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NOTES ON MANIFOLDS ADMITTING TOTALLY UMBILICAL HYPERSURFACES

1. Conformally birecurrent manifolds

Let N be a (n+1)-dimensional connected Riemannian manifold of class C^{∞} with not necessarily definite metric g_{rs}, covered by a system of coordinate neighborhoods $\{U;x^r\}$. We denote by $\left\{ {r \atop s\ t} \right\}$, R_{vrst}, R_{rs}, R the Christoffel symbols, the curvature tensor, the Ricci tensor and the scalar curvature of N, respectively. The Weyl conformal curvature tensor

$$(1.1)\quad C_{rstu} = R_{rstu} - \frac{1}{n-1}\left(g_{st}R_{ru} - g_{su}R_{rt} + g_{ru}R_{st} - g_{rt}R_{su}\right) + \frac{R}{n(n-1)}\left(g_{st}g_{ru} - g_{su}g_{rt}\right)$$

is said to be recurrent ([8]) if the condition

$$(1.2)\quad C_{r_1\ldots r_4,v}C_{s_1\ldots s_4} = C_{s_1\ldots s_4,v}C_{r_1\ldots r_4}$$

holds on N where the comma denotes covariant differentiation. A Riemannian manifold of dimension ≥ 4 will be called conformally recurrent ([1]) if its Weyl conformal curvature tensor is recurrent. The relation (1.2) states that at each point $x \in N$ such that $C_{rstu}(x) \neq 0$ there exists unique vector b_v such that

$$(1.3)\quad C_{rstu,v} = b_v C_{rstu}.$$

The tensor C_{rstu} is said to be birecurrent if the condition

$$(1.4)\qquad C_{r_1\ldots r_4,vw}C_{s_1\ldots s_4} = C_{s_1\ldots s_4,vw}C_{r_1\ldots r_4}$$

is satisfied on N. In consequence, a Riemannian manifold of dimension $\geqslant 4$ is said to be conformally birecurrent ([2]) if its Weyl conformal curvature tensor satisfies (1.4). Evidently, if at some point x, $C_{hijk}(x) \neq 0$ holds then there exists a unique tensor a_{vw} such that

$$(1.5)\qquad C_{rstu,vw} = a_{vw}C_{rstu}$$

holds good.

R e m a r k s . (i) Every conformally recurrent manifold is conformally birecurrent.

(ii) Every conformally birecurrent manifold satisfies

$$(1.6)\qquad C_{rstu,[vw]} = C_{rstu,vw} - C_{rstu,wv} = c_{vw}C_{rstu},$$

where c_{vw} is a tensor field on N.

2. Totally umbilical hypersurfaces

Let M be a hypersurface of N, defined in a local coordinate system by means of the system of parametric equations $x^r = x^r(y^a)$, where y^a are local coordinates of M. Moreover, let the induced tensor $g_{ab} = g_{rs}B^r_aB^s_b$ be a metric tensor of M, where $B^r_a = \frac{\partial x^r}{\partial y^a}$. In the sequel we will use the notation

$$B^{r_1\ldots r_p}_{a_1\ldots a_p} = B^{r_1}_{a_1}B^{r_2}_{a_2}\ldots B^{r_p}_{a_p}$$

We denote by $\left\{{a \atop b\ c}\right\}$, ∇_a, $\bar{R}_{abcd}$, $\bar{R}_{ad}$, $\bar{R}$ the Christoffel symbols, the operator of covariant differentiation, the curvature tensor, the Ricci tensor and the scalar curvature of M with respect to g_{ab}. The vector field H^r defined by

$$H^r = \frac{1}{n} g^{ab} \nabla_b B^r_a = \frac{1}{n} g^{ab} \left[\partial_b B^r_a + \left\{{r \atop s\ t}\right\} B^{st}_{ba} - B^r_c \left\{{c \atop b\ a}\right\}\right]$$

is called the mean curvature vector field of M. If the tensor $H_{ab}^{r} = \nabla_b B_a^r$ satisfies the condition

$$H_{ab}^{r} = g_{ab}H^r, \tag{2.1}$$

then M is said to be a totally umbilical hypersurface.

