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ON THE APPROXIMATE SCHEMES IN THE MODULAR SPACE

1. Introduction

The paper generalizes the theory of the approximate schemes in a normed space [2]. All definitions and theorems connected with modular and Orlicz's space can be found in [1].

Let V be an abstract nonempty set and let $\mathcal{V}$ be a filter of subsets of V.

Definition 1. A function $g : V \to R$ tends to zero with respect to $\mathcal{V}$, $g(v) \xrightarrow{\mathcal{V}} 0$, if for every $\varepsilon > 0$ there is a set $V_0 \in \mathcal{V}$ such that $|g(v)| < \varepsilon$ for all $v \in V_0$.

Let X_ϱ be a real modular space and let $(X_{\varrho_v})_{v\in V}$ be a family of real modular spaces. Let $T = (T_v)_{v\in V}$ be a family of linear operators such that $T_v(X_\varrho) = X_{\varrho_v}$ for all $v \in V$.

Definition 2. Let $x_v \in X_{\varrho_v}$ for all $v \in V$ and let $x \in X_\varrho$. The family $(x_v)_{v\in V}$ T-tends to x with respect to $\mathcal{V}$, $x_v \xrightarrow{(T,\mathcal{V})} x$, if $\varrho_v(a(x_v - T_v x)) \xrightarrow{\mathcal{V}} 0$ for every $a > 0$.

Definition 2'. Let $x_v \in X_{\varrho_v}$ for all $v \in V$ and let $x \in X_\varrho$. The family $(x_v)_{v\in V}$ (T,ϱ) - tends to x with respect to $\mathcal{V}$, $x_v \xrightarrow{(T,\varrho,\mathcal{V})} x$, if there exists $a > 0$ such that $\varrho_v(a(x_v - T_v x)) \xrightarrow{\mathcal{V}} 0$.

Remark 1. If $x_v \xrightarrow{(T,\mathcal{V})} x$, then $x_v \xrightarrow{(T,\varrho,\mathcal{V})} x$ but the reverse implication is not true.

D e f i n i t i o n 3. The family $(\varrho_v)_{v\in V}$ will be called strongly nonsingular, if from $\varrho_v(aT_vx) \xrightarrow{\vartheta} 0$ for certain $a>0$ we obtain $x = 0$.

D e f i n i t i o n 3'. The family $(\varrho_v)_{v\in V}$ will be called nonsingular, if from $\varrho_v(aT_vx) \xrightarrow{\vartheta} 0$ for every $a>0$ we obtain $x = 0$.

T h e o r e m 1. The family $(\varrho_v)_{v\in V}$ is strongly nonsingular iff from $x_v \xrightarrow{(T,\varrho,\vartheta)} x$ and $x_v \xrightarrow{(T,\varrho,\vartheta)} y$ it follows $x = y$.

P r o o f . Let $x_v \xrightarrow{(T,\varrho,\vartheta)} x$ and $x_v \xrightarrow{(T,\varrho,\vartheta)} y$. Then $\varrho_v(aT_v(x-y)) = \varrho_v(a(T_vx - x_v + x_v - T_vy)) \leqslant \varrho_v(2a(T_vx - x_v)) + \varrho_v(2a(T_vy - x_v))$ for every $a>0$. But there exists $a'>0$ such that $\varrho_v(2a'(T_vx - x_v)) \xrightarrow{\vartheta} 0$ and $\varrho_v(2a'(T_vy - x_v)) \xrightarrow{\vartheta} 0$, so $\varrho_v(a'T_v(x-y)) \xrightarrow{\vartheta} 0$. As the family $(\varrho_v)_{v\in V}$ is strongly nonsingular we have $x = y$. Now from $x_v \xrightarrow{(T,\varrho,\vartheta)} x$ and $x_v \xrightarrow{(T,\varrho,\vartheta)} y$ it follows that $x = y$. Let $\varrho_v(aT_vx) \to 0$ for certain $a>0$. As $\varrho_v(aT_vx) = \varrho_v(a(T_vx - 0))$, so $0 \xrightarrow{(T,\varrho,\vartheta)} x$. But $0 \xrightarrow{(T,\varrho,\vartheta)} 0$, so $x = 0$.

