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SOME FIXED POINT AND COINCIDENCE POINT THEOREMS FOR MULTIVALUED MAPPINGS IN TOPOLOGICAL VECTOR SPACES

0. Notations, definitions and preliminaries

In the recent time there is an increasing interest in the fixed point theory in not necessarily locally convex topological vector spaces (see [1]-[9], [11], [13], [16]).

Some further results in this direction will be proved in this paper.

First, we shall give some useful definitions and results.

D e f i n i t i o n 1 [11]. A convex space means a convex set in a vector space with any topology that induces the Euclidean topology on the convex hull of its finite subsets.

Every convex subset of a **Hausdorff topological vector** space is a convex space.

D e f i n i t i o n 2 [11]. Let X be a convex space and K a nonempty subset of X. The set K is said to be c-compact if for each finite subset $\mathfrak{F}$ of X there is a compact convex subset $K_{\mathfrak{F}}$ of X such that $K \cup \mathfrak{F} \subset K_{\mathfrak{F}}$.

If X is a convex subset of a Hausdorff topological vector space then every nonempty convex compact subset of X is c-compact. In any convex space every finite set and every convex hull of a finite set is c-compact.

D e f i n i t i o n 3 [11]. Let Y be a topological space and $B \subset Y$. The set B is said to be compactly closed (open, respectively) in Y if for every compact subset L of Y the set $B \cap L$ is closed (open, respectively) in L.

By 2^E we shall denote the family of all nonempty subsets of a nonempty set E.

In [11] the following theorem is proved:

T h e o r e m . Let X be a convex space, Y a topological space and $S:X \to 2^Y$ so that the following conditions are satisfied:

(i) For each $x \in X$, $S(x)$ is compactly open in Y.

(ii) For each $y \in Y$, $S^{-1}(y)$ is nonempty and convex.

(iii) For some c-compact subset K of X the set $Y \setminus \bigcup_{x \in K} S(x)$ is compact.

Then for each continuous mapping s of X into Y there exists an $x \in X$ such that $s(x) \in S(x)$.

D e f i n i t i o n 4 [8]. Let X be a topological vector space and $K \subset X$. The set K is said to be of Zima's type if for every neighbourhood V of zero in X there exists a neighbourhood U of zero in X such that:

$$co(U \cap (K-K)) \subset V \quad (co\ F \text{ - the convex hull of } F).$$

Some fixed point theorems for multivalued mappings which are defined on subsets of Zima's type are proved in the papers [6], [7].

In [8] an example of a non locally convex space X and $K \subset X$ is given, where K is of Zima's type. If $(E, \| \|^*)$ is a paranormed space and $K \neq \emptyset$ is such that [16]:

$$\|tx\|^* \leqslant t\ C(K)\ \|x\|^*,$$

for every $t \in [0,1]$ and every $x \in K-K$, then $co(U_{\frac{r}{C(K)}} \cap (K-K)) \subset U_r$, for every $r > 0$ (where $U_r = \{x \mid x \in X, \|x\|^* < r\}$), which means that K is of Zima's type.

If X is a Hausdorff topological vector space and K is of Zima's type then, in [6], the following implication is proved:

(1) $A \subset K$, A is precompact $\Rightarrow$ co A is precompact.

D e f i n i t i o n 5 [15]. Let X be a subset of a topological vector space, and for every $x \in X$, $Tx \subset X$. A point x_0 is said to be a maximal element of T if $Tx_0 = \emptyset$.

All topological vector spaces in this paper will be assumed to be Hausdorff. If $F \subset E$ and E is a topological vector space, by 2^F_{co} we denote the family of all nonempty convex subsets of F.

1. Using the Theorem we shall prove the following coincidence theorem.

T h e o r e m 1. Let C be a nonempty closed and convex subset of a topological vector space E, C_1 a nonempty subset of a topological vector space F, $T:C \to 2^{C_1}_{co}$, $S:C \to 2^{C_1}$ so that the following conditions are satisfied:

(i) For each $x \in C$, $S(x)$ is compactly open in C_1 and there exists $y \in C_1$ so that $x \in \operatorname{int} T^{-1}(y)$.

(ii) For each $y \in C_1$, $S^{-1}(y)$ is nonempty and convex.

(iii) For some c-compact subset K of C the set $C_1 \setminus \bigcup_{x \in K} S(x)$ is compact.