Let N^r be a local unit normal to M. Then we have

$$\text{(a)}\quad g_{rs}N^rB_a^s = 0 \quad \text{and} \quad \text{(b)}\quad g_{rs}N^rN^s = \varepsilon \tag{2.2}$$

and

$$g^{ab}B_{ab}^{rs} = g^{rs} - \varepsilon N^rN^s, \tag{2.3}$$

where $\varepsilon = \pm 1$. Since H^r is normal to M, we have a relation of the form

$$H^r = \varepsilon\, H\, N^r. \tag{2.4}$$

The scalar function H is called the mean curvature of M. For a totally umbilical hypersurface M of N the equations of Weingarten, Gauss and Codazzi respectively, can be written in the form ([7])

$$\nabla_a N^r = -H\, B_a^r, \tag{2.5}$$

$$\bar{R}_{abcd} = R_{rstu}B_{abcd}^{rstu} + \varepsilon H^2(g_{ad}g_{bc} - g_{ac}g_{bd}) \tag{2.6}$$

and

$$R_{rstu}N^rB_{bcd}^{stu} = H_d g_{bc} - H_c g_{bd}, \quad H_c = \nabla_c H. \tag{2.7}$$

Moreover, for such hypersurface the relations [7]

$$R_{rs}N^rB_a^s = (n-1)H_a \tag{2.8}$$

and

$$N^rC_{rstu}B_{bcd}^{stu} = 0 \tag{2.9}$$

hold.

3. Totally umbilical hypersurface of certain manifolds

We put

$$(3.1)\qquad Q_{ad} = -g^{bc}C_{abcd} = \varepsilon N^r N^u C_{rstu} B^{st}_{ad}$$

and

$$(3.2)\qquad P_{ad} = \varepsilon N^r N^u R_{rstu} B^{st}_{ad},$$

where

$$(3.3)\qquad C_{abcd} = C_{rstu} B^{rstu}_{abcd}.$$

Transvecting (1.1) with $B^{rstu}_{abcd} g^{bc}$, using (2.3), (3.1) and (3.2), we obtain

$$(3.4)\qquad -Q_{ad} = \frac{1}{n-1} R_{ad} - P_{ad} + S_1 g_{ad},$$

where

$$(3.5)\qquad R_{ad} = R_{rs} B^{rs}_{ad}$$

and $S_1 = \frac{1}{n} R - \frac{1}{n-1} g^{bc} R_{bc}$.

The following lemma is a generalization of Proposition 1 ([7]).

L e m m a 1. Let M be a totally umbilical hypersurface of a manifold N satisfying the condition

$$(3.6)\qquad C_{rstu,[vw]} = c_{vw} C_{rstu} + F\, Q(C)_{rstuvw},$$

where F is a function on N, c_{vw} is a tensor field on N and $Q(C)_{rstuvw}$ is defined by

$$(3.7)\qquad Q(C)_{rstuvw} = g_{rv}C_{wstu} - g_{rw}C_{vstu} - g_{sv}C_{wrtu} + g_{sw}C_{vrtu} + g_{tv}C_{wurs} - g_{tw}C_{vurs} - g_{uv}C_{wtrs} + g_{uw}C_{vtrs}.$$

Then the relation

$$(3.8)\qquad H_f C_{rstu} = 0$$

holds on M.

P r o o f . The equality (3.6), by Ricci identity and (2.3) gives

$$(3.9)\qquad (-C_{pstu}R_{qrvw} + C_{prtu}R_{qsvw} - C_{purs}R_{qtvw} + $$
$$+ C_{ptrs}R_{quvw})(B^{pq}_{ij}g^{ij} + \varepsilon N^pN^q) = c_{vw}C_{rstu} + F\,Q(C)_{rstuvw}.$$

Transvecting (3.9) with $N^rB^{stuvw}_{bcdef}$ we get, in virtue of (2.7), (2.9), (3.7), (2.2)(a) and (3.1)

$$(3.10)\quad H_f(C_{ebcd} + g_{ce}Q_{bd} - g_{de}Q_{bc}) = H_e(C_{fbcd} + g_{cf}Q_{bd} - g_{df}Q_{bc}).$$

Contracting (3.10) with g^{ed} and using (3.1) we find

$$(3.11)\qquad -H^iC_{icbf} = (n-1)H_fQ_{bc} + H^iQ_{bi}g_{cf},\ H^i = g^{ij}H_j.$$

From (3.11), by standard calculations, we obtain

$$(3.12)\qquad H_fQ_{bc} = 0.$$

The last result together with (3.10) yields $H_fC_{ebcd} = H_eC_{fbcd}$, whence

$$(3.13)\qquad H_fC_{abcd} = 0.$$

The assertion of our lemma follows now from (2.9), (3.12), (3.1), (3.13) and (3.3).

