omega \circ \tilde{\Phi}|V = \omega_\alpha \circ \tilde{\Phi}|V$$

holds. Hence

$$\omega|\tilde{\Phi}(V) = \omega_\alpha|\tilde{\Phi}(V).$$

So we have showed that for an arbitrary point $q \in \Phi(M)$ there exists an open neighboruhood $\tilde{\Phi}(V)$ of q as well as there exists a function $\omega \in \varepsilon_n$ such that

$$\omega | \tilde{\Phi}(V) = \omega_\alpha | \tilde{\Phi}(V).$$

Hence it follows that $\omega_\alpha \in \varepsilon_{n\Phi(M)}$. In this way we have proved that $\tilde{\Phi}^*$ is an isomorphism between the rings as well as $\tilde{\Phi}^*(\omega_\alpha) = \alpha$.

C o r o l l a r y 2.3. A differential space (M,C) is finitely generated by n real functions if and only if there exists a mapping $\Phi : (M,C) \to (R^n, \varepsilon_n)$ such that $\Phi^*: \varepsilon_{n\Phi(M)} \to C$ is an isomorphism between the rings. Moreover, the set of functions $\{\Phi^*(\pi_1|\Phi(M)), \ldots, \Phi^*(\pi_n|\Phi(M))\}$ is a set of generators for the differential structure C.

P r o o f . If (M,C) is a finitely generated differential space then by Theorem 2.2 there exists a mapping $\Phi : (M,C) \to (R^n, \varepsilon_n)$ such that $\Phi^*: \varepsilon_{n\Phi(M)} \to C$ is an isomorphism between rings.

Conversely, let $\Phi : (M,C) \to (R^n, \varepsilon_n)$ be a smooth mapping such that $\Phi^* : \varepsilon_{n\Phi(M)} \to C$ is an isomorphism between rings. From the lemma 1.3 it follows that the set $\{\Phi^*(\pi_1|\Phi(M)), \ldots \ldots \Phi^*(\pi_n|\Phi(M))\}$ generates the differential structure C. Hence the differential space (M,C) is finitely generated.

Now let us go to the locally finitely generated differential space. It is easy to prove

C o r o l l a r y 2.4. A differential space (M,C) is locally finitely generated by n functions iff for each point $p \in M$ there exists an open neighbourhood V of p as well as a smooth mapping $f: (V, C_V) \to (f(V), \varepsilon_{nf(V)})$ such that $f^*: \varepsilon_{nf(V)} \to C_V$ is an isomorphism between linear rings. If (M,C) is a Hausdorff differential space then (M,C) is a differential space of class $D_0$ [4].

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INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF WARSAW,
00-661 WARSZAWA
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