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A PROLONGATION OF THE REAL ALMOST-PRODUCT STRUCTURE OF A DIFFERENTIABLE MANIFOLD

One of the most interesting real G-Structure of the first order is the real almost-product structure (π_R-structure) defined by G.Legrande ([7]) by means of a linear operator J acting on the tangent vector space T_x at each point x of a differentiable manifold V_m and satisfying a relation of the form

$$J^2 = \text{Identity},$$

which gives a decomposition of this space in a direct sum.

To this structure there is found a prolongation, in the sense, that there is defined a real G-structure of order 2, called almost product structure of order 2 (π_R^2-structure), by means of a linear operator J satisfying the same relation on the vector space T_x^2 of the second order tangent vectors.

For this purpose, there is given a brief discussion of the fibre bundle of the second order tangent vectors and its associated principal prolongation of order 2 of a differentiable manifold, from the standpoint that will be used throughout this paper. Then there is defined the π_R^2-structure, its adapted basis, connection, characteristic cohomology class and holonomy group.

1. Let V_m be an m-dimensional differentiable manifold of class C^∞ and $T^2_{1,x}{}^*(V_m)$ the set of all 2-jets of the functions

on V_m with source $x \in V_m$ and target 0. According to C.Ehresmann ([4], [6]), the set $T_1^{2*}(V_m) = \bigcup_{x \in V_m} T^2_{1,x}{}^*(V_m)$ has the structure of a fibre bundle with basis V_m, structural group L^2_m and fibre $L^2_{1,m}$.

The structural group L^2_m is ([6]) the set $j^2_0 f$ of all invertible 2-jets with source and target $0 \in R^m$ of a 2-mapping f at the point $0 \in R^m$, which can be expressed by the form,

$$y^i = a^i_{j_1} x^{j_1} + \frac{1}{2!} a^i_{j_1 j_2} x^{j_1} x^{j_2},$$

where $\{y^i\}$ is a coordinate system of $y = f(x)$ in the neighborhood of the point $0 \in R^m$, $\{x^j\}$ is a coordinate system of x in the neighborhood of the point $0 \in R^m$, $a^i_{j_1} = \left(\frac{\partial y^i}{\partial x^{j_1}}\right)_0$ with $\det(a^i_{j_1}) \neq 0$ and $a^i_{j_1 j_2} = \left(\frac{\partial^2 y}{\partial x^{j_1} \partial x^{j_2}}\right)_0$.

Hence, each $a \in L^2_m$ can be written in the form,

(1.1) $a = (a^i_{j_1}, a^i_{j_1 j_2}), \quad i, j_1, j_2 = 1,2,\dots,m, \ \det (a^i_{j_1}) \neq 0$

and $a^i_{j_1 j_2}$ is symmetric with respect to j_1, j_2.

If

$$\beta = \left(\beta^{j_1}_{k_1}, \beta^{j_1}_{k_1 k_2}\right) \in L^2_m$$

then, from the composition of 2-jets ([6]), it follows that the product of the two elements a and β of L^2m,

$$a\beta = c = \left(c^i_{k_1}, c^i_{k_1 k_2}\right),$$

can be defined by the relations

$$(1.2)\quad \begin{cases} c^i_{k_1} = a^i_{j_1}\,\beta^{j_1}_{k_1}, \\ c^i_{k_1k_2} = a^i_{j_1}\,\beta^{j_1}_{k_1k_2} + a^i_{j_1j_2}\,\beta^{j_1}_{k_1}\,\beta^{j_2}_{k_2}. \end{cases}$$

Also,

$$(1.3)\qquad \dim L^2_m = N = m\binom{m+2}{2} - m = m^2 + m\binom{m+1}{2}.$$

The Lie algebra $\underline{L^2_m}$ of the Lie group L^2_m is ([8]) defined by

$$(1.4)\quad \underline{L^2_m} = \Big\{\lambda/\lambda = \big(\lambda^i_{j_1},\ \lambda^i_{j_1j_2}\big),\ i,j_1,j_2=1,2,\dots,m,\ \lambda^i_{j_1}\in R^m\otimes R^{m*},\ \lambda^i_{j_1j_2}\in R^m\otimes S^2(R^{m*})\Big\},$$

where $S^2(R^{m*})$ is the set of the 2-linear symmetric forms on R^m.