As an immediate consequence of Lemma 1 we get the following generalization of Theorem 5 ([7]).

T h e o r e m 1. Let M be a totally umbilical hypersurface of manifold N satisfying the condition (1.6). If $C_{rstu} \neq 0$ everywhere on M, then the mean curvature of M is constant.

Theorem 1, together with Remarks, gives

C o r o l l a r y 1. Let M be a totally umbilical hypersurface of a conformally recurrent or conformally birecurrent manifold N. If $C_{rstu} \neq 0$ everywhere on M, then the mean curvature of M is constant.

L e m m a 2. Let M be a totally umbilical hypersurface of a manifold N satisfying (3.6). The following relation

$$(3.14)\qquad -\frac{1}{n-1}R_{bc,[ef]} = c_{ef}Q_{bc} + g_{bf}U_{ce} + g_{cf}U_{be} - g_{ce}U_{bf} - g_{be}U_{cf}$$

holds on M, where

$$(3.15)\qquad c_{ef} = c_{rs}B^{rs}_{ef},$$

$$(3.16)\qquad \begin{cases} U_{da} = U_{ad} = (\varepsilon H^2 - F)Q_{ad} + \bar{U}_{ad}, \\ \bar{U}_{ad} = (S_2+S_1)P_{ad} - \frac{1}{n-1}R_{ia}P^i{}_d + F\,Q_{ad},\ P^i{}_d = g^{ij}P_{jd} \\ \text{and}\quad S_2 = \frac{R}{n(n-1)} - \varepsilon H^2. \end{cases}$$

P r o o f . Transvecting (3.9) with $N^rN^wB^{stuv}_{bcde}$ we obtain, in virtue of (2.9), (3.1)-(3.3), (3.5), (3.7) and (2.2)

$$(3.17)\qquad C_{ibcd}P^i{}_e + Q_{bd}P_{ce} - Q_{bc}P_{de} = F(-C_{ebcd} - g_{ce}Q_{bd} + g_{de}Q_{bc}).$$

On other hand, transvecting (3.3) with $P^a{}_e$ and using (1.1), (2.6) and (3.5), we find

$$(3.18)\qquad C_{ibcd}P^i{}_e = P^i{}_e\bar{R}_{ibcd} - \frac{1}{n-1}(P_{ed}R_{bc} - P_{ec}R_{bd} + g_{bc}R_{id}P^i_e - g_{bd}R_{ic}P^i{}_e) + S_2(P_{ed}g_{bc} - P_{ec}g_{bd}).$$

Symmetrizing (3.18) in (e,b) we obtain

$$(3.19)\qquad C_{ibcd}P^i{}_e + C_{iecd}P^i{}_b = P^i{}_e\bar{R}_{ibcd} + P^i{}_e R_{iecd} + S_2(P_{ed}g_{bc} + P_{bd}g_{ec} - P_{ec}g_{bd} - P_{bc}g_{ed}) - \frac{1}{n-1}(P_{ed}R_{bc} + P_{bd}R_{ec} - P_{ec}R_{bd} - P_{bc}R_{ed}) - \frac{1}{n-1}(g_{bc}R_{id}P^i_e + g_{ec}R_{id}P^i{}_b - g_{bd}R_{ic}P^i{}_e - g_{ed}R_{ic}P^i{}_b).$$

Moreover, symmetrizing (3.17) in (e,b), applying (3.19) and Ricci identity we get

$$(3.20)\quad P_{eb,[cd]} = S_2(g_{bc}P_{ed} + g_{ec}P_{bd} - g_{bd}P_{ec} - g_{ed}P_{bc}) - \frac{1}{n-1}(g_{bc}R_{id}P^i{}_e + g_{ec}R_{id}P^i{}_b - g_{bd}R_{ic}P^i{}_e - g_{ed}R_{ic}P^i{}_b) + Q_{bd}P_{ce} + Q_{ed}P_{bc} - Q_{bc}P_{de} - Q_{ec}P_{bd} - F(g_{de}Q_{bc} - g_{ce}Q_{bd} + g_{bd}Q_{ec} - g_{bc}Q_{ed}) - \frac{1}{n-1}(P_{ed}R_{bc} + P_{bd}R_{ec} - P_{ec}R_{bd} - P_{bc}R_{ed}).$$

