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OPERATIONS WITH DISTRIBUTION VECTORS

The space of distributions D' is isomorphically embedded in the space of distribution vectors $\underset{\sim}{D}'$ (1) and this larger space $\underset{\sim}{D}'$ is equipped with operations of multiplication and integration. Several formulae for $\delta^2(x)$, $\delta^{(p)}.\delta^{(q)}$, $x_+^p.\delta^{(q)}(x)$, etc., are derived which, as we know, are significant for some applications, in particular, in quantum field theory but they do not make sense in D' itself. The paper is a continuation of a previous work (1) but it could be read independently.

In the following we let C be the field of complex numbers, D be the space of infinitely differentiable functions defined in the real line with compact support and D' be the space of all distributions on D.

Definition 1. Let α_r be in C for $r = 0,1,\dots$. We say that

$$\underset{\sim}{\alpha} = [\alpha_0, \alpha_1, \dots]$$

is a number vector (2). We denote the vector space of all number vectors, with the usual definition of the sum and product by a scalar, by $\underset{\sim}{C}$.

Definition 2. Let h_r be in D' for $r=0,1,\dots$. We say that

$$\underset{\sim}{h} = [h_0, h_1, \dots, h_r, \dots]$$

is a distribution vector.

If $h_{r+i} = 0$ for $i = 1,2,\dots$, we write

$$\underset{\sim}{h} = [h_0,h_1,\dots,h_r,0,0,\dots] = [h_0,h_1,\dots,h_r]$$

and if $h_i = 0$ for $i = 1,2,\dots$ we write

$$\underset{\sim}{h} = [h_0] = h_0.$$

We denote the vector space of all distribution vectors, with the usual definition of sum and product by a scalar, by $\underset{\sim}{D}'$.

Definition 3. Let $\underset{\sim}{h} = [h_0,h_1,\dots,h_r,\dots]$ be in $\underset{\sim}{D}'$ and let φ be in D. We define $(\underset{\sim}{h},\varphi)$ to be the number vector

$$(\underset{\sim}{h},\varphi) = [(h_0,\varphi),(h_1,\varphi),\dots,(h_r,\varphi),\dots] .$$

Definition 4. Let $\underset{\sim}{h} = [h_0,h_1,\dots,h_r,\dots]$ be in $\underset{\sim}{D}'$. We define the derivative $\underset{\sim}{h}'$ of $\underset{\sim}{h}$ by

$$\underset{\sim}{h}' = [h_0',h_1',\dots,h_r',\dots] .$$

Theorem 1. Let $\underset{\sim}{h} = [h_0,h_1,\dots,h_r,\dots]$ be in $\underset{\sim}{D}'$ and let φ be in D. Then

$$(\underset{\sim}{h}',\varphi) = -(\underset{\sim}{h},\varphi').$$

The proof of the theorem follows easily.

Definition 5. Let ρ be a fixed function in D having the properties:

(i) $$\rho(x) = 0 \quad \text{for} \quad |x| \leqslant 1,$$

(ii) $$\rho(x) \geqslant 0,$$

(iii) $$\rho(x) = \rho(-x),$$

(iv) $$\int_{-1}^{1} \rho(x)dx = 1.$$

We define the function δ_ν by $\delta_\nu(x) = \nu\rho(\nu x)$ for all $\nu > 0$.

Definition 6. Let f and g be in D' and let $g_\nu = g * \delta_\nu$. If there exist $h_0, h_1, \ldots, h_r$ in D' such that

$$(f, g_\nu \varphi) = \sum_{i=0}^{r} (h_i, \varphi)\nu^i + \Delta(\nu),$$

for arbitrary φ in D, where

$$\lim_{\nu \to \infty} \Delta(\nu) = 0$$

(Δ could depend on φ as well), we define the product $f \circ g$ in $\underset{\sim}{D}'$ by

$$f \circ g = [h_0, h_1, \ldots, h_r] .$$

We say that h_0 is the finite part of $f \circ g$. If $h_i \neq 0$ for some $i \geq 1$, we write

$$\text{p.f.}(f \circ g) = h_0$$

and if $h_i = 0$ for $i = 1, 2, \ldots, r$, we write

$$f \circ g = h_0.$$

Theorem 2. The product "$\circ$" is a generalization of the usual product in D' when one of the distributions is a smooth function, i.e. $f \circ g = f.g$ for all $f \in D'$ and all $g \in C^\infty$.