Similarly, $L^2_{1,m}$ is ([3], [6]) the set $j^2_0 g$ of all 2-jets with source $0\in R^m$ and target $0\in R$ of a 2-mapping g at $0\in R^m$, which can be expressed by the form,

$$t = y_{i_1}\,x^{i_1} + \frac{1}{2!}\,y_{i_1i_2}\,x^{i_1}x^{i_2},$$

where $\{x^i\}$ is a coordinate system at x in the neighborhood of $0\in R^m$, $t = g(x)$ is its image by g in the neighborhood of $0\in R$, $y_{i_1} = \left(\frac{\partial t}{\partial x^{i_1}}\right)_0$ and $y_{i_1i_2} = \left(\frac{\partial^2 t}{\partial x^{i_1}\partial x^{i_2}}\right)_0$.

Therefore, each $y\in L^2_{1,m}$ can be written in the form,

$$(1.5)\quad y = y_{i_1}, y_{i_1i_2},\quad i_1,i_2=1,2,\dots,m,\ y_{i_1i_2} \text{ is symmetric with respect to } i_1,i_2.$$

Also,

$$(1.6)\qquad \dim L^2_{1,m} = v(m,2) = \binom{m+2}{m} - 1 = m + \binom{m+1}{2}.$$

Let $\{x^i\}_{i=1,2,\dots,m}$ be a system of local coordinates at $x \in V_m$ for a given chart. Then, according to definition ([4]) of the prolongation of a chart on $T_1^{2*}(V_m)$, the element $\omega \in T_1^{2*}(V_m)$ can be expressed by the form,

(1.7) $\omega = \left(x^i, y_{i_1}, y_{i_1 i_2}\right)$, $i, i_1, i_2 = 1,2,\dots,m$, $y_{i_1 i_2}$ is symmetric with respect to i_1, i_2.

If

$$\left(x^{j'}, y_{j_1'}, y_{j_1' j_2'}\right)$$

is the expression of ω in a new coordinate system $\{x^{j'}\}_{j=1,2,\dots,m}$ at $x \in V_m$, then from relations given by C.Ehresmann ([6]), the transformation law for the local coordinates of ω is given by the equations,

$$(1.8)\qquad \begin{cases} x^{j'} = \varphi^{j'}(x^i) \\ y_{j_1'} = y_{j_1} a^{i_1}_{j_1'} \\ y_{j_1' j_2'} = y_{i_1} a^{i_1}_{j_1' j_2'} + y_{i_1 i_2} a^{i_1}_{j_1'} a^{i_2}_{j_2'}, \end{cases}$$

where the functions $\varphi^{j'}$ have continuous partial derivatives of every kind up to order 2 and $\left(a^{i}_{j_1'}\ a^{i}_{j_1' j_2'}\right) \in L^2_m$.

The equations (1.8) provides also the left action of L^2_m on the vector space $L^2_{1,m}$ which is isomorphic to the fibres $T^{2}_{1,x}{}^*(V_m)$ over $x \in V_m$.

The dual fibre bundle $T^2(V_m)$ of the fibre bundle $T_1^{2*}(V_m)$, has ([2]) a basis V_m, structural group L^2_m and fibre $F^2 = (L^2_{1,m})^*$. Hence, each $X \in T^2_x$ (where T^2_x is the fibre over $x \in V_m$ of the fibre bundle $T^2(V_m)$) can be defined by the linear mapping,

$$X : T^2_{1,x} \longrightarrow R.$$

C o n c l u s i o n 1.1. The fibre bundle $T^2(V_m)\times$ $\times(V_m, L^2_m, F^2)$ is ([1]) the fibre bundle of all tangent vectors of order 2 and $X \in T^2_x$ is a tangent vector of order 2 (or 2-tangent vector) at the point $x \in V_m$.

Let

$$(1.9)\quad e = \left(e_{i_1}, e_{i_1 i_2}\right),\ i_1, i_2 = 1,2,\dots,m,\ e_{i_1 i_2} \text{ symmetric in the indices } i_1, i_2,$$

be a basis of T^2_x (therefore and of its isomorphic vector space F^2) with respect to the local system $\{x^i\}_{i=1,2,\dots,m}$ at the point $x \in V_m$.

It is evident that every $X \in T^2_x$ can be expressed uniquely in the form

$$(1.10)\qquad X = \sum_{i_1=1}^{m} X^{i_1} e_{i_1} + \sum_{1\leq i_1 \leq i_2 \leq m} X^{k_1 i_2} e_{i_1 i_2},$$

where $\left(X^{i_1}, X^{i_1 i_2}\right)$ are functions of the neighborhood (U, x^i) on V_m and $X^{i_1 i_2}$ is symmetric in the indices i_1, i_2.