(iv) There exists a finite set $M \subset C_1$ and a compact set $L \subset C$ such that:

$$\bigcap_{x \in M} \left[\operatorname{int} T^{-1}(x)\right]^c \subset L .$$

Then there exists $x_0 \in C$ such that:

$$S(x_0) \cap T(x_0) \neq \emptyset.$$

P r o o f . First, we shall prove that there exists a continuous mapping $s:C \to C_1$ such that $s(x) \in T(x)$ for every $x \in C$. From (i) it follows that $L \subset \bigcup_{y \in C_1} \operatorname{int} T^{-1}(y)$, and since L is compact there exists a finite set $N \subset C_1$ such that $L \subset \bigcup_{y \subset N} \operatorname{int} T^{-1}(y)$. Further, (iv) implies that $C = \bigcup_{y \in N \cup M} \operatorname{int} T^{-1}(y)$; let $N \cup M = \{y_1, y_2, \dots, y_n\}$.

By $\{f_1, f_2, \ldots, f_n\}$ we shall denote a partition of unity subordinated to the open covering $\{\text{int } T^{-1}(y_i)\}_{i=1}^{n}$ and let

$$s(x) = \sum_{i=1}^{n} f_i(x) y_i$$

for every $x \in C$. Then $s(x) = \sum_{i: f_i(x) \neq 0} f_i(x) y_i$ and since $f_i(x) \neq 0$ implies that x is in int $T^{-1}(y_i)$, we obtain that $f_i(x) \neq 0$ implies that $y_i \in T(x)$. From this we have that $s(x) \in \text{co } T(x)$. Since $T(x)$ is convex for every $x \in C$ it follows that $s(x) \in T(x)$ for every $x \in C$. From the Theorem we conclude that there exists $x_0 \in C$ such that $s(x_0) \in S(x_0)$, since s is a continuous mapping from C into C_1. Then $s(x_0) \in S(x_0) \cap T(x_0)$ and so Theorem 1 is proved.

Corollary 1. Let E, F, C and S be as in Theorem 1, C_1 a nonempty, convex subset of F and for every $x \in C$, $Tx \subset C_1$ so that the following conditions are satisfied:

(i) For every $x \in C$ such that $Tx \neq \emptyset$ there exists $y \in C_1$ so that $x \in \text{int}(\text{co } T)^{-1}(y)$, where $(\text{co } T)^{-1}(y) = \{x \mid y \in \text{co } Tx\}$.

(ii) There exists a finite set $M \subset C_1$ and a compact set $L \subset C$ such that

$$\bigcap_{x \in M} [\text{int } (\text{co } T)^{-1}(x)]^c \subset L.$$

(iii) For every $x \in C$, $Sx \cap \text{co } Tx = \emptyset$.

Then there exists $x_0 \in C$ so that $Tx_0 = \emptyset$.

Proof. Suppose that $Tx \neq \emptyset$ for every $x \in C$. Then the mapping $G: x \mapsto \text{co } Tx$ $(x \in C)$ satisfies all the conditions for T in Theorem 1 and so there exists $x \in C$ such that $Sx \cap \text{co } Tx \neq \emptyset$. But this contradicts to (iii). So $\{x \mid Tx = \emptyset\} \neq \emptyset$.

Remark. If $E = F$ and $C_1 \subset C$, Corollary 1 is, in fact, a result on the existence of a maximal element of T ([15]).

C o r o l l a r y 2. Let E, F, C, C_1 and T be as in Theorem 1, A an arbitrary set in a space Z and $g: C\times C_1 \mapsto Z$ so that the following conditions are satisfied:

(i) For each fixed $x \in C$, the set $\{y | y \in C_1,\ g(x,y) \in A\}$ is compactly open in C_1.

(ii) For each fixed $y \in C_1$, the set $\{x | x \in C,\ g(x,y) \in A\}$ is convex.

(iii) For some c-compact subset K of C the set: $\{y | y \in C_1,\ g(x,y) \notin A$ for all $x \in K\}$ is compact.

Then the mapping g satisfies at least one of the following properties:

(1) There exists $y_0 \in C_1$ such that $g(x,y_0) \notin A$ for all $x \in C$.

(2) There exists $x_0 \in C$ and $y_0 \in T(x_0)$ such that $g(x_0,y_0) \in A$.

P r o o f . Let $S(x) = \{y | y \in C_1,\ g(x,y) \in A\}$ for every $x \in C$. Suppose that (1) is not satisfied. This implies that $S^{-1}(y) \neq \emptyset$ for every $y \in C_1$, and so all the conditions of Theorem 1 are satisfied. Let $y_0 \in C$ and $y_0 \in Tx_0 \cap Sx_0$. Then $g(x_0,y_0) \in A$, where $y_0 \in Tx_0$.

C o r o l l a r y 3. Let C be a compact and convex subset of a topological vector space E, C_1 a nonempty subset of a topological vector space F, Z a regular space, C_2 a closed subset of Z and $\vartheta(C_2)$ a basis of open neighbourhoods of C_2 in Z. Let $T: C \to 2^{C_1}_{co}$ be as in Theorem 1 and let $g: C\times C_1 \to Z$ be a continuous mapping such that for every $y \in C_1$ and for any $U \in \vartheta(C_2)$ the set $\{x | x \in C,\ g(x,y) \in U\}$ is nonempty.