Finally, substituting in (3.20), (3.4) and (3.16) we obtain

$$(3.21)\quad P_{eb,[cd]} = g_{bc}\bar{U}_{de} + g_{ec}\bar{U}_{db} - g_{bd}\bar{U}_{ce} - g_{ed}\bar{U}_{cb}.$$

Now we prove, that

$$(3.22)\quad R_{bi}P^i{}_d = R_{di}P^i{}_b.$$

Contracting (3.17) with g^{ed} and making use of (3.1) we obtain $C_{ibcj}P^{ij} + Q_{bi}P^i{}_c - g^{ij}P_{ij}Q_{bc} = nFQ_{bc}$, $P^{ij} = g^{aj}P^i{}_a$, whence it follows $Q_{bi}P^i{}_c = Q_{ci}P^i{}_b$. Transvecting (3.4) with $P^a{}_e$ and using the last relation we get (3.22). Making use of the definition of $\bar{U}_{ad}$ and (3.22) we state that U_{ad} is symmetric.

The following relation

$$(3.23)\quad Q_{bc,[ef]} = (\varepsilon H^2 - F)(g_{bf}Q_{ec} - g_{be}Q_{cf} + g_{cf}Q_{eb} - g_{ce}Q_{fb}) + c_{ef}Q_{bc}$$

holds on M. In fact, transvecting (3.9) with $\varepsilon N^r N^u B^{stvw}_{bcef}$ and applying (2.9), (2.6), (3.1), (3.7) and Ricci identity, we find (3.23).

On the other hand, from (3.4) we obtain $-\frac{1}{n-1}R_{ad,[ef]} = Q_{ad,[ef]} - P_{ad,[ef]}$. This together with (3.23) and (3.21) leads to (3.14) Our lemma is thus proved.

L e m m a 3. Let M be a totally umbilical hypersurface of a manifold N satisfying (3.6). Then the relation

$$c_{vw}C_{rstu} = 0 \tag{3.24}$$

holds on M.

P r o o f . Transvecting (3.9) with B^{rstuvw}_{abcdef}, making use of (3.3), (2.6), (2.9), (3.15), (3.7) and the Ricci identity we obtain

$$C_{abcd,[ef]} = c_{ef}C_{abcd} + (F - \varepsilon H^2)Q(C)_{abcdef},$$

whence, in view of (3.3), (1.2), (2.6) and (3.5) we get

$$\begin{aligned}\bar{R}_{abcd,[ef]} - \frac{1}{n-1}(g_{ad}R_{bc,[ef]} - g_{ac}R_{bd,[ef]} + \\ + g_{bc}R_{ad,[ef]} - g_{bd}R_{ac,[ef]}) = c_{ef}C_{abcd} + \\ + (F - \varepsilon H^2)Q(C)_{abcdef}.\end{aligned} \tag{3.25}$$

Permuting (3.25) in pairs of indices (a,b), (c,d), (e,f) adding the resulting equations to (3.25) and using (3.14) we obtain

$$c_{ef}W_{abcd} + c_{ab}W_{cdef} + c_{cd}W_{efab} = 0, \tag{3.26}$$

where

$$W_{abcd} = C_{abcd} - (g_{ad}Q_{bc} - g_{ac}Q_{bd} + g_{bc}Q_{ad} - g_{bd}Q_{ac}), \tag{3.27}$$

whence

$$c_{ef}W_{abcd} = 0. \tag{3.28}$$

Contracting (3.28) with g^{bc} we get, in virtue of (3.27) and (3.1), $c_{ef}Q_{ad} = 0$. Hence (3.27) and (3.28) yield $c_{ef}C_{abcd} = 0$. Finally (2.9), (3.3) and (3.1) lead immediately to

$$c_{ef}C_{rstu} = 0. \tag{3.29}$$

Assume now that at some point $x \in M$

$$C_{rstu}(x) \neq 0 \tag{3.30}$$

Then it readily follows that the tensors $c_{vw}B^{vw}_{ef}$, $c_{vw}B^{v}_{e}N^{w}$ and $c_{vw}N^{v}N^{w}$ vanish at x. Thus $c_{vw}(x) = 0$. The last remark completes the proof.