T h e o r e m 1'. The family $(\varrho_v)_{v\in V}$ is nonsingular iff from $x_v \xrightarrow{(T,\vartheta)} x$ and $x_v \xrightarrow{(T,\vartheta)} y$ it follows $x = y$.

We omit the proof quite analogous to that of Theorem 1.

R e m a r k 2. If ϱ_v are norms for all $v \in V$, then the nonsingularity and strong nonsingularity of the family $(\varrho_v)_{v\in V}$ are equivalent.

R e m a r k 3. In the whole paper $D(A)$ is the domain of the operator A and $R(A)$ is the image of A.

Let X_{ϱ_1} and X_{ϱ_2} be real modular spaces. Let $(X_{\varrho_v})_{v\in V}$ and $(X_{\varrho'_v})_{v\in V}$ be families of real modular spaces. Let $\underline{T} = (\underline{T}_v)_{v\in V}$ and $\underline{T}' = (\underline{T}'_v)_{v\in V}$ be two families of linear operators such that $\underline{T}_v(X_{\varrho_1}) = X_{\varrho_v}$ and $\underline{T}'_v(X_{\varrho_2}) = X_{\varrho'_v}$ for every $v \in V$.

Let A be an operator such that $D(A) \subset X_{\rho_1}$ and $R(A) \subset X_{\rho_2}$. Let $\mathcal{A} = (A_v)_{v\in V}$ be a family of operators such that $D(A_v) \subset X_{\rho_v}$, $R(A_v) \subset X_{\rho_v}$, $\underline{T}_v(D(A)) \subset D(A_v)$ for every $v \in V$.

D e f i n i t i o n 4. We say that the approximate property holds on $x \in D(A)$, if for every $a > 0$

$$\rho_v'(a(A_v(\underline{T}_v x) - \underline{T}_v' A(x))) \xrightarrow{\mathcal{V}} 0.$$

D e f i n i t i o n 4'. We say that the ρ-approximate property holds on $x \in D(A)$, if there exists $a > 0$ such that

$$\rho_v'(a(A_v(\underline{T}_v x) - \underline{T}_v' A(x))) \xrightarrow{\mathcal{V}} 0.$$

2. The linear approximate scheme

Let now A be a linear operator and let $\mathcal{A}$ be a family of linear operators.

D e f i n i t i o n 5. We say that the family $\mathcal{A}$ is stable, if there exists a continuous, strongly increasing function $w(t)$ such that $w(0) = 0$, $w(+\infty) = +\infty$ and $\rho_v'(A_v x_v) \geqslant \geqslant w(\rho_v(x_v))$ for every $v \in V$ and every $x_v \in D(A_v)$.

R e ma r k 4. If ρ_v and ρ_v' are norms for all $v \in V$, then $w(t) = t$ (see [2]).

Let us consider the equation

$$(1) \qquad Ax = y, \quad y \in R(A).$$

We will call it the exact equation. It has a solution, since $y \in R(A)$, which is called the exact solution.

Let us consider now a family of equations

$$(2) \qquad A_v x_v = y_v, \quad \text{for every } v \in V \text{ and } y_v \in R(A_v).$$

The solution of (2) exists for every $v \in V$ and will be called the approximate solution. The family of equations (2) will be called the approximate scheme for solution of the equation (1). It will be said that the approximate scheme is

(T,ρ,ϑ)-convergent ((T,ϑ)-convergent), if every family of approximate solutions (T,ρ)-tends (T-tends) to the exact solution with respect to ϑ.

Theorem 2. If:

(a) the family $(\rho_v)_{v\in V}$ is strongly nonsingular,

(b) the ρ-approximate property holds on every exact solution,

(c) the family $\mathcal{A}$ is stable and w^{-1} is a convex or subadditive function,

(d) $y_v \xrightarrow{(T,\rho,\vartheta)} y$,

then:

(a′) the exact solution is only one,

(b′) the approximate solution is only one for every $v \in V$,

(c′) the approximate scheme is (T,ρ,ϑ)-convergent.

Proof.