The above theorem is just an interpretation of Definition 6, having in mind as well that $g_\nu \varphi \underset{\nu \to \infty}{\longrightarrow} g\varphi$ in the test-topology of D when g is a smooth function and φ is in D.

Theorem 3. Let f and g be in D' and suppose that the products $f' \circ g$ (or $f \circ g'$) and $f \circ g$ are in $\underset{\sim}{D}'$. Then the product $f \circ g'$ (or $f' \circ g$) is in $\underset{\sim}{D}'$ and

$$(f \circ g)' = f' \circ g + f \circ g'.$$

P r o o f . Suppose

$$(f, g_\nu \varphi) = \sum_{i=0}^{r} (h_i, \varphi)\nu^i + \Delta(\nu),$$

$$(f', g_\nu \varphi) = \sum_{i=0}^{r} (k_i, \varphi)\nu^i + \Delta_1(\nu),$$

for arbitrary φ in D, so that

$$f \circ g = [h_0, h_1, \dots, h_r] ,$$

$$f' \circ g = [k_0, k_1, \dots, k_r] .$$

Then

$$((fg_\nu)', \varphi) = -(fg_\nu, \varphi') = (fg'_\nu + f'g_\nu, \varphi)$$

and so

$$(f, g'_\nu \varphi) = -(f, g_\nu \varphi') - (f', g_\nu \varphi) =$$

$$= -\sum_{i=0}^{r} (h_i, \varphi')\nu^i - \Delta_2(\nu) - \sum_{i=0}^{r} (h_i, \varphi)\nu^i - \Delta_1(\nu) =$$

$$= \sum_{i=0}^{r} (h'_i - k_i, \varphi)\nu^i - (\Delta_1 + \Delta_2)(\nu)$$

for some function Δ_2, where

$$\lim_{\nu \to \infty} \Delta_2(\nu) = \lim_{\nu \to \infty} (\Delta_1 + \Delta_2)(\nu) = 0.$$

It follows that the product $f \circ g'$ is in $\underset{\sim}{D}'$ and

$$f \circ g' = [h'_0 - k_0, h'_1 - k_1, \dots, h'_r - k_r] = (f \circ g)' - f' \circ g.$$

The results of the theorem follows.

We now put for simplicity

$$\rho_i = \rho^{(i)}(0)$$

for i = 0,1,... so that in particular

$$\rho_i = 0$$

for odd i.

T h e o r e m 4. The product $\delta^{(p)} \circ \delta^{(q)}$ is in $\underset{\sim}{D}'$ and

$$\delta^{(p)} \circ \delta^{(q)} = \underset{\sim}{h}(p,q) = \left[h_0(p,q), h_1(p,q), \ldots, h_{p+q}(p,q)\right]$$

for p,q = 0,1,2,..., where

$$h_i(p,q) = \begin{cases} 0, & 0 \leq i \leq q, \\ (-1)^{i-q-1} \binom{p}{i-q-1} \rho_{i-1} \delta^{(p+q+1-i)}, & q < i \leq p+q+1, \end{cases}$$

and $\binom{p}{q}$ denotes the binomial coefficient

$$\binom{p}{q} = \frac{p!}{q!(p-q)!} .$$

In particular

$$\delta^2 = \delta \circ \delta = \left[0, \rho_0 \delta\right],$$

$$\delta' \circ \delta = \left[0, \rho_0 \delta'\right],$$

$$\delta \circ \delta' = 0.$$

So, we see that the multiplication operation "o" is a non-commutative operation.

