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ON THE CONVERGENCE OF SUCCESSIVE APPROXIMATIONS FOR FUNCTIONAL-DIFFERENTIAL EQUATIONS OF HYPERBOLIC TYPE

1. Introduction

The present note is devoted to the study of the convergence of the sequence of successive approximations for the solution of the Goursat problem for the equation

$$z_{xy}(x,y) = f(x,y,z,z_x,z_y)$$

where z, z_x, z_y denote appropriate functions on any subset of the space R^2, while $z_{xy}(x,y)$ is the value of z_{xy} at $(x,y) \in R^2$ and f satisfies some Carathèodory type conditions.

The main result of this paper says that nonconvergence of successive approximations of such type equations is in any sense a rare case. This type property is said to be generic.

The study of generic properties for hyperbolic equations was started by A.Alexiewicz and W.Orlicz [1], who proved that under some very natural conditions on f, the uniquenes of solutions of the Darboux problem is a generic property. Lasota and Yorke [4] studied generic properties concerning existence and uniqueness of solutions for differential equations in a Banach space. More recently M.Kisielewicz [2], [3] studied generic properties for functional differential equations of neutral type of the form

$$\dot{x}(t) = f(t,x_t,\dot{x}_t)$$

where $x_t(\theta) = x(t+\theta)$ for fixed $t \in R$, $\theta \in [-r,0]$ and in the most general form

$$\dot{x}(t) = f(t,x,\dot{x}).$$

Other generic properties have been studied in [5]. Further references can be found in [3].

2. Notations and preliminaries

For given positive numbers α, β, a, b and non-decreasing functions $y = g(x)$ and $x = h(y)$ of class C' defined on intervals $[0,a]$ or $[0,b]$ respectively, having the only one common point at zero, let $0 \leqslant g(x) \leqslant b$, $0 \leqslant h(y) \leqslant a$ and $P = [-\alpha,a] \times [-\beta,b]$.

Let $D = \{(s,t): h(t) < s \leqslant a,\ g(s) < t \leqslant b\}$ and $D_{xy} = \{(s,t) : h(t) < s \leqslant x,\ g(s) < t \leqslant y\}$ for $x \in [0,a]$ and $y \in [0,b]$.

Furthermore, let $G = P \setminus D$, $D_x = \{y \in [g(x),b] : (x,y) \in D\}$ and $D_y = \{x \in [h(y),a] : (x,y) \in D\}$. Similarly we define G_x and G_y.

We introduce the following notations:

R^n - the n-dimensional space with the norm $\|\cdot\|$;

$C_0(P,R^n)$ - the space of all continuous functions with the supremum norm $|\cdot|_0$;

$C_1(P,R^n)$ - the Banach space of equivalence classes of all functions p such that $p(\cdot,y)$ is measurable and $p(x,\cdot)$ is continuous for a.a $x \in [-\alpha,a]$ and such that

$$|p|_1 = \int_{-\alpha}^{\alpha} \max_{y \in [-\beta,b]} \|p(x,y)\|\, dx < \infty;$$

$C_2(P,R^n)$ - the Banach space of equivalence classes of all functions q such that $q(\cdot,y)$ is continuous for a.a $y \in [-\beta,b]$ and $q(x,\cdot)$ is measurable and so that

$$|q|_2 = \int_{-\beta}^{b} \max_{x \in [-\alpha,a]} \|q(x,y)\|\, dy < \infty;$$

$A(P,R^n)$ - the space of all absolutely continuous functions such that $z_x \in C_1$, $z_y \in C_2$ and $z_{xy} \in L$. We shall consider $A(P,R^n)$ together with the norm $|z|_P = |z|_0 + |z_x|_1 + |z_y|_2$ and $W_P = C_0 \times C_1 \times C_2$ with the norm $|(z,p,q)|_P = |z|_0 + |p|_1 + |q|_2$.