(a′) Let x_1 and x_2 be two exact solutions. So $x_1, x_2 \in X_{\rho_1}$, $Ax_1 = Ax_2 = y$. From (c) we have

$$\rho'_v(2a(A_v\underline{T}_v x_1 - \underline{T}'_v Ax_1)) + \rho'_v(2a(A_v\underline{T}_v x_2 - \underline{T}'_v Ax_2)) \geqslant$$

$$\geqslant \rho'_v(aA_v\underline{T}_v(x_1 - x_2)) \geqslant w(\rho_v(a\underline{T}_v(x_1 - x_2)))$$

for every $a > 0$. Hence, from (b) there exists $b > 0$ such that

$$w(\rho_v(b\underline{T}_v(x_1 - x_2))) \xrightarrow{\vartheta} 0$$

which implies, by the properties of w, $\rho_v(b\underline{T}_v(x_1 - x_2)) \xrightarrow{\vartheta} 0$. So from (a) we obtain $x_1 = x_2$.

(b′) Let x_v^1 and x_v^2 be two approximate solutions. From (c) we have $0 = \rho'_v(A_v(x_v^1 - x_v^2)) \geqslant w(\rho_v(x_v^1 - x_v^2))$ for every $v \in V$, hence $\rho_v(x_v^1 - x_v^2) = 0$, so $x_v^1 = x_v^2$.

(c′) Let x be the exact solution and let x_v be an approximate solution. From (c) we have for every $a > 0$:

if w^{-1} is a subadditive function, then

$$\rho_v(a(x_v - \underline{T}_v x)) \leqslant w^{-1}(\rho'_v(A_v(a(x_v - \underline{T}_v x)))) =$$
$$= w^{-1}(\rho'_v(a(y_v - A_v\underline{T}_v x))) \leqslant w^{-1}(\rho'_v(2a(y_v - \underline{T}'_v y))) +$$
$$+ w^{-1}(\rho'_v(2a(\underline{T}'_v Ax - A_v\underline{T}_v x))).$$

Hence, from (b), (d) we obtain that $x_v \xrightarrow{(T,\rho,\vartheta)} x$; if w^{-1} is a convex function we have

$$\rho_v(a(x_v - \underline{T}_v x)) \leq \frac{1}{2} w^{-1}(2\rho'_v(2a(y_v - \underline{T}'_v y))) +$$

$$+ \frac{1}{2} w^{-1}(2\rho'_v(2a(\underline{T}'_v Ax - A_v \underline{T}_v x))).$$

Hence, from (b), (d) we obtain $x_v \xrightarrow{(T,\delta,\vartheta)} x$.

T h e o r e m 2'. If:

(a) the family $(\rho_v)_{v\in V}$ is nonsingular,

(b) the approximate property holds on every exact solution

(c) the family $\mathcal{A}$ is stable and w^{-1} is a convex or subadditive function,

(d) $y_v \xrightarrow{(T,\vartheta)} y$,

then:

(a') the exact solution is only one,

(b') the approximate solution is only one for every $v \in V$,

(c') the approximate scheme is (T,ϑ)-convergent.

We omit the proof quite analogous to that of Theorem 2.

R e m a r k 5. If $y = y_v = 0$ for all $v \in V$, then w^{-1} does not be a convex or subadditive function.

C o r o l l a r y 1. If the assumptions of Theorem 2 hold, then $A_v^{-1} y_v \xrightarrow{(T,\rho,\vartheta)} x$. If moreover, $\underline{T}'_v y \in R(A_v)$ for every $v \in V$, then $A_v^{-1}\underline{T}_v y \xrightarrow{(T,\rho,\vartheta)} x$.

P r o o f . From (c) we obtain that A_v^{-1} exists for every $v \in V$. So $A_v^{-1} y_v \xrightarrow{(T,\rho,\vartheta)} x$, because $A_v^{-1} y_v = x_v$. Let $a > 0$. From the assumptions we obtain

$$\rho_v(a(A_v^{-1}\underline{T}_v y - \underline{T}'_v x)) \leq \rho_v(2a(A_v^{-1}\underline{T}'_v y - A_v^{-1} y_v)) +$$

$$+ \rho_v(2a(A_v^{-1} y_v - \underline{T}_v x)) \leq w^{-1}(\rho'_v(2a(\underline{T}'_v y - y_v))) +$$

$$+ \rho_v(2a(A_v^{-1} y_v - \underline{T}_v x)) \xrightarrow{\vartheta} 0$$

for sufficiently small $a > 0$.