For another system of local coordinates $\{x^{j'}\}_{j=1,2,\dots,m}$ at $x \in V_m$ the second order tangent vectors,

$$\left(e_{j_1'},\ e_{j_1' j_2'}\right),$$

of the new basis of T^2_x are transformed to the basis (1.9) as follows,

$$(1.11)\qquad \begin{cases} e_{i_1} = e_{j_1'}\, a^{j_1'}_{i_1} \\ e_{i_1 i_2} = e_{j_1'}\, a^{j_1'}_{i_1 i_2} + e_{j_1' j_2'}\, a^{j_1'}_{i_1}\, a^{j_2'}_{i_2}, \end{cases}$$

where $a = \left(a^{j_1'}_{i_1},\ a^{j_1'}_{i_1 i_2}\right) \in L^2_m$.

Then, the corresponding transformation law for the local coordinates of $X \in T^2_x$ will be,

$$(1.12) \qquad \begin{cases} X^{j'_1} = a^{j'_1}_{i_1} X^{i_1} + a^{j'_1}_{i_1 i_2} X^{i_1 i_2} \\ X^{j'_1 j'_2} = a^{j'_1}_{i_1} a^{j'_2}_{i_2} X^{i_1 i_2}. \end{cases}$$

The equations (1.12) establishes also the left action of L^2_m on the fibre F^2.

We consider the fibre bundle of all 2-frames of the manifold V_m, $H^2(V_m) = \bigcup_{x \in V_m} H^2_x(V_m)$, where H^2_x is ([3]) the set of all invertible 2-jets of R^m into V_m with source $0 \in R^m$ and target $x \in V_m$.

The above fibre bundle is ([4]) a principal fibre bundle with basis V_m, structural group L^2_m and is called principal prolongation of order 2 of the manifold V_m. It is also associated ([6]) to the fibre bundle $T^{2*}_1(V_m)(V_m, L^2_m, L^2_{1,m})$. Hence, to each h $H^2(V_m)$ there corresponds the isomorphism

$$h^{-1}: z \in L^2_{1,m} \rightarrow zh^{-1} \in T^{2*}_{1,x}(V_m),$$

or

$$h: T^{2*}_{1,x}(V_m) \rightarrow L^2_{1,m}.$$

If we consider their dual spaces, it follows that each $h \in H^2(V_m)$ can be identified with the isomorphism,

$$(1.12) \qquad h: y \in F^2 \rightarrow hy \in T^2_x.$$

Therefore, $H^2(V_m)(V_m, L^2_m)$ is also the associated principal fibre bundle to the fibre bundle $T^2(V_m)(V_m, L^2_m, F^2)$ of the tangent vectors of order 2.

From (1.13) it follows that h can be identified with the image

(1.14) $\left(h_{i_1}, h_{i_1 i_2}\right)$, $i_1, i_2 = 1,2,\dots,m$, $h_{i_1}, h_{i_1 i_2}$ linear forms on $T^{2*}_{1,x}$ and $h_{i_1 i_2}$ symmetric in the indices i_1, i_2,

by means of h of a basis (1.9) of the vector space F^2.

C o n c l u s i o n 1.2. $H^2(V_m)$ can be identified with the space of bases of the vector spaces T^2_x at $x \in V_m$.

2. A real almost product structure of second order (briefly a π^2_R-structure) can be defined on a manifold V_m by means of two fields of class C^∞ of proper supplementary subspaces of T^2_x, T_1 and T_2 with $\dim T_1 = n_1 + \binom{n_1+1}{2} + \binom{n_2+1}{2}$, $\dim T_2 = n_2 + n_1 n_2$ ($n_1 \neq 0$, $n_2 \neq 0$, $n_1 + n_2 = m$).

If J_x is a linear operator on T^2_x such that,

$$J_x v = v_1 - v_2, \quad v_1 \in T_1, \ v_2 \in T_2,$$

then

$$J^2_x = \text{Identity on } T^2_x = I_{v(m,2)}. \tag{2.1}$$

To this operator there corresponds the element F of the tensor product $T^2_x \otimes (T^2_x)^*$,

$$F = \left(F^{j_1}_{i_1}, F^{j_1 j_2}_{i_1}, F^{j_1}_{i_1 i_2}, F^{j_1 j_2}_{i_1 i_2}\right) \quad i_1, i_2, j_1, j_2 = 1,2,\dots,m, \tag{2.2}$$

$F^{j_1 j_2}_{i_1}$ symmetric with respect to j_1, j_2, $F^{j_1}_{i_1 i_2}$ symmetric with respect to i_1, i_2 and $F^{j_1 j_2}_{i_1 i_2}$ symmetric in the indices i_1, i_2 and j_1, j_2.