Let F be a space of all functions $f : \bar{D} \times C_0 \times C_1 \times C_2 \to R^n$ satisfying the Carathèodory type conditions:

(i) $f(\cdot,\cdot,z,p,q)$ is measurable for fixed $(z,p-q) \in C_0 \times C_1 \times C_2$

(ii) $f(x,y,\cdot,\cdot,\cdot)$ is continuous for fixed $(x,y) \in \bar{D}$

(iii) there exists a Lebesgue integrable function $m:\bar{D} \to R_+$ such that $\|f(x,y,z,p,q)\| \leq m(x,y)$ for $(z,p,q) \in W_P$.

Let us introduce in F the equivalence relation "$\sim$" by setting $f_1 \sim f_2$ iff $f_1(x,y,z,p,q) = f_2(x,y,z,p,q)$ for a.a $(x,y) \in D$ and $(z,p,q) \in W_P$. We denote by $\mathcal{F}$ the space of all equivalence classes of F with the norm

$$(*) \quad \|f\|_{\mathcal{F}} = \iint_D \sup\left\{\|f(x,y,z,p,q)\| : (z,p,q) \in W_P\right\} dxdy, \quad f \in \mathcal{F}.$$

A mapping $f \in \mathcal{F}$ is called to be locally Lipschitzean with respect to $(z,p,q) \in W_P$, iff for every (z,p,q) there exists a neighbourhood U of (z,p,q) and a Lebesgue integrable function $k_{zpq}:\bar{D} \to R_+$ such that

$$\|f(x,y,z_1,p_1,q_1) - f(x,y,z_2,p_2,q_2)\| \leq$$

$$\leq k_{zpq}(x,y)\left(|z_1-z_2|_0^{xy} + |p_1-p_2|_1^{xy} + |q_1q_2|_2^{xy}\right)$$

for (z_1,p_1,q_1), $(z_2,p_2,q_2) \in U$ and $(x,y) \in \bar{D}$, where

$$|z_1-z_2|_0^{xy} = \sup_{P_{xy}} \|z_1-z_2\|,$$

$$|p_1-p_2|_1^{xy} = \int_{-\alpha}^{x} \max_{t \in [-\beta,y]} \|p_1(s,t) - p_2(s,t)\| ds$$

and

$$|q_1-q_2|_2^{xy} = \int_{-\beta}^{y} \max_{s \in [-\alpha,x]} \|q_1(s,t) - q_2(s,t)\| dt.$$

Similarly to proofs in the paper [6] we can prove the following Lemmas

Lemma 1. $(A(P,R^n), |\cdot|_P)$ is a Banach space.

Lemma 2. $(\mathcal{F}, \rho)$ is a complete metric space with ρ defined by (*).

Lemma 3. Suppose that $z : G \to R^n$ is absolutely continuous function, then there exists an absolutely continuous extension $\tilde{z}$ of the function z on P such that $|\tilde{z}|_P = |z|_G$.

Lemma 4. For every $f \in \mathcal{F}$ and $\varepsilon > 0$ there exists $f_\varepsilon \in \mathcal{F}$, locally Lipschitzean and such that $\rho(f_\varepsilon, f) \leq \varepsilon$.

3. Convergence of successive approximations

Let us consider the following functional-differential equation

$$\text{(I)} \quad \begin{cases} z(x,y) = \varphi(x,y) & \text{for } (x,y) \in G \\ z_{xy}(x,y) = f(x,y,z,z_x,z_y) & \text{for a.a } (x,y) \in \bar{D} \end{cases}$$

where $\varphi \in A(G,R^n)$ is given, $f : \bar{D} \times C_0 \times C_1 \times C_2 \to R^n$ and $f \in \mathcal{F}$. The problem consisting in finding a solution of equation (I) will be called the Goursat problem.

It is easy to verify that (I) is equivalent to the integral equation

$$\text{(II)} \quad z(x,y) = \begin{cases} \varphi(x,y) & \text{for } x,y) \in G \\ \lambda(x,y) + \iint\limits_{D_{xy}} f(s,t,z,z_x,z_y)dsdt & \text{for } (x,y) \in D \end{cases}$$

where

$$\lambda(x,y) = \varphi(0,0) + \int_0^x \varphi_x(s,g(s))ds + \int_0^y \varphi_y(h(t),t)dt.$$