Lemma 3, in virtue of Remarks, implies

T h e o r e m 2. Let N be a conformally birecurrent (resp. conformally recurrent) manifold. If N admits a totally umbilical hypersurface M, then on M the relation (1.1) is satisfied.

4. Totally umbilical hypersurfaces of conformally birecurremtn manifolds

If a totally umbilical submanifold M (dim $M \geqslant 4$) of a conformally birecurrent manifild N is also conformally birecurrent, then the relation

$$g_{rs}H^{r}H^{s}\,\bar{C}_{abcd} = 0 \tag{4.1}$$

holds on M ([3]), Corollary 1), where $\bar{C}_{abcd}$ is Weyl conformal curvature tensor of M. If M is a conformally birecurrent totally umbilical hypersurface of a conformally birecurrent manifold, then the condition (4.1), by (2.4), is reduced to

$$H\,\bar{C}_{abcd} = 0. \tag{4.2}$$

In this section we prove, that (4.2) yields

$$H\,C_{rstu} = 0. \tag{4.3}$$

We put

$$S_{abcdef} = R_{rstu,v}\nabla_{f}(B^{rstuv}_{abcde}) \tag{4.4}$$

and

$$D_{abcd} = R_{rstu,v}B^{rstv}_{abcd}H^{u}. \tag{4.5}$$

The tensors S_{abcdef} and D_{abcd} satisfy the following relations ([3], formulas (2.16) and (2.19))

$$(4.6)\quad S_{abcdef} = g_{fa}D_{dcbe} + g_{fb}D_{cdae} + g_{fc}D_{bade} + g_{fd}D_{abce} + + g_{ef}(D_{abcd} - D_{abdc})$$

and

$$(4.7)\quad D_{abcd} = g_{bc}E_{ad} - g_{ac}E_{bd} + \varepsilon H^2(\bar{R}_{abcd} - - \varepsilon H^2(g_{ad}g_{bc} - g_{ac}g_{bd})) + H^2(g_{bd}P_{ac} - g_{ad}P_{bc}),$$

where $E_{ad} = \frac{1}{2}\varepsilon\nabla_d\nabla_a(H^2)$.

L e m m a 4. Let M be a conformally birecurrent totally umbilical hypersurface of a conformally birecurrent manifold N. If the condition

$$(4.8)\quad C_{rstu}(x) \neq 0$$

holds at a certain point $x \in M$, then $H(x) = 0$.

P r o o f . The following equation ([3], Lemma 4) is satisfied on some neighborhood $V \subset M$ of x

$$(4.9)\quad \nabla_f\nabla_e\bar{C}_{abcd} - a_{ef}\bar{C}_{abcd} = T_{abcdef},$$

where $a_{ef} = a_{vw}B^{vw}_{ef}$, a_{vw} is the tensor of birecurrency of C_{rstu},

$$(4.10)\quad T_{abcdef} = S_{abcdef} - \frac{1}{n-2}(g_{ad}S_{bcef} - g_{ac}S_{bdef} - - g_{bd}S_{acef} + g_{bc}S_{adef}) + \frac{1}{(n-1)(n-2)}S_{ef}(g_{ad}g_{bc}-g_{ac}g_{bd}),$$

where $S_{adef} = g^{bc}S_{abcdef}$ and $S_{ef} = g^{bc}S_{bcef}$.

We suppose that

$$(4.11)\quad H(x) \neq 0.$$

Then, by (4.2), we obtain on the neighborhood $V' \subset V$

(4.12) $$\bar{C}_{abcd} = 0$$

whence, in virtue of (4.9), it follows that

(4.13) $$T_{abcdef} = 0.$$

Lemma 1, in view of Remark (ii) and (4.8), yields on V'

(4.14) $$\nabla_f H = 0.$$

Now the equalities (4.7), (4.10), (4.13) and (4.14) give

(4.15) $$D_{abcd} = \varepsilon H^2(\bar{R}_{abcd} - \varepsilon H^2(g_{ad}g_{bc} - g_{ac}g_{bd})) + $$
$$+ H^2(g_{bd}P_{ac} - g_{ad}P_{bc})$$

and

$$S_{abcdef} - \frac{1}{n-2}(g_{ad}S_{bcef} - g_{ac}S_{bdef} - g_{bd}S_{acef} + g_{bc}S_{adef}) +$$
$$+ \frac{1}{(n-1)(n-2)} S_{ef}(g_{ad}g_{bc} - g_{ac}g_{bd}) = 0,$$

whence, substituting (4.6) and (4.15), by transvection with $g^{af}g^{ed}$ we find

