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Zbigniew Łuszczki, Bogdan Rzepecki

AN EXISTENCE THEOREM FOR ORDINARY DIFFERENTIAL EQUATIONS OF ORDER $\alpha \in (0,1]$

Let $0 < \alpha \leq 1$ and let $I = [0,T]$, $T > 0$. In this note we study the existence of the unique solution on I to the Cauchy problem for ordinary differential equations with a derivative of order α.

First, we need the concepts of fractional integration and differentiation. Let u be a function on the positive real axis. The integral of order α of u is defined by the convolution integral

$$D^{-\alpha}u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} u(s)ds,$$

where Γ denotes the Gamma function. Obviously, if u belongs to $L^1(I)$ then $D^{-\alpha}u$ exists almost everywhere and belongs to the same space.

The derivative $D^{\alpha}u$ of order α of a function u is now defined indirectly through fractional integration. More precisely,

$$D^{\alpha}u(t) = \frac{d}{dt} (D^{-(1-\alpha)}u(t))$$

whenever it exists. Additional details and properties may be found e.g. in [3] or [4].

The definitions of integration and differentiation of fractional order of real or complex-valued functions go back to J.Liouville, B.Riemann, and H.Weyl. In connection with Laplace transform theory, these notations are intensively treated in [2], see also [6]. For recent literature see the papers in [1]. Moreover, in [1] the reader will obtain some ideas on the theory of fractional integration and differentiation in connection with semi-group theory and linear partial differential equations.

Our result reads as follows.

T h e o r e m 1. Let $x_0 \in \mathbb{R}$ and let f be a continuous function defined on $I \times \mathbb{R}$. If, moreover, f satisfies in the second variable the Lipschitz condition, then the equation

$$D^{\alpha}x(t) = f(t, x(t))$$

has a unique solution existing on $(0,T]$ such that

$$\lim_{t \to 0+} D^{-(1-\alpha)}x(t) = x_0.$$

P r o o f . Without loss of generality we may suppose that $T > 1$. Denote by X the set of all continuous functions on $(0,T]$ which are Lebesgue integrable on I. We endow the vector space X with the sequence of seminorms:

$$p_0(x) = \int_0^T |x(t)|dt, \quad p_n(x) = \sup_{n^{-1} \leq t \leq T} |x(t)|, \quad n=1,2,\ldots;$$

under this topology, X becomes a Fréchet space. The proof is based on the following fixed point theorem [5].

T h e o r e m . Let F and I be mappings of X into itself such that $p_m(Fu-Fv) \leq k \cdot p_m(Iu-Iv)$, $m=0,1,\ldots$, for all $u,v \in X$, where k is a constant less than 1. If I as a one-to-one transformation for which $F[X] \subset I[X]$ and $I[X]$ is a closed set, then the equation $Ix = Fx$ has a unique solution.

Suppose $|f(t,x_1) - f(t,x_2)| \leq L|x_1-x_2|$ for $t \in [0,T]$ and $x_1,x_2 \in \mathbb{R}$. Let $r > 0$ be a constant with $r^{\alpha} > L$. Define mappings I and F by putting

$$(Ix)(t) = e^{-rt}\, x(t),$$

$$(Fx)(t) = e^{-rt}\left(\frac{x_0}{\Gamma(\alpha)}\, t^{\alpha-1} + D^{-\alpha}f(t,x(t))\right)$$

for $x \in X$. It is clear that $F[X] \subset X = I[X]$.