It is defined by,

$$\begin{cases} (J_x v)^{j_1} = F^{j_1}_{i_1} v^{i_1} + F^{j_1}_{i_1 i_2} v^{i_1 i_2} \\ (J_x v)^{j_1 j_2} = F^{j_1 j_2}_{i_1} v^{i_1} + F^{j_1 j_2}_{i_1 i_2} v^{i_1 i_2}, \end{cases} \tag{2.3}$$

where $v = \left(v^{i_1},\ v^{i_1 i_2}\right)$ is the 2-tangent vector at $x \in V_m$.

R e m a r k 2.1. The relation (1.12) for the element v of T_x^2 in the overlap of two neighborhoods (U, x^i) and $(V, x^{j'})$, by means of matrices, can be written in the form,

$$(2.4)\qquad \begin{bmatrix} v^{j'_1} & v^{j'_1 j'_2} \end{bmatrix} = \begin{bmatrix} v^{i_1} & v^{i_1 i_2} \end{bmatrix} \begin{bmatrix} a^{j'_1}_{i_1} & 0 \\ a^{j'_1}_{i_1 i_2} & a^{j'_1}_{i_1} a^{j'_2}_{i_2} \end{bmatrix},$$

or briefly,

$$V_x^v = A_u^v(x)\ V_x^u,$$

where

$$(2.5)\qquad \begin{cases} A_u^v(x) = \begin{bmatrix} a^{j'_1}_{i_1} & 0 \\ a^{j'_1}_{i_1 i_2} & a^{j'_1}_{i_1} a^{j'_2}_{i_2} \end{bmatrix}, \quad V_x^u = \begin{bmatrix} v^{i_1} & v^{i_1 i_2} \end{bmatrix}, \\ V_x^v = \begin{bmatrix} v^{j'_1} & v^{j'_1 j'_2} \end{bmatrix}. \end{cases}$$

Hence, the element $a = \left(a^{j'_1}_{i_1},\ a^{j'_1}_{i_1 i_2}\right)$ of L_m^2 can be identified with the matrice A of the relation (2.5).

Similarly, according to the relation (1.8), the transformation law for each element ω of T_x^{2*} (that is of $T^{2\,*}_{1,x}(V_m)$) by means of matrices, can be written by the form,

$$(2.6)\qquad \begin{bmatrix} \omega_{j'_1} \\ \omega_{j'_1 j_2} \end{bmatrix} = \begin{bmatrix} a^{i_1}_{j'_1} & 0 \\ a^{i_1}_{j'_1 j'_2} & a^{i_1}_{j'_1} a^{i_2}_{j'_2} \end{bmatrix} \begin{bmatrix} \omega_{i_1} \\ \omega_{i_1 i_2} \end{bmatrix}$$

where $\left(a^{i_1}_{j'_1},\ a^{i_1}_{j'_1 j'_2}\right) \in L_m^2$ is the inverse element of $a = \left(a^{j'_1}_{i_1},\ a^{j'_1}_{i_1 i_2}\right) \in L_m^2$.

Briefly,

$$\omega^x_v = \omega^x_u \, A^u_v(x), \quad A^u_v(x) = \left[A^v_u(x)\right]^{-1}.$$

For convenience in calculation, from now on, we will keep using matrices.

Thus, the tensor F of the relation (2.2) can be represented by the following matrix

$$F = \begin{bmatrix} F^{j_1}_{i_1} & F^{j_1 j_2}_{i_1} \\ F^{j_1}_{i_1 i_2} & F^{j_1 j_2}_{i_1 i_2} \end{bmatrix}. \tag{2.7}$$

