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ON THE NONEQUIVALENCY OF THE THEOREM OF MIOUEL M_8 WITH A FOUR-POINT CASE OF THE THEOREM M_4^2

The configurational Theorem of Miquel has analogous significance in inversive planes as the Theorem of Pappus in projective planes.

An inversive (Möbius) plane is an incidence structure $I = (\mathcal{P}, \mathfrak{B}, Z)$, such that for every point $P \in \mathcal{P}$, the internal structure I_p is an affine plane and whose blocks are called circles ([1], 6,1).

In the fundamental paper on the axiomatics of the geometry of circles [2] van der Waerden and Smid proved that an inversive plane is miquelian (satisfies the Theorem of Miquel) if, and only if, it can be represented by a quadratic extension of a commutative field (or it is an ovoidal plane $I(\sigma)$, with σ a non-ruled quadric in a 3-dimensional geometry over a commutative field).

There are different formulations of the configurational proposition known as the Theorem of Miquel [1], [2], [3], [4].

We call "configuration of Miquel" the set of eight points A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4 and six circles c_1, c_2, c_3, c_4, c_A, c_B in an inversive plane, such that:

c_i is incident with A_i, B_i, A_{i+1}, B_{i+1} with the subscripts taken mod 4;

c_A is incident with A_1, A_2, A_3, A_4;

c_B is incident with B_1, B_2, B_3, B_4.

Here we use the following formulation of Theorem of Miquel:

If eight points are incident with five of the circles of a miquelian configuration, then the sixth circle of the configuration is incident with four of the points.

The case where A_i, B_i are eight distinct points is called Theorem of Miquel - M_8.

In [2] van der Waerden and Smid consider the case M_8 (VM) along with the case when A_i coincide with B_i.

Yi Chen in [4] proves that the last case is a consequence of M_8. He considers the different nontrivial coincidences of the points A_i, B_i, called k-point degenerations of the Theorem of Miquel according to the number k of the points in the configuration. There exist exactly nine such nonisomorphic miquelian configurations.

We make the following convention: when $A_i = B_i$, c_i is tangent to c_{i+1}; when $A_i = A_{i+1}$, c_{i+1} is tangent to c_A, and when $B_i = B_{i+1}$, c_{i+1} is tangent to c_B.

There are different notations for these configurations [4], [5], [6]. Here we denote by:

M_7 : $A_i = B_i$

M_6^1 : $A_i = B_i$; $A_{i+1} = B_{i+1}$

M_6^2 : $A_i = B_i$; $A_{i+2} = B_{i+2}$

M_6^3 : $A_i = B_i$; $A_{i+1} = A_{i+2}$

M_5^1 : $A_i = B_i$; $A_{i+1} = B_{i+1}$; $A_{i+2} = B_{i+2}$

M_5^2 : $A_i = B_i$; $A_{i+1} = A_{i+2}$; $B_{i+1} = B_{i+2}$

M_5^3 : $A_i = B_i$; $A_{i+1} = A_{i+2}$; $B_{i+2} = B_{i+3}$

M_4^1 : $A_i = B_i$; $A_{i+1} = B_{i+1}$; $A_{i+2} = B_{i+2}$; $A_{i+3} = B_{i+3}$

M_4^2 : $A_i = B_i$; $A_{i+1} = B_{i+1}$; $A_{i+2} = A_{i+3}$; $B_{i+2} = B_{i+3}$.

In all cases i is one of 1, 2, 3, 4 and the subscripts taken mod 4.

In the papers [4], [5], [6] is investigated the equivalency of the above configurations. H. Shaeffer proves that M_7 implicates M_8, and both M_6^1 and M_6^3 implicate M_8 [6].

So it is natural to investigate the problem of realization of miquelian configurations in non-miquelian inversive planes, i.e. inversive planes which do not satisfy the Theorem of Miquel M_8. The possibility of realization of some miquelian configuration in such plane enables us to come to conclusion about its equivalency with M_8.

There are known two classes of finite inversive planes M(q) and S(q), which are isomorphic to an egglike inversive plane $I(\sigma)$, where σ is non-ruled quadric in P(3,q) and ovoid $t(\psi)$ respectively ([1], 2,4).

We investigate in [7] the realization of the configuration M_4^1 in finite non-miquelian inversive planes S(q). In that paper we call M_4^1 "Little Theorem of Miquel".