Hence we obtain

$$z_x(x,y) = \begin{cases} \varphi_x(x,y) & \text{for a.a } x \in [-\alpha,a] \text{ and } y \in G_x \\ \varphi_x(x,g(x)) + \int\limits_{g(x)}^{y} f(x,t,z,z_x,z_y)dt & \text{for a.a } x \in [0,a] \text{ and } y \in D_x \end{cases}$$

and

$$z_y(x,y) = \begin{cases} \varphi_y(x,y) & \text{for } x \in G_y \text{ and a.a } y \in [-\beta,b] \\ \varphi_y(h(y),y) + \int\limits_{h(y)}^{x} f(s,y,z,z_x,z_y)ds & \text{for } x \in D_y \text{ and a.a } y \in [0,b]. \end{cases}$$

Let $X = A(G,R^n) \times \mathcal{F}$ and let us denote by S the set of all (φ,f) with locally Lipschitzean f.

In virtue of Lemma 4, we have $\bar{S} = X$.

Let (z_n) be a sequence from $A(P,R^n)$ defined by

$$(**) \quad z_n(x,y) = \begin{cases} \varphi(x,y) & \text{for } (x,y) \in G \\ \lambda(x,y) + \iint\limits_{D_{xy}} f(s,t,z_{n-1},(z_{n-1})_x,(z_{n-1})_y)dsdt & \text{for } (x,y) \in D, \end{cases}$$

where $z_0(x,y) = \varphi(x,y)$ for $(x,y) \in G$ and $z_0(x,y) = \tilde{\varphi}(x,y)$, for $(x,y) \in D$. Here $\tilde{\varphi}$ denote an absolutely continuous extension of the function φ on P.

T h e o r e m 1. For every $(\varphi,f) \in S$ the sequence (z_n) defined by $(**)$ is convergent in $A(P,R^n)$ to the unique solution of (I).

P r o o f . Let $z_0(x,y) = \varphi(x,y)$ for $(x,y) \in G$ and $z_0(x,y) = \tilde{\varphi}(x,y)$ for $(x,y) \in D$.

Suppose U_0 and $k_0 : \bar{D} \to R_+$ are a neighbourhood of $(z_0,(z_0)_x,(z_0)_y)$ and Lebesgue integrable function, respectively such that

$$\| f(x,y,z_1,p_1,q_1) - f(x,y,z_2,p_2,q_2) \| \leqslant$$

$$\leqslant k_0(x,y)(|z_1-z_2|_0^{xy} + |p_1-p_2|_1^{xy} + |q_1-q_2|_2^{xy})$$

for $(z_1,p_1,q_1),(z_2,p_2,q_2) \in U_0$ and $(x,y) \in \bar{D}$.

Let S_0 be a closed ball of W_p with the center $(z_0,(z_0)_x,(z_0)_y)$ and a radius $r_0 > 0$ such that $S_0 \subset U_0$.

Select $(x_0,y_0) \in D$ such that

$$\iint\limits_{D_0} m(x,y)dxdy \leqslant \frac{r_0}{6}; \quad \iint\limits_{D_0} k_0(s,t)dsdt < \frac{1}{3}, \quad |\lambda - z_0|_{D_0} \leqslant \frac{r_0}{6},$$

where $D_0 = \{(s,t) : h(t) \leqslant s \leqslant x_0,\ g(s) \leqslant t \leqslant y_0\}$. This is possible, because $|\lambda - z_0|_{D_{xy}} \to 0$, $\iint\limits_{D_{xy}} m(s,t)dst \to 0$ and $\iint\limits_{D_{xy}} k_0(s,t)dsdt \to 0$ as $(x,y) \to (0,0)$.

Let

$$u_1(x,y) = \begin{cases} \varphi(x,y) & \text{for } (x,y) \in G \\ \lambda(x,y) + \iint\limits_{D_{xy}} f(s,t,z_0,(z_0)_x,(z_0)_y)dsdt & \text{for } (x,y) \in D. \end{cases}$$

We observe that $u_1 \in A(P,R^n)$ and

$$|u_1 - z_0|_{0D_0} \leqslant \frac{r_0}{3}, \quad |(u_1)_x - (z_0)_x|_{1D_0} \leqslant \frac{r_0}{3}, \quad |(u_1)_y - (z_0)_y|_{2D_0} \leqslant \frac{r_0}{3}.$$

Then $|u_1 - z_0|_{D_0} \leqslant r_0$.