(4.16) $$\varepsilon P_{bc} = \frac{1}{n-2}(\bar{R}_{bc} - \frac{\bar{R}}{n} g_{bc}) + \varepsilon \frac{P}{n} g_{bc},$$

where $P = g^{bc}P_{bc}$. On the other hand, contracting (2.6) with g^{ad} using (3.2), (3.5) and (4.16) we obtain

(4.17) $$\frac{1}{n-2}\bar{R}_{ad} = \frac{1}{n-1} R_{ad} + \frac{1}{n-1} Z g_{ad},$$

where $Z = \frac{\bar{R}}{n(n-2)} - \varepsilon \frac{P}{n} + (n-1)\varepsilon H^2$. Now the equality (4.12), together with (2.6), (4.17), (1.1) and (3.3), gives

(4.18) $$C_{abcd} = K(g_{ad}g_{bc} - g_{ac}g_{bd}),$$

whence $C_{abcd} = 0$ and $Q_{ad} = 0$.

Since (2.9) holds, $C_{rstu}(x) = 0$, which is a contradiction. This completes the proof.

Lemma 4 implies

T h e o r e m 3. Let M be a conformally birecurrent totally umbilical hypersurface of a conformally birecurrent manifold N. Then on M the relation (4.3) is satisfied.

As it is known ([6]) every totally umbilical submanifold of a conformally recurrent manifold is also conformally recurrent. Using this fact, Remark (i) and Theorem 3 we obtain

C o r o l l a r y 2. (cf. [7], Proposition 1). Let M be a totally umbilical hypersurface of a conformally recurrent manifold N. Then on M the relation (4.3) is satisfied.

5. Example

In the last section we give an example of a totally umbilical hypersurface satisfying the condition (4.3) with non-zero function H.

Let $\{V;x^\alpha\}$ be a locally chart on an n-dimensional manifold $(\overset{*}{M},\overset{*}{g})$ of constant curvature, $\alpha,\beta \in \{2,3,\dots,n+1\}$, $n \geq 3$. We denote by $\overset{*}{g}_{\alpha\beta}$ the components of $\overset{*}{g}$ on V. Moreover, let on $\mathbb{R}$, with identity map x^1, be given the metric tensor $\bar{g}_{11} = 1$.

Now we define on the manifold $N = \mathbb{R}\times V$ the metric $\tilde{g}_{rs}$ by

$$(5.1)\qquad \tilde{g}_{rs} = \begin{cases} \bar{g}_{11} & \text{if } r = s = 1 \\ \sigma\overset{*}{g}_{\alpha\beta} & \text{if } r = \alpha \text{ and } s = \beta, \\ 0 & \text{otherwise} \end{cases}$$

where $\sigma = \sigma(x^1)$ is a differentiable function different from zero in every point of N. The manifold N with the metric $\tilde{g}_{rs}$ is conformally flat ([4], pp.176, 179).

The manifold M defined by the equations $x^1 = C = \text{const}$, $x^2 = y^1$, $x^3 = y^2,\dots,x^{n+1} = y^n$, with metric $\sigma(C)\overset{*}{g}_{\alpha\beta}$ induced from $\tilde{g}_{rs}$ is a totally umbilical submanifold of N ([5], Theorem 1). The mean curvature vector field H^r of M is given by ([5], p.107)

$$H^r = \begin{cases} -\frac{1}{2}\bar{g}^{11}\ \partial_1 \ln|\sigma| & \text{if } r = 1 \\ 0 & \text{otherwise} \end{cases} \tag{5.2}$$

The above relation, together with (2.4) and (2.2)(b) gives

$$N^r = \begin{cases} 1 & \text{if } r = 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and } \varepsilon = 1. \tag{5.3}$$

From (2.11), in virtue of (5.2) and (5.3), we obtain the equality $H = -\frac{1}{2}\partial_1 \ln|\sigma|$. If σ is a non-constant function on N, then H is non-zero on M. Since N is conformally flat, the condition (4.3) holds on M.

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