Here we shall investigate the realization of the other possible four-point miquelian configuration M_4^2 in the same plane.

We use for our investigation a representation of S(q), which we introduced before [8]. The points of S(q) are the points (x,y) of the corresponding affine plane A(2,q) and the symbol (∞). The circles of S(q) are the special ovals $D\psi(x,y) + Ax + By + C = 0$ in A(2,q), where $\psi(x,y) = x^{\sigma+2} + y^{\sigma} + xy$ and $D,A,B,C \in GF(q)$-Galois field of q elements, $q = 2^e$, e odd and > 1, and σ is the unique automorphism of GF(q) satisfying $x^{\sigma^2} = x^2$ for all $x \in GF(q)$. The point (∞) is incident only with circles with $D = 0$.

The elements of Aut S(q) are explicitly given by

$$\varphi^{\alpha}_{abpsr} : \begin{cases} x' = \left[a + r\,\dfrac{g(p,s,x,y)}{f(p,s,x,y)}\right]^{\alpha} \\[2ex] y' = \left[b + r\,\dfrac{r^{\sigma}h(p,x) + a^{\sigma}g(p,s,x,y)}{f(p,s,x,y)}\right]^{\alpha} \end{cases}$$

for $(x,y) \neq (p,s)$ and $\varphi^{\alpha}_{abpsr}(p,s) = (\infty)$, $\varphi^{\alpha}_{abpsr}(\infty) = (a^{\alpha},b^{\alpha})$

$$\tau^{\beta}_{klm} : \begin{cases} x' = [k\ h(1,x)]^{\beta} \\ y' = [k^{\sigma+1} g(1,m,x,y)]^{\beta}, \quad \tau^{\beta}_{klm}(\infty) = (\infty). \end{cases}$$

We denote by $h(p,x) = p+x$, $g(p,s,x,y) = p^{\sigma}h(p,x) + h(s,y)$, $f(p,s,x,y) = \psi(x,y) + sx + py + \psi(p,s)$, also $a,b,p,s,r,k,l,m \in GF(q)$, $r \neq 0$, $k \neq 0$, and α, β are inner automorphism of $GF(q)$.

Aut $S(q)$ is doubly transitive on the points of $S(q)$ and transitive on the circles of $S(q)$. Note that in every set of pencils (parabolic pencils) with carrier L - Π^L, there exists unique pencil π^L_0 such that the stabilizer of Π^L is transitive on the circles of π^L_0 [9]. We call π^L_0 special pencil in the point L.

It is convenient to denote by:

c (ABC...) - the circle c is incident with points A,B,C,... ; c A c' - the circles c and c' are tangent in the point A.

The Theorem of Miquel for the configuration M^2_4 is:

If A, B, C, D are four distinct points nonincident with a circle and

i) c_1(ACD), c_2 A c_1, c_2 B c_3, c_3(BCD), c_A(ABC), c_A C c_4, $D \in c_4$, then c_B(ABD) and c_B D c_4,

ii) c_1(ACD), c_2 A c_1, c_2 B c_3, c_3(BCD), c_A(ABC), c_B(ABD), then c_4 C c_A and c_4 D c_B.

We shall prove the following

T h e o r e m : Every non-miquelian finite inversive plane $S(q)$ satisfies the Theorem of Miquel for the configuration M^2_4.

P r o o f . Aut $S(q)$ is doubly transitive on the points of $S(q)$, so without restriction we can take $A = (\infty)$ and $B = (0,0)$.

The circles c_1 and c_2 determine a pencil π^A with a carrier A.

I. Let π^A is the special pencil in the point A. Then

$$c_2 : x = 0 \quad \text{and} \quad c_1 : x = m, \quad m \neq 0.$$

But in the subgroup of Aut S(q) which fixes the points A and B, exists automorphism $\tau^1_{m^{-1}_{oo}}$ such that

$$\tau^1_{m^{-1}_{oo}}(c_1) = c_1' : x = 1 \quad \text{and} \quad \tau^1_{m^{-1}_{oo}}(c_2) = c_2.$$

So without restriction we put $c_1 : x = 1$.

As $C \in c_1$, $D \in c_1$, so

$$C = (1,c), \quad D = (1,d); \quad c,d \in GF(q).$$

From the condition c_2 B c_3 we have

$$c_3 : \psi(x,y) + \rho x = 0.$$

But $C \in c_3$, $D \in c_3$ therefore

(1) $$\rho = \psi(1,c) = \psi(1,d)$$

i.e. $(c+d)^{\sigma-1} = 1$ or $d = c+1$.