Now let

$$z_1(x,y) = \begin{cases} u_1(x,y) & \text{for } (x,y) \in G \cup D_0 \\ z_0(x,y) + (\widetilde{u_1 - z_0}) & \text{for } (x,y) \in D \setminus D_0. \end{cases}$$

Using induction we define sequences: (u_n) and (z_n) in $A(P,R^n)$ by setting

$$u_n(x,y) = \begin{cases} \varphi(x,y) & \text{for } (x,y) \in G \\ \lambda(x,y) + \iint\limits_{D_{xy}} f(s,t,z_{n-1},(z_{n-1})_x,(z_{n-1})_y)\,dsdt & \text{for } (x,y) \in D \end{cases}$$

$$z_n(x,y) = \begin{cases} u_n(x,y) & \text{for } (x,y) \in G \cup D_0 \\ z_0(x,y) + (\widetilde{u_n - z_0}) & \text{for } (x,y) \in D \setminus D_0 \end{cases}$$

for $n = 1,2,\ldots$.

It is easy to see that

$$|z_n - z_0|_P = |u_n - z_0|_{D_0} \leqslant r_0$$

for $n = 1,2,\ldots$ and $(z_n,(z_n)_x,(z_n)_y) \in S_0 \in U_0$. Therefore,

$$|z_2 - z_1|_P = |z_2 - z_1|_{0D_0} + |(z_2)_x - (z_1)_x|_{1D_0} + |(z_2)_y - (z_1)_y|_{2D_0} \leqslant$$

$$\leqslant \sup_{D_0} \iint\limits_{D_{xy}} k_0(s,t)|z_1 - z_0|^{st}\,dsdt + 2\iint\limits_{D_0} k_0(s,t)|z_1 - z_0|^{st}dsdt \leqslant$$

$$\leqslant |z_1 - z_0|_P \left(3\iint\limits_{D_0} k_0(s,t)dsdt\right).$$

Similarly, we get

$$|z_{n+1} - z_n|_P \leqslant 3|z_n - z_{n-1}|_P \iint\limits_{D_0} k_0(s,t)dsdt$$

for $n = 1,2,\ldots$. Hence, by the completeness of $A(P,R^n)$ the existence of $z^0 \in A(P,R^n)$ satisfying $\lim\limits_{n\to\infty} |z_n - z^0|_P = 0$ follows. Of course $(z^0,z^0_x,z^0_y) \in S_0$. Then

$$\|z^0(x,y)-\lambda(x,y) - \iint_{D_{xy}} f(s,t,z^0,z^0_x,z^0_y)dsdt\| \leqslant |z^0-z_n|_{oP} +$$

$$+ \iint_{D_{xy}} \|f(s,t,z^0,z^0_x,z^0_y) - f(s,t,z_{n-1},(z_{n-1})_x,(z_{n-1})_y)dsdt \leqslant$$

$$\leqslant |z-z_n|_P \left(1 + \iint_{D_0} k_0(s,t)dsdt\right)$$

for $(x,y) \in D_0$ and $n = 1,2,\dots$. Furthermore $z^0(x,y) = \varphi(x,y)$ for $(x,y) \in G$. Thus

$$z^0(x,y) = \begin{cases} \varphi(x,y) & \text{for } (x,y) \in G \\ \lambda(x,y) + \iint_{D_{xy}} f(s,t,z^0,z^0_x,z^0_y)dsdt & \text{for } (x,y) \in D_0 \\ z_0(x,y) + (\widetilde{z^0-z_0}) & \text{for } (x,y) \in D \setminus D_0. \end{cases}$$

We shall show now that there exists exactly one function z^0 satisfying above equality. Suppose that $\bar{z} \in A(P,R^n)$ satisfies it too. Since

$$|z^0-\bar{z}|_P = |z^0-\bar{z}|_{oD_0} + |z^0_x-\bar{z}_x|_{1D_0} + |z^0_y-\bar{z}_y|_{2D_0} \leqslant$$

$$\leqslant 3|z^0-\bar{z}|_P \iint_{D_0} k_0(s,t)dsdt \leqslant |z^0-\bar{z}|_P$$

then $|z^0-\bar{z}|_P = 0$.