C o r o l l a r y 1′. If the assumptions of Theorem 2 hold, then $A_v^{-1}y_v \xrightarrow{(T,\vartheta)} x$. If, moreover, $T_v'y \in R(A_v)$ for every $v \in V$, then $A_v^{-1}T_v'y \xrightarrow{(T,\vartheta)} x$.

3. Nonlinear approximate scheme

Let now A be a nonlinear operator and let $\mathcal{A}$ be a family of nonlinear operators.

D e f i n i t i o n 6. We say that the family $\mathcal{A}$ is stable, if there exists a continuous, strongly increasing function w(t) such that w(0) = 0, w(+∞) = +∞ and for every $x_v^1, x_v^2 \in D(A_v)$, $v \in V$ and $a > 0$ there is

$$\text{(a)} \quad \rho_v'(a(A_v(x_v^1) - A_v(x_v^2))) \geqslant w(\rho_v(a(x_v^1 - x_v^2))).$$

If there exists $M > 0$ such that (a) holds only for $a \in (0,M]$, then we say that the family $\mathcal{A}$ is ρ-stable.

R e m a r k 6. If $\mathcal{A}$ is a family of linear operators, then ρ-stability and stability are equivalent, since $D(A_v)$ is a linear space for every $v \in V$.

Now we must solve the equation

$$\text{(3)} \qquad A(x) = 0.$$

The solution of (3) will be called the exact solution. Let us consider now the family of equations

$$\text{(4)} \qquad A_v(x_v) = 0 \quad \text{for every} \quad v \in V.$$

The solution of (4) will be called the approximate solution for every $v \in V$.

T h e o r e m 3. If:

(a) the family $(\rho_v)_{v \in V}$ is strongly nonsingular,

(b) the exact solution exists,

(c) the approximate solution exists for every $v \in V$,

(d) the ρ-approximate property holds on every exact solution,

(e) the family $\mathcal{A}$ is ρ-stable,

then:

(a′) the exact solution is only one,
(b′) the approximate solution is only one for every $v \in V$,
(c′) the family $(x_v)_{v \in V}$ of approximate solutions of (4) (T, ϱ) - tends to the exact solution x of (3) with respect to ϑ.

P r o o f .

(a′) Let x_1, x_2 be two exact solutions. From modular properties and from (e) we obtain

$$\varrho_v(a(\underline{T}_v(x_1 - x_2))) \leqslant w^{-1}(\varrho'_v(A_v(\underline{T}_v x_1) - A_v(\underline{T}_v x_2))) \leqslant$$

$$\leqslant w^{-1}(\varrho'_v(2aA_v(\underline{T}_v x_1)) + \varrho'_v(2aA_v(\underline{T}_v x_2)))$$

for $a \in (0, \frac{M}{2}]$. Hence, from (d) we obtain $\varrho_v(a(\underline{T}_v(x_1 - x_2))) \xrightarrow{\vartheta} 0$. So from (a) it follows that $x_1 = x_2$.

(b′) Let $A_v(x_v^1) = A_v(x_v^2) = 0$ for $v \in V$. We have $0 = \varrho'_v(a(A_v(x_v^1) - A_v(x_v^2))) \geqslant w(\varrho_v(a(x_v^1 - x_v^2)))$ for $a \in (0, M]$, so $x_v^1 = x_v^2$.

(c′) Let x be an exact solution and x_v be an approximate solution. From (e) we obtain

$$\varrho_v(a(x_v - \underline{T}_v x)) \leqslant w^{-1}(\varrho'_v(a(A_v(x_v) - A_v(\underline{T}_v x)))) =$$

$$= w^{-1}(\varrho'_v(aA_v(\underline{T}_v x))),$$

for $a \in (0, M]$. Hence, from (d) we obtain $x_v \xrightarrow{(T, \varrho, \vartheta)} x$.

T h e o r e m 3′. If:
(a) the family $(\varrho_v)_{v \in V}$ is nonsingular,
(b) the exact solution exists,
(c) the approximate solution exists for every $v \in V$,
(d) the approximate property holds on every exact solution,
(e) the family $\mathcal{A}$ is stable,

then:
(a′) the exact solution is only one,
(b′) the approximate solution is only one for every $v \in V$,
(c″) the family $(x_v)_{v \in V}$ of approximate solutions of (4) T - tends to the exact solution x of (3) with respect to ϑ.