From (2.1) and (2.3) one verifies easily the following equations,

$$\begin{cases} F^{j_1}_{i_1} F^{i_1}_{k_1} + F^{j_1}_{i_1 i_2} F^{i_1 i_2}_{k_1} = \delta^{j_1}_{k_1} \\ F^{j_1}_{i_1} F^{i_1}_{k_1 k_2} + F^{j_1}_{i_1 i_2} F^{i_1 i_2}_{k_1 k_2} = 0 \\ F^{j_1 j_2}_{i_1} F^{i_1}_{k_1} + F^{j_1 j_2}_{i_1 i_2} F^{i_1 i_2}_{k_1} = 0 \\ F^{j_1 j_2}_{i_1} F^{i_1}_{k_1 k_2} + F^{j_1 j_2}_{i_1 i_2} F^{i_1 i_2}_{k_1 k_2} = \delta^{j_1}_{k_1} \delta^{j_2}_{k_2}. \end{cases} \tag{2.8}$$

Conversely, let us assume that there is given on the manifold V_m, a tensor field on T^2_x of type $\binom{1}{1}$ and of class C^∞, which satisfies the equations (2.8). Then, to each point $x \in V_m$, the linear operator J_x on T^2_x, which is defined by the tensor F from the relations (2.3), has proper values 1 and -1. Let T_1 (respectively T_2) the subspace of T^2_x derived from the proper 2-tangent vectors which corresponds to the proper value 1 (respectively -1). If $v \in T^2_x$ and $v_1 = v + J_x v \in T_1$, $v_2 = v - J_x v$, then $v = \frac{v_1 + v_2}{2}$, that is, T_1 and T_2 are supplementary spaces and V_m is equipped with a π^2_R-structure.

3. A basis $e = \left(e_{i_1}, e_{i_1 i_2}\right)$ of T_x^2 will be called adapted to the π_R^2-structure with respect to x (briefly π_R^2-adapted basis) if $(e_{\alpha_1}, e_{\alpha_1\alpha_2}, e_{A_1A_2}) \in T_1$ et $(e_{A_1}, e_{\alpha_1 A_1}) \in T_2$
(i and every latin index takes its value in 1,2,...,m, α and every greek index takes its value in $1,2,\dots,n_1$ and finally, A and every capital latin index takes its value in $n_1+1,\dots,m$).

If $(e_{j'_1}, e_{j'_1 j'_2}) = \left\{(e_{\beta'_1}, e_{\beta'_1\beta'_2}, e_{\beta'_1\beta'_2}), (e_{\beta'_1}, e_{\beta'_1\beta'_1})\right\}$
is another π_R^2-adapted basis, the element $k = \left(k_{i_1}^{j'_1}, k_{i_1 i_2}^{j'_1}\right) \in L_m^2$ of the transformation law between π_R^2-adapted bases may be written,

$$
(3.1)\qquad \begin{cases} k_{i_1}^{j'_1} = \left(k_{\alpha_1}^{\beta'_1}, k_{A_1}^{\beta'_1}\right) \in L_{(n_1,n_2)} \\ k_{i_1 i_2}^{j'_1} = \left(k_{\alpha_1\alpha_2}^{\beta'_1}, k_{A_1A_2}^{\beta'_1}, k_{\alpha_1 A_1}^{\beta'_1}\right), \end{cases}
$$

where $L_{(n_1,n_2)}$ is ([7]) the structural group of the real almost product structure of order 1.

Using matrices, k can be written by the matrix,

$$
(3.2)\qquad K = \begin{bmatrix} \begin{bmatrix} k_{\alpha_1}^{\beta'_1} & 0 \\ 0 & k_{A_1}^{B'_1} \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} k_{\alpha_1\alpha_2}^{\beta'_1} & 0 \\ 0 & k_{\alpha_1 A_1}^{\beta'_1} \\ k_{A_1A_2}^{\beta'_1} & 0 \end{bmatrix} & \begin{bmatrix} k_{\alpha_1}^{\beta'_1} k_{\alpha_2}^{\beta'_2} & 0 & 0 \\ 0 & k_{\alpha_1}^{\beta'_1} k_{A_1}^{B'_1} & \\ 0 & 0 & k_{A_1}^{B'_1} k_{A_2}^{B'_2} \end{bmatrix} \end{bmatrix}
$$

Let $L^2_{(n_1,n_2)}$ be the subgroup of L^2_m consisting of all elements of the form (3.1) with corresponding matrix of the form (3.2).