Let now $z^{(1)}_0(x,y) = z^0(x,y)$ for $(x,y) \in G \cup D_0$ and $z^{(1)}_0(x,y) = \widetilde{z^0(x,y)}$ for $(x,y) \in D \setminus D_0$ where z^0 is a solution of (I) on D_0.

Simiarly we can prove that there exists a closed ball $S_1 \subset U_1$ with the center $(z^{(1)}_0, (z^{(1)}_0)_x, (z^{(1)}_0)_y)$ and a radius $r_1 > 0$ and a point $(x_1,y_0)(x_0 < x_1 \leqslant a)$ such that

$$\iint\limits_{D_1} m(x,y) \leq \frac{r_1}{6}\ , \quad \iint\limits_{D_1} k_1(s,t)dsdt < \frac{1}{3}\ , \quad |\lambda_1 - z_0^{(1)}|_{D_1} \leq \frac{r_0}{6}$$

where

$$\lambda_1(x,y) = z_0^{(1)}(x_0,g(x_0)) + \int_{x_0}^{x} (z_0^{(1)})_x(s,g(s))ds +$$

$$+ \int_{g(x_0)}^{y} (z_0^{(1)})_y(x_0,t)dt$$

and $D_1 = \{(x,y) : x_0 < x \leq x_1,\ g(x) < y \leq y_0\}$.

Analogously, we define

$$u_n(x,y) = \begin{cases} z_0^{(1)}(x,y) & \text{for} \quad (x,y) \in G \cup D_0 \\ \lambda_1(x,y) + \iint\limits_{D_{1xy}} f(s,t,z_{n-1},(z_{n-1})_x,(z_{n-1})_y)dsdt & (x,y) \in D_1 \end{cases}$$

and

$$z_n(x,y) = \begin{cases} u_n(x,y) & \text{for} \quad (x,y) \in G \cup D_0 \cup D_1 \\ z_0^{(1)}(x,y) + (\widetilde{u_n - z_0^{(1)}}) & \text{for} \quad (x,y) \in D \setminus (D_0 \cup D_1). \end{cases}$$

Now, in a similar way as above we can define a function $z^1 \in A(P,R^n)$ such that $\lim_{n\to\infty} |z_n - z^1|_P = 0$.

Continuing this process we can define a unique function $z \in A(P,R^n)$ satisfying (I) on the whole P. This completes the proof.

3. Non-convergence of successive approximations for the equation (I)

Let $(\varphi,f) \in X$ and let $(z_n^{(\varphi,f)})$ be a sequence of $A(P,R^n)$ defined by

$$(***)\qquad z_n^{(\varphi,f)}(x,y) =$$

$$= \begin{cases} \varphi(x,y) & \text{for } (x,y)\in G \\ \lambda(x,y) + \iint\limits_{D_{xy}} f(s,t,z_{n-1}^{(\varphi,f)},(z_{n-1}^{(\varphi,f)})_x,(z_{n-1}^{(\varphi,f)})_y)\,ds\,dt & \text{for } (x,y)\in D \end{cases}$$

In this section we shall show that non-convergence of a sequence $(z_n^{(\varphi,f)})$ is in any sense a rare case.

We will use here the following, not published yet result of Lasota (the proof is given in [2]).

L e m m a 5. Let (X,d) be a complete metric space and S a dense subset of (X,d). Suppose a function $\chi:X \to [0,\infty)$ is such that $\chi(x_n) \to 0$ as $n \to \infty$ for any sequence (x_n) of X such that $x_n \to x \in S$ as $n \to \infty$. Then the set $\mathfrak{X} = \{x\in X:\chi(x)=0\}$ is a residual subset of (X,d).