T h e o r e m 5. The products $x_+^p \circ \delta^{(q)}$ and $\delta^{(q)} \circ x_+^p$ are in $\underset{\sim}{D}'$ and

$$x_+^p \circ \delta^{(q)} = \underset{\sim}{h}(p,q) = \left[h_0(p,q), h_1(p,q), \ldots, h_{q-p}(p,q)\right]$$

for p = 0,1,...,q and q = 0,1,2,..., where

$$h_i(p,q) = \begin{cases} \frac{1}{2}(-1)^p \binom{q}{p} p!\, \delta^{(q-p)}, & i = 0, \\ (-1)^{p-1} \binom{q-i}{p} p!\, \rho_{i-1}\, \delta^{(q-p-i)}, & 1 \leqslant i \leqslant q-p \end{cases}$$

and

$$\delta^{(q)} \circ x_+^p = \underset{\sim}{k}(p,q) = \left[k_0(p,q), k_1(p,q), \ldots, k_{q-p}(p,q)\right]$$

for $p = 0,1,\ldots,q$ and $q = 0,1,\ldots,$ where

$$k_i(p,q) = \begin{cases} \frac{1}{2}(-1)^p \binom{q}{p} p!\, \delta^{(q-p)}, & i = 0 \\ (-1)^{p-1} \binom{q}{p+i} p!\, \rho_{i-1}\, \delta^{(q-p-i)}, & 1 \leqslant i \leqslant q-p. \end{cases}$$

In particular

$$x_+^p \circ \delta^{(p)} = \delta^{(p)} \circ x_+^p = \frac{1}{2}(-1)^p\, p!\, \delta$$

for $p = 0,1,2,\ldots$.

These theorems are equivalent to Theorem 3 and 4 proved in [1].

We now consider the product of two distribution vectors. For convenience we note that $\underset{\sim}{D}'$ is isomorphic to the space of power series in an indeterminate ν having distributions as coefficients. Under this natural isomorphism we write

$$\underset{\sim}{f} = [f_0, f_1, \ldots, f_r, \ldots] \equiv \sum_{r=0}^{\infty} f_r \nu^r.$$

D e f i n i t i o n 7. Let

$$\underset{\sim}{f} = [f_0, f_1, \ldots, f_r, \ldots] \equiv \sum_{r=0}^{\infty} f_r \nu^r$$

and

$$\underset{\sim}{g} = [g_0, g_1, \ldots, g_r, \ldots] \equiv \sum_{r=0}^{\infty} g_r \nu^r$$

be in $\underset{\sim}{D}'$ and suppose that $f_r \circ g_s$ exist for all $r,s = 0,1,\ldots$ (in the sense of Definition 6) and

$$f_r \circ g_s = \sum_{m=0}^{\mu_{rs}} h_{rsm}\nu^m$$

for $r,s = 0,1,\ldots$ for some distributions h_{rsm} and some integers $\mu_{rs} \geqslant 0$. Let now put

$$h_n = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{m=0}^{\mu_{rs}} h_{rsm} \qquad (r+s+m = n)$$

for $n = 0,1,2,\ldots$. We define the product $\underset{\sim}{f} \circ \underset{\sim}{g}$ in $\underset{\sim}{D}'$ by

$$\underset{\sim}{f} \circ \underset{\sim}{g} = \left(\sum_{r=0}^{\infty} f_r \nu^r\right) \circ \left(\sum_{s=0}^{\infty} g_s \nu^s\right) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (f_r \circ g_s)\, \nu^{r+s} =$$

$$\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \left(\sum_{m=0}^{\mu_{rs}} h_{rsm}\nu^m\right) \nu^{r+s} =$$

$$= \sum_{n=0}^{\infty} h_n \nu^n \equiv [h_0, h_1, \ldots, h_n, \ldots]$$

and say that h_0 is the finite part of $\underset{\sim}{f} \circ \underset{\sim}{g}$.

T h e o r e m 6. Let $\underset{\sim}{f}$ and $\underset{\sim}{g}$ be in $\underset{\sim}{D}'$ and suppose that the products $\underset{\sim}{f} \circ \underset{\sim}{g}$ and $\underset{\sim}{f}' \circ \underset{\sim}{g}$ (or $\underset{\sim}{f} \circ \underset{\sim}{g}'$) are in $\underset{\sim}{D}'$. Then the product $\underset{\sim}{f} \circ \underset{\sim}{g}'$ (or $\underset{\sim}{f}' \circ \underset{\sim}{g}$) is in $\underset{\sim}{D}'$ and

$$(\underset{\sim}{f} \circ \underset{\sim}{g})' = \underset{\sim}{f}' \circ \underset{\sim}{g} + \underset{\sim}{f} \circ \underset{\sim}{g}'.$$