One can easily verify that with respect to an π^2_R-adapted basis the tensor F associated to the operator J can be represented by the matrix,

$$(3.3)\qquad F = \begin{bmatrix} \begin{bmatrix} \delta^{\beta_1}_{\alpha_1} & 0 \\ 0 & -\delta^{B_1}_{A_1} \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} \delta^{\beta_1}_{\alpha_1}\delta^{\beta_2}_{\alpha_2} & 0 & 0 \\ 0 & -\delta^{\beta_1}_{\alpha_1}\delta^{B_1}_{A_1} & 0 \\ 0 & 0 & \delta^{B_1}_{A_1}\delta^{B_2}_{A_2} \end{bmatrix} \end{bmatrix}.$$

The group $L^2_{(n_1,n_2)}$ can be characterized as the subgroup of L^2_m defined by all elements of L^2_m which commute with F.

4. The set $E_{\pi^2_R}(V_m)$ of all π^2_R-adapted bases at the different points of V_m is equipped with a structure of principal fibre bundle of basis V_m and structural group $L^2_{(n_1,n_2)}$.

π^2_R-connection in V_m is called any infinitesimal connection ([8]) defined on the principal fibre bundle $E_{\pi^2_R}(V_m)$.

We consider a covering of V_m by open neighborhoods endowed with local cross sections of $E_{\pi^2_R}(V_m)$. Any π^2_R-connection may be defined in each neighborhood U by a local form π with values in the Lie algebra $\underline{L}^2_{(n_1,n_2)}$ of the group $L^2_{(n_1,n_2)}$.

Hence an π^2_R-connection is represented by the matrix,

$$(4.1)\quad \Pi = \begin{bmatrix} \begin{bmatrix} \pi^{\alpha_1}_{\beta_1} & 0 \\ 0 & \pi^{A_1}_{B_1} \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} \pi^{\alpha_1}_{\beta_1\beta_2} & 0 \\ 0 & \pi^{A_1}_{\beta_1 B_1} \\ \pi^{\alpha_1}_{B_1 B_2} & 0 \end{bmatrix} & \begin{bmatrix} \pi^{\alpha_1}_{\beta_1}\pi^{\alpha_2}_{\beta_2} & 0 & 0 \\ 0 & \pi^{\alpha_1}_{\beta_1}\,\pi^{A_1}_{\beta_1} & 0 \\ 0 & 0 & \pi^{A_1}_{B_1}\pi^{A_2}_{B_2} \end{bmatrix} \end{bmatrix},$$

where $\pi^{i_1}_{j_1} = \left(\pi^{\alpha_1}_{\beta_1}, \pi^{A_1}_{B_1}\right)$ and $\pi^{i_1}_{j_1 j_2} = \left(\pi^{\alpha_1}_{\beta_1\beta_2}, \pi^{\alpha_1}_{B_1 B_2}, \pi^{A_1}_{\beta_1 B_1}\right)$ are linear differential forms on U.

It can be verified that

$$(4.2)\begin{cases} \nabla F^{\alpha_1}_{\beta_1}=0, \quad \nabla F^{\alpha_1}_{B_1}=-2\pi^{\alpha_1}_{B_1}=0, \quad \nabla F^{\alpha_1}_{\beta_1\beta_2}=0, \quad \nabla F^{\alpha_1}_{\beta_1\beta_1}=-2\pi^{\alpha_1}_{\beta_1 B_1}=0, \\ \nabla F^{\alpha_1}_{B_1 B_2}=0, \quad \nabla F^{A_1}_{\beta_1}=2\pi^{A_1}_{\beta_1}=0, \quad \nabla F^{A_1}_{B_1}=0, \quad \nabla F^{A_1}_{\beta_1\beta_2}=2\pi^{A_1}_{\beta_1\beta_2}=0, \\ \nabla F^{A_1}_{\beta_1 B_1}=0, \quad \nabla F^{A_1}_{B_1 B_2}=2\pi^{A_1}_{B_1 B_2}=0, \quad \nabla F^{\alpha_1\alpha_2}_{\beta_1}=0, \quad \nabla F^{\alpha_1\alpha_2}_{B_1}=0 \\ \nabla F^{\alpha_1\alpha_2}_{\beta_1\beta_2}=0, \quad \nabla F^{\alpha_1\alpha_2}_{\beta_1 B_1}=-2\pi^{\alpha_1}_{\beta_1}\pi^{\alpha_2}_{B_1}=0, \quad \nabla F^{\alpha_1\alpha_2}_{B_1 B_2}=0, \quad \nabla F^{\alpha_1 A_1}_{\beta_1}=0 \\ \nabla F^{\alpha_1 A_1}_{B_1}=0, \quad \nabla F^{\alpha_1 A_1}_{\beta_1\beta_2}=2\pi^{\alpha_1}_{\beta_1}\pi^{A_1}_{\beta_2}=0, \quad \nabla F^{\alpha_1 A_1}_{\beta_1 B_1}=0, \quad \nabla F^{\alpha_1 A_1}_{B_1 B_2}=2\pi^{\alpha_1}_{B_1}\pi^{A_1}_{B_2}=0 \\ \nabla F^{A_1 A_2}_{B_1}=0, \quad \nabla F^{A_1 A_2}_{B_1}=0, \quad \nabla F^{A_1 A_2}_{\beta_1\beta_2}=0, \quad \nabla F^{A_1 A_2}_{\beta_1 B_1}=-2\pi^{A_1}_{\beta_1}\pi^{A_2}_{B_1}=0, \\ \nabla F^{A_1 A_2}_{B_1 B_2}=0. \end{cases}$$