We omit the proof quite analogous to that of Theorem 3.

Remark 7. If ρ_1, ρ_2, $(\rho_v)_{v\in V}$, $(\rho'_v)_{v\in V}$ are norms, then from Theorems 2, 2′, 3, 3′ we obtain the suitable theorems from [2].

Remark 8. The assumption that $\mathcal{A}$ is stable is very useful.

Definition 7. Let X_{ρ_1} and X_{ρ_2} be modular spaces. An operator $A : X_{\rho_1} \to X_{\rho_2}$ will be called (ρ_1,ρ_2)-closed, if for all sequences $\{x_n\} \subset X_{\rho_1}$ from $x_n \xrightarrow{\rho_1} x_0$ and $A(x_n) \xrightarrow{\rho_2} y_0$ we obtain $x_0 \in D(A)$ and $y_0 = A(x_0)$.

Lemma 1. If X_{ρ_1} is ρ_1-complete and A is (ρ_1,ρ_2)-closed and stable, then R(A) is ρ_2-closed.

Proof. Let $\{y_n\} \subset R(A)$ and $y_n \xrightarrow{\rho_2} y_0$ as $n \to \infty$. We must see that $y_0 \in R(A)$. If $y_n \in R(A)$, then there exists $x_n \in D(A)$ such that $y_n = A(x_n)$ for every $n \in N$. We have

$$\rho_2(a(y_i - y_j)) = \rho_2(a(A(x_i) - A(x_j))) \geqslant w(\rho_1(a(x_i - x_j))),$$

for $i,j \in N$ and $a > 0$, hence

$$\rho_1(a(x_i - x_j)) \leqslant w^{-1}(\rho_2(a(y_i - y_j))).$$

Since $y_n \xrightarrow{\rho_2} y_0$, then there exists $b > 0$ such that $\rho_2(b(y_i - y_j)) \to 0$ as $i,j \to \infty$. So $\rho_1(b(x_i - x_j)) \to 0$ as $i,j \to \infty$. Since X_{ρ_1} is ρ_1-complete, then there exists $x_0 \in X_{\rho_1}$ such that $x_n \xrightarrow{\rho_1} x_0$. Since A is (ρ_1,ρ_2)-complete, we obtain $x_0 \in D(A)$ and $A(x_0) = y_0 \in R(A)$, so R(A) is ρ_2-complete.

Definition 8. An operator $A : X_{\rho_1} \to X_{\rho_2}$ will be called (ρ_1,ρ_2)-continuous, if from $x_n \xrightarrow{\rho_1} x$ we obtain $A(x_n) \xrightarrow{\rho_2} A(x)$, where $\{x_n\} \subset X_{\rho_1}$ and $x \in X_{\rho_1}$.

Theorem 4. If the assumptions of Lemma 1 hold, then A^{-1} exists and is (ρ_1,ρ_2)-continuous.

P r o o f . A^{-1} exists because A is one-to-one. Let $A(x_1) = A(x_2)$. We have

$$0 = \rho_2(0) = \rho_2(a(A(x_1) - A(x_2))) \geq w(\rho_1(a(x_1 - x_2)))$$

for $a > 0$, so $x_1 = x_2$. Let $\{y_n\} \subset R(A)$ and $y_2 \xrightarrow{\rho_2} y$. By Lemma 1, $y \in R(A)$. So there exists $\{x_n\} \subset X_{\rho_1}$ such that $x_n = A^{-1}(y_n)$ for every $n \in N$. Let $x = A^{-1}(y)$, so $x \in D(A)$ and we must prove that $x_n \xrightarrow{\rho_1} x$. For $a > 0$ we have

$$\rho_1(a(A^{-1}(y_n) - A^{-1}(y))) = \rho_1(a(x_n - x)) \leq$$

$$\leq w^{-1}(\rho_2(a(A(x_n)-A(x)))) = w^{-1}(\rho_2(a(y_n - y))) \to 0 \text{ as } n \to \infty .$$

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