From the condition c_A (A B C) we obtain

$$c_A : y = cx.$$

i) From c_4 C c_A, $D \in c_4$ and (1) it follows that

$$c_4 : \psi(x,y) + \psi(1,c) = 0.$$

On the other hand c_B (A B D) i.e. $c_B : y = dx$.
The common points of c_B and c_4 we obtain from

$$\psi(x,dx) + \psi(1,c) = 0.$$

But $d = c+1$ i.e. $(x+d^{\frac{\sigma}{2}})^{\sigma+2} + c^{\sigma+1} = 0$ whence $x = (c+d)^{\frac{\sigma}{2}} = 1$.
Therefore c_B D c_4, q.e.d.
ii) From the condition $c_B(A\ B\ D)$ we have

$$c_B : y = dx.$$

But c_4 C c_A, $D \in c_4$ and therefore

$$c_4 : \psi(x,y) + \psi(1,c) = 0.$$

From (1), as well as in i) we prove that c_4 D c_3, q.e.d.

II. Let π^A is a non-special pencil in the point A. Then

$$c_2 : y = l\,x \quad \text{and} \quad c_1 : y = l\,x+n, \quad n\neq 0.$$

But in the subgroup of Aut S(q) which fixes the points A and B exists automorphism τ^1_{koo} with $k = n^{1-\sigma}$ such that

$$\tau^1_{koo}(c_2) = c_2' : y = mx,$$

$$\tau^1_{koo}(c_1) = c_1' : y = mx + 1 \quad \text{with} \quad m = n^{\sigma-2}\,l.$$

So without restriction we put

$$c_1 : y = mx + 1, \quad c_2 : y = mx, \quad m \in GF(q).$$

Since $C \in c_1$, $D \in c_1$:

$$C = (c,\ mc+1), \quad D = (d,\ md+1), \quad c,d \in GF(q).$$

From the condition c_3 B c_2 it follows that

$$c_3 : \psi(x,y) + \lambda(mx + y) = 0.$$

But $C \in c_3$, $D \in c_3$ and we obtain

(2) $$\lambda = \psi(c,\ mc+1) = \psi(d,\ md+1).$$

Since $c_A(A\ B\ C)$, then

$$c_A : (mc+1)x + cy = 0.$$

i) From $c_4 \subset c_A$ and $D \in c_4$, and (2) we have

$$c_4 : \psi(x,y) + \lambda = 0.$$

But $c_B(A\ B\ D)$, and therefore

$$c_B : (md+1)x + dy = 0.$$

Let us find the common points of c_B and c_4. If $d = 0$, then $\lambda = 1$, $D = (0,1)$ and $c_4 : \psi(x,y) + 1 = 0$, $c_B : x = 0$. So $\psi(0,y) + 1 = 0$ i.e. $y = 1$.

Therefore $D = (0,1)$ is the unique common point of c_B and c_4, i.e. $c_B\ D\ c_4$. If $d \neq 0$ then $c_B : y = \frac{md+1}{d}$ and $\psi(x, \frac{md+1}{d} x) + \lambda = 0$, or

$$\left[x + \frac{md+1}{d}^{6/2}\right]^{6+2} = \left(\frac{md+1}{d}\right)^{6+1} + \lambda$$

from where

$$(3) \qquad x + \left(\frac{md+1}{d}\right)^{6/2} = \left[\left(\frac{md+1}{d}\right)^{6+1} + \psi(d,md+1)\right]^{\frac{2-6}{2}} .$$

The equation (3) has unique solution $x = d$ and therefore $c_4 D\ c_B$.

ii) From $c_B(A\ B\ D)$ it follows that

$$c_B : (md+1)x + dy = 0.$$

From $c_4 \subset c_A$ and $D \in c_4$ we obtain

$$c_4 : \psi(x,y) + \lambda = 0.$$

Hence as in i) we prove that $c_4\ D\ c_B$, q.e.d.

Since the planes S(q) do not satisfy the Theorem of Miquel M_8 [1], [8] we obtain the following:

C o r o l l a r y : M_4^2 does not imply M_8, i.e. M_4^2 is not equivalent to M_8.

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