P r o p o s i t i o n 4.1. With respect to an π_R^2-connection, the absolute differential of the tensor F is zero.

$E_{\pi_R^2}(V_m)$ may be considered as a sub-bundle of the fibre bundle $H^2(V_m)$ of 2-frames that is of bases of vector spaces $\{T_x^2\}_{x\in V_m}$ (conclusion (1.2)). An π_R^2-connection defines canonically a special affine connection of order 2 ([5]) on V_m which it may be identified.

Conversely, let us consider a special affine connection of order 2 and a covering of V_m by open neighborhoods equipped with local cross sections of $E_{\pi_R^2}(V_m)$. This connection may be defined on each neighborhood by a local form ω with values in the Lie algebra of L_m^2,

$$(4.3)\begin{cases}\omega = \left(\omega^{i_1}_{j_1},\ \omega^{i_1}_{j_1j_2}\right),\ i_1,j_1,j_2=1,2,\dots,m,\ \omega^{i_1}_{j_1}\in R^m\otimes R^{m*},\\ \omega^{i_1}_{j_1j_2}\in R^m\otimes s^2(R^{m*}) \text{ and } \left(\omega^{i_1}_{j_1}\right),\ \left(\omega^{i_1}_{j_1j_2}\right) \text{ are local linear differential forms.}\end{cases}$$

In order that the given connection may be identified with an π_R^2-connection it is necessary and sufficient that the form (4.3) belongs in the Lie algebra of the structural group $L^2_{(n_1,n_2)}$ of $E_{\pi_R^2}(V_m)$. That is,

$$(4.4)\qquad \omega^{\alpha_1}_{B_1} = \omega^{A_1}_{\beta_1} = \omega^{\alpha_1}_{\beta_1B_1} = \omega^{A_1}_{\beta_1\beta_2} = \omega^{A_1}_{B_1B_2} = 0.$$

Comparing with (4.2) we obtain the following

P r o p o s i t i o n 4.2. In order that a special affine connection of order 2 may be identified with an π_R^2-connection, it is necessary and sufficient that the absolute differential of the tensor F is zero with respect to this connection.

5. Given an π^2_R-connection Υ, the curvature form of this connection is the tensor 2-form,

$$\Omega = \nabla\pi = d\pi + \pi\wedge\pi, \tag{5.1}$$

which is defined by the matrix,

$$\Omega = \begin{bmatrix} \begin{bmatrix} \Omega^{\alpha_1}_{\beta_1} & 0 \\ 0 & \Omega^{A_1}_{B_1} \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} \Omega^{\alpha_1}_{\beta_1\beta_2} & 0 \\ 0 & \Omega^{A_1}_{\beta_1 B_1} \\ \Omega^{\alpha_1}_{B_1B_2} & 0 \end{bmatrix} & \begin{bmatrix} \Omega^{\alpha_1}_{\beta_1}\Omega^{\alpha_2}_{\beta_2} & 0 & 0 \\ 0 & \Omega^{\alpha_1}_{\beta_1}\Omega^{A_1}_{B_1} & 0 \\ 0 & 0 & \Omega^{A_1}_{B_1}\Omega^{A_2}_{B_2} \end{bmatrix} \end{bmatrix}. \tag{5.2}$$