Then there exists $x \in C$ and $y \in Tx$ such that $g(x,y) \in C_2$.

P r o o f . As in [11], let $U \in \vartheta(C_2)$ and $A = U$. Then all the conditions of Corollary 2 are satisfied and since (1) in Corollary 2 is not satisfied, we conclude that there exists $x_U \in C$ such that $g(x_U, s(x_U)) \in U$. The continuous mapping $s: C \to C_1$ does not depend on U and $s(x) \in T(x)$, $x \in C$ (the existence of s is proved in Theorem 1). As in [11] it follows that there exists $x \in C$ such that $g(x,s(x)) \in C_2$ and $s(x) \in T(x)$.

T h e o r e m 2. Let E, F and C_1 be as in Theorem 1, C be a nonempty, convex paracompact subset of E, $T:C \to 2^{C_1}_{co}$ and let $S:C \to 2^{C_1}$ satisfy conditions (i)-(iii) of Theorem 1. Then there exists $x_0 \in C$ such that $T(x_0) \cap S(x_0) \neq \emptyset$.

P r o o f . Since C is paracompact, from $C = \bigcup_{y \in C_1} \text{int}\, T^{-1}(y)$ it follows that there exists an open locally finite refinement $\vartheta = \{V_i\}_{i \in I}$ of the covering $\{\text{int}\, T^{-1}(y)\}_{y \in C_1}$. Let $\{f_i\}_{i \in I}$ be a partition of unity subordinated to the covering ϑ. For every $i \in I$ there exists $y_i \in C_1$ so that $V_i \subset \text{int}\, T^{-1}(y_i)$; let $s(x) = \sum_{i \in I} f_i(x)\, y_i$ for every $x \in C$. Further, $f_i(x) \neq 0$ implies that $y_i \in T(x)$, and since ϑ is locally finite it follows that $s:C \to C_1$ is a continuous mapping such that $s(x) \in T(x)$. Then from the Theorem it follows that $\{x \mid x \in C,\ T(x) \cap S(x) \neq \emptyset\} \neq \emptyset$.

2. The following fixed point theorem is a generalization of a fixed point theorem proved by Himmelberg [10].

T h e o r e m 3. Let K be a nonempty convex subset of a complete topological vector space L, let $F:K \to 2^K$ be an upper semicontinuous multifunction such that F(x) is closed and convex for all $x \in K$ and suppose $F(K) \subset C \subset K$, where C is a compact subset of K. If K is of Zima's type there exists a fixed point of the mapping F.

P r o o f . The proof is similar to that of Theorem 2 in [10]. Let $A = \text{co}\, C$ and $\tilde{K} = \bar{A}$. Since K is of Zima's type from (1) it follows that $\tilde{K}$ is compact. Let $H = F \cap (A \times A)$ and $\bar{H}$ be the closure of H into $\tilde{K} \times \tilde{K}$. As in [10] it follows that there exists $x \in \tilde{K}$ such that $x \in \bar{H}(x)$ and since $x \in A$ we have that $x \in F(x)$.

T h e o r e m 4. Let X be a nonempty convex and paracompact subset of Zima's type of a topological vector space, D a compact subset of X and $T:X \to 2^D_{co}$ so that for every $x \in X$ there exists $y \in D$ such that $x \in \text{int}\, T^{-1}(y)$. Then there exists $x_0 \in D$ such that $x_0 \in Tx_0$.

P r o o f . As in Theorem 2 it follows that there exists a continuous mapping $s:X \to D$ such that $s(x) \in Tx$. From Theorem 3 it follows that $\{x \mid x \in D,\ x = s(x)\} \neq \emptyset$ and so $\{x \mid x \in D,\ x \in Tx\} \neq \emptyset$.

Let us give an application of Theorem 3 in economy.

Let $\{X_i\}_{i\in I}$ be a family of nonempty subsets. An abstract economy or qualitative game E is a family $\{(X_i, A_i, P_i)\}_{i\in I}$ of ordered triples (X_i, A_i, P_i) where $A_i : \prod_{j\in I} X_j \to 2^{X_i}$ and $P_i : \prod_{j\in I} X_j \to 2^{X_i}$. An equilibrium for E is an $x^* \in X = \prod_{j\in I} X_j$ such that for each $i \in I$:

(i) $$x_i^* \in A_i(x^*),$$

(ii) $$P_i(x^*) \cap A_i(x^*) = \emptyset.$$

T h e o r e m 5. Let $\{(X_i, A_i, P_i)\}_{i\in N}$ be an abstract economy so that the following conditions are satisfied for every $i \in I$:

(i) X_i is a convex, compact, metrizable subset of Zima's type of a topological vector space.