P r o o f . Suppose

$$\underset{\sim}{f} \equiv \sum_{r=0}^{\infty} f_r \nu^r, \qquad \underset{\sim}{g} \equiv \sum_{r=0}^{\infty} g_r \nu^r.$$

Then

$$\underset{\sim}{f} \circ \underset{\sim}{g} \equiv \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (f_r \circ g_s)\nu^{r+s}$$

and

$$\underset{\sim}{f}' \circ \underset{\sim}{g} \equiv \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (f_r' \circ g_s)\nu^{r+s}$$

so that the products $f_r \circ g_s$ and $f_r' \circ g_s$ are in $\underset{\sim}{D}'$. By Theorem 3 the product $f_r \circ g_s'$ is in $\underset{\sim}{D}'$ and

$$(f_r \circ g_r)' = f_r' \circ g_s + f_r \circ g_s'.$$

Thus

$$(\underset{\sim}{f} \circ \underset{\sim}{g})' \equiv \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (f_r \circ g_s)'\nu^{r+s} =$$

$$= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (f_r' \circ g_s + f_r \circ g_s')\nu^{r+s}$$

which implies the existence of $\underset{\sim}{f} \circ \underset{\sim}{g}'$ and

$$(\underset{\sim}{f} \circ \underset{\sim}{g})' = \underset{\sim}{f}' \circ \underset{\sim}{g} + \underset{\sim}{f} \circ \underset{\sim}{g}'.$$

Example 1. $(\delta, \delta_\nu \varphi) = (\delta, \nu\rho(\nu x)\varphi(x)) = \nu\rho(0)\varphi(0) = \nu\rho_0(\delta, \varphi)$ for arbitrary φ in D and so

$$\delta^2 \equiv \delta \circ \delta = [0, \rho_0 \delta].$$

Example 2. $\delta^3 \equiv \delta \circ \delta \circ \delta = \nu^2\rho_0^2\delta = [0,0,\rho_0^2\delta]$.
More general for the n-th power of the delta-function δ we obtain

$$\delta^n = \nu^{n-1}\rho_0^{n-1}\delta = [0,0,\dots,\rho_0^{n-1}\delta].$$

E x a m p l e 3. $(\delta, \delta_\nu' \varphi) = (\delta, \nu^2 \rho'(\nu x)\varphi(x)) = -\nu^2 \rho'(0)\varphi(0) = 0$ for arbitrary φ in D and so

$$\delta \circ \delta' = 0.$$

E x a m p l e 4. Using Theorem 3 we see that $\delta' \circ \delta$ is in $\underset{\approx}{D}'$ and

$$\delta' \circ \delta = (\delta \circ \delta)' - \delta \circ \delta' = \left[0, \rho_0 \delta'\right].$$

E x a m p l e 5. $(\delta', \delta_\nu' \varphi) = (\delta', \nu^2 \rho'(\nu x)\varphi(x)) = -\nu^3 \rho''(0)\varphi(0) + \nu^2 \rho'(0)\varphi'(0) = -\nu^3 \rho''(0)(\delta, \varphi)$ for arbitrary φ in D and so

$$\delta' \circ \delta' = \left[0, 0, 0, -\rho_2 \delta\right].$$

These four results are, of course, particular cases of Theorem 4.

E x a m p l e 6. $[\delta, \delta'] \circ [\delta', \delta] \equiv (\delta + \nu\delta') \circ (\delta' + \nu\delta) = \delta \circ \delta' + \nu\delta \circ \delta + \nu\delta' \circ \delta' + \nu^2 \delta' \circ \delta = [0, 0, \rho_0 \delta, \rho_0 \delta', -\rho_2 \delta$.

We finally consider integration in D$'$ and $\underset{\sim}{D}'$.