It may be seen from (5.1) that

$$\Omega^{\alpha_1}_{\beta_1} = d\pi^{\alpha_1}_{\beta_1} + \pi^{\alpha_1}_{\Upsilon_1}\wedge\pi^{\Upsilon_1}_{\beta_1},$$

$$\Omega^{A_1}_{B_1} = d\pi^{A_1}_{B_1} + \pi^{A_1}_{\Gamma_1}\wedge\pi^{\Gamma_1}_{\beta_1},$$

$$\Omega^{\alpha_1}_{\beta_1\beta_2} = d\pi^{\alpha_1}_{\beta_1\beta_2} + \pi^{\alpha_1}_{\Upsilon_1}\wedge\pi^{\Upsilon_1}_{\beta_1\beta_2} + \pi^{\alpha_1}_{\Upsilon_1\Upsilon_2}\wedge\pi^{\Upsilon_1}_{\beta_1}\pi^{\Upsilon_2}_{\beta_2},$$

$$\Omega^{\alpha_1}_{B_1B_2} = d\pi^{\alpha_1}_{B_1B_2} + \pi^{\alpha_1}_{\Upsilon_1}\wedge\pi^{\Upsilon_1}_{B_1B_2} + \pi^{\alpha_1}_{\Gamma_1\Gamma_2}\wedge\pi^{\Gamma_1}_{B_1}\pi^{\Gamma_2}_{B_2},$$

$$\Omega^{A_1}_{\beta_1B_1} = d\pi^{A_1}_{\beta_1B_1} + \pi^{A_1}_{\Gamma_1}\wedge\pi^{1}_{\beta_1B_1} + \pi^{A_1}_{\Upsilon_1\Gamma_1}\wedge\pi^{\Upsilon_1}_{\beta_1}\pi^{\Gamma_1}_{B_1}.$$

In particular, it can be verified that

$$\Omega^{\alpha_1}_{\alpha_1} = d\pi^{\alpha_1}_{\alpha_1}, \quad \Omega^{A_1}_{A_1} = d\pi^{A_1}_{A_1}, \quad \Omega^{\alpha_1}_{\alpha_1\alpha} = d\pi^{\alpha_1}_{\alpha_1\alpha}, \quad \Omega^{A_1}_{A_1\alpha} = d\pi^{A_1}_{A_1 A}.$$

If we consider a covering of V_m by neighborhoods equiped with local cross sections of $E_{\pi^2_R}(V_m)$, then

$$\psi_1 = \Omega^{\alpha_1}_{\alpha_1}, \quad \psi_2 = \Omega^{A_1}_{A_2},$$

are closed scalar 2-forms and

$$\psi^1_\alpha = \Omega^{\alpha_1}_{\alpha_1\alpha}, \quad \psi^2_\alpha = \Omega^{A_1}_{A_1\alpha}, \quad \alpha = 1,2,\dots,n_1,$$

are closed vector 2-forms on $E_{\pi^2_R}(V_m)$.

We call ψ_k, k=1,2, the k-th characteristic form of order 1 and ψ^k_α, k=1,2, $\alpha=1,2,\dots,n_1$, the k-th characteristic from of order 2 of the π^2_R-connection $Y^{\bullet}$.

P r o p o s i t i o n 5.1. The characteristic 2-forms of order 1 (respectively of order 2) of all the π^2_R-connections have the same cohomology class of degree 2 (characteristic cohomology class of order 1 and of order 2 for the π^2_R-structure).

6. The holonomy group of an π^2_R-connection on V_m is ([9]) subgroup of the structural group $L^2_{(n_1,n_2)}$ of the fibre bundle $E_{\pi^2_R}(V_m)$.

Conversely, let $V^{\bullet}_m$ be a differentiable manifold endowed with a special affine connection of order 2. Let us assume that $x \in V_m$ there exists a 2-frame z_x such that the holonomy group ψ_{z_x} of the connection at z_x is a subgroup of $L^2_{(n_1,n_2)}$. Let us consider now, at the point x, the tensor whose components with respect to the basis z_x of T^2_x is of the form (3.3). This tensor is invariant under transformations by the elements of ψ_{z_x} and its components satisfy the relations (2.8). From this tensor we obtain ([9]) by parallel transport in V_m a tensor field on V_m with absolute differential equal to zero. The relations (2.8) remains true at every point of V_m and thus an π_R^2-structure is defined on V_m. Since the absolute differential of this tensor field is equal to zero by proposition (4.2), the given connection may be identified with a π_R^2-connection.

P r o p o s i t i o n 6.1. A necessary and sufficient condition in order that a special affine connection in a manifold V_m be an π_R^2-connection of an π_R^2-structure is that the holonomy group of the connection be a subgroup of $L^2_{(n_1,n_2)}$.

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