Let $u,v \in X$. Since

$$\int_0^T \left|\int_0^t u(t-s)v(s)ds\right| dt \leqslant p_0(u)p_0(v)$$

and

$$e^{-rt}\left|D^{-\alpha}f(t,u(t)) - D^{-\alpha}f(t,v(t))\right| \leqslant$$

$$\leqslant \frac{1}{\Gamma(\alpha)}\, e^{-rt}\int_0^t (t-s)^{\alpha-1}\,|f(s,u(s)) - f(s,v(s))|\, ds \leqslant$$

$$\leqslant \frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\, e^{r(s-t)}\,|(Iu)(s) - (Iv)(s)|ds,$$

so

$$\int_0^T |(Fu)(t)-(Fv)(t)|dt \leqslant \frac{L}{\Gamma(\alpha)}\int_0^T\left(\int_0^t (t-s)^{\alpha-1}e^{-r(t-s)}|(Iu)(s) -\right.$$

$$\left. - (Iv)(s)ds|\right)dt \leqslant \frac{L}{\Gamma(\alpha)}\cdot p_0(Iu-Iv)\cdot\int_0^T s^{\alpha-1}e^{-rs}ds \leqslant$$

$$\leqslant \frac{1}{r^\alpha}\cdot\frac{L}{\Gamma(\alpha)}\cdot p_0(Iu-Iv)\cdot\int_0^\infty s^{\alpha-1}e^{-s}ds = r^{-\alpha}L\cdot p_0(Iu-Iv)$$

and

$$|(Fu)(t) - (Fv)(t)| \leqslant$$

$$\leqslant \frac{L}{\Gamma(\alpha)}\cdot p_n(Iu-Iv)\cdot\int_0^t (t-s)^{\alpha-1}\, e^{-r(t-s)}\, ds \leqslant$$

$$\leqslant \frac{1}{r^\alpha}\cdot\frac{L}{\Gamma(\alpha)}\cdot p_n(Iu-Iv)\cdot\int_0^\infty s^{\alpha-1}e^{-s}ds = r^{-\alpha}\, L\cdot p_n(Iu-Iv)$$

for a positive integer n and $n^{-1} \leq t \leq T$. This implies

$$p_m(Fu-Fv) \leq r^{-\alpha} L \cdot p_m(Iu-Iv), \quad m = 0,1,\ldots$$

for all u, v in X.

Consequently, there exists a unique $x \in X$ such that

$$x(t) = \frac{x_0}{\Gamma(\alpha)} t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s))\, ds.$$

We have

$$D^{-(1-\alpha)}\left(\frac{x_0}{\Gamma(\alpha)} t^{\alpha-1}\right) = x_0,$$

$$D^{-(1-\alpha)}(D^{-\alpha} f(t,x(t))) = D^{-1} f(t,x(t)) = \int_0^t f(s,x(s))ds.$$

Therefore

$$D^{-(1-\alpha)} x(t) = x_0 + \int_0^t f(s,x(s))ds$$

and

$$D^{\alpha} x(t) = \frac{d}{dt}\left(D^{-(1-\alpha)} \frac{x_0}{\Gamma(\alpha)} t^{\alpha-1}\right) +$$

$$+ \frac{d}{dt}(D^{-(1-\alpha)}(D^{-\alpha} f(t,x(t))) = f(t,x(t)),$$

and the proof is complete.

R e m a r k . Let E be a Banach algebra with norm $\|\cdot\|$. By $\mathcal{H}$ we shall denote the set of all continuous functions $v: (0,T] \to E$ such that $\int_0^T \|v(t)\| dt < \infty$.

Assume that $g \in \mathcal{H}$, $K \in \mathcal{H}$ and $h: I \times E \to E$ is a continuous function. Modifying the above reasoning we obtain the following generalization of Theorem 1.

T h e o r e m 2. If h is a function satisfying in the second variable the Lipschitz condition and $\|K(t)\| = O(t^{\alpha-1})$, $0 \leq t \leq T$, then the integral equation

$$x(t) = g(t) + \int_0^t K(t-s)h(s,x(s))ds$$

has a unique solution in the set $\mathcal{K}$.

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DEPARTMENT OF MATHEMATICS, HIGHER SCHOOL OF ENGINEERING,
65-246 ZIELONA GÓRA, POLAND
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