(ii) $A_i(x) = \mathrm{co}\, A_i(x) \neq \emptyset$ for all $x \in X$.

(iii) The mapping $\bar{A}_i : X \to 2^{X_i}$, where $\bar{A}_i(x) = \overline{A_i(x)}$ $(x \in X)$, is upper semicontinuous.

(iv) For each $x \in X$ such that $g_i(x) \neq \emptyset$ there exists $y \in X_i$ such that $x \in \mathrm{int}\, g_i^{-1}(y)$, where

$$g_i(x) = A_i(x) \cap \mathrm{co}\, P_i(x), \quad x \in X.$$

(v) The set $U_i = \{x \mid x \in X,\ g_i(x) \neq \emptyset\}$ is open in X.

(vi) $x_i \notin \mathrm{co}\, P_i(x)$, for all $x \in X$.

Then E has an equilibrium.

P r o o f . As in Theorem 2 it follows that there exists a continuous mapping $f_i : U_i \to X_i$ such that $f_i(x) \in g_i(x)$ for all $x \in U_i$, $i \in I$. Let for every $i \in I$

$$F_i(x) = \begin{cases} f_i(x), & x \in U_i \\ \bar{A}_i(x), & x \notin U_i \end{cases}$$

Then $F_i : X \to 2^{X_i}$ is an upper semicontinuous mapping, let us denote for every $x \in X$

$$F(x) = \prod_{i \in I} F_i(x).$$

Then $F : x \mapsto F(x)$ is an upper semicontinuous mapping of X into the family of nonempty, closed and convex subsets of X, which is of Zima's type [8]. From Theorem 3 it follows that there is an $x \in X$ such that $x \in F(x)$. Such an element x is also an equilibrium point of E.

REFERENCES

[1] F.E. B r o w d e r : The fixed point theory of multi-valued mappings in topological vector spaces, Math. Ann., 117 (1968) 283-301.

[2] B.C. C u ō n g : Some fixed point theorems for multi-functions in topological vector spaces (announcement of results), Bull. Acad. Polon. Sci. Sér. Sci. Math. 32 (1984) 215-221.

[3] O. H a d ž i ć : Some fixed point and almost fixed point theorems for multivalued mappings in topological vector spaces, Nonlinear Anal. Theory, Methods, Appl. 5 (1981) 1009-1019.

[4] O. H a d ž i ć : On Kakutani's fixed point theorem in topological vector spaces, Bull. Acad. Polon. Sci., Sér. Sci. Math., 30 (1982) 141-144.

[5] O. H a d ž i ć : A fixed point theorem for the sum of two mappings, Proc. Amer. Math. Soc., 85 (1982), 37-41.

[6] O. H a d ž i ć : On Sadovski's fixed point theorem in topological vector spaces, Comment. Math. Prace Mat. 27 (1983) 51-55.

[7] O. H a d ž i ć : Fixed point theory in topological vector spaces, University of Novi Sad, Institute of Mathematics, Novi Sad (1984) 337 pp.

[8] O. H a d ž i ć , Lj. G a j i ć : Some applications of fixed point theorems for multivalued mappings on minimax problems in topological vector spaces, Math. Operationsforsch. Statist., Ser. Optimization, 15 (1984) 193-201.

[9] S. H a h n , F. P ö t t e r : Über Fixpunkte kompakter Abbildungen in topologischen Vektorräumen, Studia Math. 50 (1974) 1-16.

[10] C.J. H i m m e l b e r g : Fixed points of compact multifunctions, J. Math. Anal., Appl., 38 (1972) 205-207.

[11] M. L a s s o n d e : On the use of KKM multifunctions in fixed point theory and related topics, J. Math. Anal. Appl., 97 (1983) 151-201.

[12] G. M e h t a : Existence of fixed points, equilibria and maximal elements in topological vector spaces, 1985 (to appear).

[13] B. R z e p e c k i : Remarks on Schauder's fixed point principle and its applications, Bull. Acad. Polon. Sci. Sér. Sci. Math., 27 (1979) 473-480.

[14] E. T a r a f d a r , G. M e h t a : The existence of quasi-equilibrium in a competitive economy, Int. J. Sci. Eng., 1 (1984) 1-12.

[15] N. Y a n n e l i s , N. P r a b h a r : Existence of maximal elements and equilibria in linear topological spaces, J. Math. Econ., 12 (1983) 233-245.

[16] K. Z i m a : On the Schauder fixed point theorem with respect to paranormed spaces, Comment. Math. Prace Mat. 19 (1977) 421-423.

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