Let $\chi : X \to [0,\infty)$ be defined by $\chi(\varphi,f) = \lim\limits_{m\to\infty} \operatorname{diam} E_m^{(\varphi,f)}$ where $E_m^{(\varphi,f)} = \left\{z_m^{(\varphi,f)}, z_{m+1}^{(\varphi,f)}, \dots\right\}$ for $m = 1,2,\dots$ and $\operatorname{diam} E_m^{(\varphi,f)}$ denotes the diameter of $E_m^{(\varphi,f)}$. We have $E_{m+1}^{(\varphi,f)} \subset E_m^{(\varphi,f)}$ and then $\operatorname{diam} E_{m+1}^{(\varphi,f)} \leqslant \operatorname{diam} E_m^{(\varphi,f)}$. Since

$$0 \leqslant \operatorname{diam} E_1^{(\varphi,f)} \leqslant 6 \iint\limits_D m(x,y)\,dx\,dy$$

for every $(\varphi,f)\in X$, then $\lim\limits_{m\to\infty} \operatorname{diam} E_m^{(\varphi,f)}$ exists for each $(\varphi,f)\in X$. Now, by definition of $\chi(\varphi,f)$ it at once follows that $\chi(\varphi,f) = 0$ iff $(z_m^{(\varphi,f)})$ converges in $A(P,R^n)$.

L e m m a 6. Let $(\varphi,f)\in S$ and (φ_n,f_n) be a sequence in X such that $|\varphi_n - \varphi_G| + \rho(f_n,f) \to 0$ as $n \to \infty$. Then $|z_m^{(\varphi_n,f_n)} - z_m^{(\varphi,f)}|_P \to 0$ as $n \to \infty$, uniformly with respect to $m \geqslant 1$.

P r o o f . Let $z_0(x,y) = \varphi(x,y)$ for $(x,y) \in G$ and $z_0(x,y) = \tilde{\varphi}(x,y)$ for $(x,y) \in D$. Select a neighborhood U_0 of $(z_0,(z_0)_x,(z_0)_y)$ and $k_0 \in L(D)$ such that

$$\| f(x,y,z_1,p_1,q_1) - f(x,y,z_2,p_2,q_2) \| \leqslant$$

$$\leqslant k_0(x,y)(|z_1-z_2|_0^{xy} + |p_1-p_2|_1^{xy} + |q_1-q_2|_2^{xy})$$

for $(x,y) \in \bar{D}$ and (z_1,p_1,q_1), $(z_2,p_2,q_2) \in U_0$.

Let B_0 be a closed ball of W_p with the center $(z_0,(z_0)_x,(z_0)_y)$ and a radius $r_0 > 0$ such that $B_0 \subset U_0$.

Similarly as in the proof of Theorem 1 we can select $(x_0,y_0) \in D$ and $N \geqslant 1$ such that

$$|\varphi_n - \varphi|_G \leqslant \frac{r_0}{3}, \quad \iint_{D_0} m(x,y)dxdy \leqslant \frac{r_0}{9}, \quad \iint_{D_0} k_0(s,t)dsdt < \frac{1}{3}$$

$$|\lambda_n - z_0|_{D_0}^{\iint} \leqslant \frac{r_0}{3} \quad \text{and} \quad |\lambda - z_0|_{D_0}^{\iint} \leqslant \frac{r_0}{3} \quad \text{where}$$

$$\lambda_n(x,y) = \varphi_n(0,0) + \int_0^x (\varphi_n)_x(s,g(s))ds + \int_0^y (\varphi_n)_y(h(t),t)dt.$$

Denote

$$j_{x_0y_0}(z_m^{(\varphi,f)})(s,t) = \begin{cases} z_m^{(\varphi,f)}(s,t) & \text{for } (s,t) \in G \cup D_0 \\ z_m^{(\varphi,f)}(s,t) & \text{for } (s,t) \in D \setminus D_0, \end{cases}$$

$$k_{x_0y_0}((z_m^{(\varphi,f)})_x)(s,t) = \begin{cases} (z_m^{(\varphi,f)})_x(s,t) & \text{for } (s,t) \in G \cup D_0 \\ 0 & \text{for } (s,t) \in D \setminus D_0, \end{cases}$$

$$l_{x_0y_0}((z_m^{(\varphi,f)})_y)(s,t) = \begin{cases} (z_m^{(\varphi,f)})_y(s,t) & \text{for } (s,t) \in G \cup D_0 \\ 0 & \text{for } (s,t) \in D \setminus D_0. \end{cases}$$