D e f i n i t i o n 8. Let f be in D$'$, let μ be a measure in R and let σ be a measurable subset of R. We say that f is integrable on σ if there exists an integer $m \geq 0$ and complex coefficients $\alpha_0, \alpha_1, \ldots, \alpha_m$ for which

$$\int_\sigma f_\nu(x)\,d\mu(x) = \sum_{i=0}^{m} \alpha_i \nu^i + \Delta(\nu)$$

where $f_\nu = f * \delta_\nu$ and

$$\lim_{\nu \to \infty} \Delta(\nu) = 0.$$

We then write

$$\int_\sigma f(x)\,d\mu(x) = \left[\alpha_0, \alpha_1, \ldots, \alpha_m\right] \equiv \sum_{i=0}^{m} \alpha_i \nu^i$$

and say that α_0 is the finite part of the integral.

E x a m p l e 7. $\int_{-\infty}^{\infty} \delta(x)dx = 1.$

E x a m p l e 8. $\int_0^1 \delta_\nu(x)dx = \int_0^1 \nu\rho(\nu x)dx = \frac{1}{2}$ and so $\int_0^1 \delta(x)dx = \frac{1}{2}$.

E x a m p l e 9. $\int_{-\infty}^{\infty} \delta'(x)dx = 0.$

E x a m p l e 10. $\int_0^1 \delta_\nu'(x)dx = \int_0^1 \nu^2\rho'(\nu x)dx = -\nu\rho_0$ and so $\int_0^1 \delta'(x)dx = [0,-\rho_0]$.

T h e o r e m 7. For all $f \in D'$ and all $\varphi \in D$ we have:

$$\int_{-\infty}^{\infty} (f \circ \varphi)dx \equiv \int_{-\infty}^{\infty} f(x)\varphi(x) = (f,\varphi).$$

P r o o f . It is well known that $(f\varphi)_\nu \xrightarrow[\nu\to\infty]{} f\varphi$ in the topology of $\mathcal{E}'$ ($\mathcal{E}'$ is the space of all distributions with compact supports) so that

$$\lim_{\nu\to\infty} \int_{-\infty}^{\infty} (f\varphi)_\nu dx = (f\varphi,1) = (f,\varphi).$$

The proof is finished.

D e f i n i t i o n 9. Let

$$\underset{\sim}{f} = [f_0,f_1,\ldots,f_r,\ldots] \equiv \sum_{r=0}^{\infty} f_r\nu^r$$

be in $\underset{\sim}{D}'$ and suppose that f_r is integrable on σ with

$$\int_\sigma f_r(x)d\mu(x) \equiv \sum_{i=0}^{m_r} \alpha_{ri}\nu^i$$

for $r = 0,1,\ldots$. We say that $\underset{\sim}{f}$ is integrable on σ and write

$$\int_{\delta} \underset{\sim}{f}(x)d\mu(x) \equiv \sum_{r=0}^{\infty} \sum_{i=0}^{m_r} dr_i \nu^{r+i} \equiv [d_0, d_1, \ldots, d_n, \ldots]$$

where

$$d_n = \sum_{r=0}^{\infty} \sum_{i=0}^{m_r} dr_i, \qquad (r+i = n)$$

for $n = 0,1,\ldots$ and say that d_0 is the finite part of the integral.

E x a m p l e 11. $\int_0^1 [\delta(x), \delta'(x)]\,dx = [1/2, 0, -\rho_0]$.

We see that the integral of a given distribution vector (if exists) is a number vector.

R e m a r k . The reader could remain disappointed at the fact that the multiplication operation introduced in our paper is nonassociative which follows directly from the example

$$\delta' \circ (\delta \circ \delta') \neq (\delta' \circ \delta) \circ \delta' .$$

Recall, however, that according to the well-known interpretation of the Schwartz example

$$(x^{-1} \cdot x)\delta(x) \neq x^{-1}(x \cdot \delta(x))$$

it is principally impossible to supply the distribution space or any of its enlargements (in particular, the space of distribution vectors $\underset{\sim}{D}'$) with an associative multiplication operation.

REFERENCES

[1] B. F i s h e r : Products of distributions defined by a vector, Proceedings of the Third International Conference on Complex Analysis and Applications, Varna (1985), (to appear).

[2] Chr. Ya. Christov, T.D. Todorov: Asymptotic numbers - algebraic operations with them, Serdica, Bulgaricae Mathematicae Publicationes, vol.2 (1976) 87-102.

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