Let us observe that

$$\left| j_{x_0y_0}(z_m^{(\varphi_n,f_n)}) - j_{x_0y_0}(z_m^{(\varphi,f)}) \right|_0 +$$

$$+ \left| k_{x_0y_0}((z_m^{(\varphi_n,f_n)})_x) - k_{x_0y_0}((z_m^{(\varphi,f)})_x) \right|_1 +$$

$$+ \left| l_{x_0y_0}((z_m^{(\varphi_n,f_n)})_y) - l_{x_0y_0}((z_m^{(\varphi,f)})_y) \right|_2 \leqslant$$

$$\leqslant \frac{1}{1 - 3 \iint\limits_{D_0} k_0(x,y)\,dxdy} \left[|\varphi_n - \varphi|_G + |\lambda_n - \lambda|_{D_0} + 3\varrho(f_n,f) \right]$$

for $n \geqslant N$, which proves that

$$|z_m^{(\varphi_n,f_n)} - z_m^{(\varphi,f)}|_{D_0} \to 0 \quad \text{as} \quad n \to \infty,$$

uniformly with respect to $m \geqslant 1$. Continuing this process we can get

$$|z_m^{(\varphi_n,f_n)} - z_m^{(\varphi,f)}|_P \to 0 \quad \text{as} \quad n \to \infty \,,$$

uniformly with respect to $m \geqslant 1$. This proves Lemma 6.

We shall show now that $\chi(\varphi_n,f_n) \to 0$ for every sequence (φ_n,f_n) of X such that $|\varphi_n - \varphi_G| + \varrho(f_n,f) \to 0$ as $n \to \infty$, where $(\varphi,f) \in S$.

L e m m a 7. Let (φ_n,f_n) be a sequence of X such that $|\varphi_n - \varphi_G| + \varrho(f_n,f) \to 0$ as $n \to \infty$ and $(\varphi,f) \in S$. Then $\chi(\varphi_n,f_n) \to 0$ as $n \to \infty$.

P r o o f . Suppose that (φ_n, f_n) is a sequence of X convergent to $(\varphi, f) \in S$ such that $\chi(\varphi_n, f_n)$ is not convergent to zero. Then there are $\eta > 0$ and a subsequence (φ_k, f_k) of (φ_n, f_n) such that $\varphi_k - \varphi_G + \varrho(f_k, f) \to 0$ as $k \to \infty$ and $\chi(\varphi_k, f_k) \geq \eta$ for each $k \geq 1$. Hence, it follows that $\operatorname{diam} Z_m^{(\varphi_k, f_k)} \geq \eta$ for $k, m \geq 1$. But in view of the Lemma 6 a sequence $(z_m^{(\varphi_k, f_k)})$ is convergent to $z_m^{(\varphi, f)}$ in $A(P, R^n)$, uniformly with respect to $m \geq 1$. Therefore, for $n, m \geq 1$ and sufficiently large k, say $k \geq N_k$, we have

$$\left| z_m^{(\varphi_k, f_k)} - z_n^{(\varphi_k, f_k)} \right|_P \leq \frac{1}{k} + \left| z_m^{(\varphi, f)} - z_n^{(\varphi, f)} \right|_P.$$

Thus

$$\eta \leq \operatorname{diam} Z_m^{(\varphi_k, f_k)} \leq \frac{1}{k} + \operatorname{diam} Z_m^{(\varphi, f)} = \frac{1}{k}$$

for $k \geq N_k$, because $(\varphi, f) \in S$. This contradicts to $\eta > 0$.

T h e o r e m 2. The set $\mathfrak{X}$ of all $(\varphi, f) \in X$ for which a sequence $(z_n^{(\varphi, f)})$ defined by the formula (***) is convergent in $A(P, R^n)$ is residual subset of X.

P r o o f . In virtue of Lemma 7 the mapping $\chi : X \to [0, \infty)$ satisfies the assumptions of Lemma 5. This implies that the set $\Omega = \{(\varphi, f) \in X : \chi(\varphi, f) = 0\}$ is a residual subset of X. Then, $\mathfrak{X}$ is a residual subset of X, too and the proof has been completed.

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