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IMPROVEMENT THEOREMS FOR POLYADIC GROUPS H-DERIVED FROM GROUPS

1. Introduction

The papers [15]-[17], [6] have been devoted to the basis of a general theory of derived polyadic groups (for the notion of a derived polyadic group (in various meanings) cf. also e.g. [3], [18], [19], [8], [4]-[7], [9]-[13]). In particular, in [17] we introduced the notion of a C-modification of a system with respect to a group. As will be pointed out in this paper, the so-called modification theorems related with that notion enable us to strengthen some basic results of the polyadic group theory, such as the Hosszú theorem and the Dörnte criterion for an n-group to be derived from a group.

2. Preliminaries

We shall use the notions, the terminology and notation of [15]-[17] where one can find the definitions of C-systems over (k+1)-groups, of C-derived (n+1)-groups and of other relevant notions. Most of the theorems to which we refer in this paper can be found in [15]-[17].

The symbol (G,f) will always denote an (n+1)-group, and $(G,\cdot)$ and $(G,\circ)$ will denote groups. By x^i we mean the i-th power of x in $(G,\cdot)$, whereas $x^{\langle i\rangle}$ denotes the i-th power of x in $(G,\circ)$. By $\bar{x}$ we mean the element skew to x in (G,f). We denote by $\mathrm{Cent}(G,\cdot)$ the center of $(G,\cdot)$.

Let a be an element of an (n+1)-group (G,f) and let i be an integer. The symbol $a^{[i]}$ is due to Post [18]. Namely, $a^{[i]} = f_{(i+2|i|)}(\overset{((i+2|i|)n+1-2|i|)}{a}, \overset{(2|i|)}{\bar{a}})$ for $n > 1$, and $a^{[i]} = a^{i+1}$ for a = 1, where by a^{i+1} we understand the usual power in the (binary) group (G,f) (cf. also [16]).

3. Improvement theorems for the condition H and certain its strengthenings

To any condition C one can assign the class of (n+1)-groups C-derived from groups. This class will be denoted by $\underline{D}(C)$. For distinct conditions those classes are usually distinct. If the condition C is stronger than C′, then the class $\underline{D}(C)$ is obviously a subclass of $\underline{D}(C')$. The class $\underline{D}(C)$ often turns out to be a proper subclass, as it takes place for H and E, or E and PE (for the definition of various conditions, e.g. H, E, P etc., see [15], [16]). But it may happen that though $C \neq C'$, the classes $\underline{D}(C)$ and $\underline{D}(C')$ are nevertheless equal. A condition C is said to be an (essential) improvement of the condition C′ if C is (essentially) stronger than C′ and $\underline{D}(C) = \underline{D}(C')$. It is worthwhile to add that this notion is meaningful for (n+1)-groups C-derived from (k+1)-groups with $k > 1$. But in this paper we are interested only in the case $k = 1$. Modification theorems formulated in [17] give the necessary and sufficient conditions for two C-systems $\langle\gamma;b\rangle$ and $\langle\delta;c\rangle$ to be n-C-creating systems of the same group (G,f). Fixing one of the system, say $\langle\gamma;b\rangle$, we may ask in what way the second system (i.e., $\langle\delta;c\rangle$) depends on $\langle\gamma;b\rangle$. From Theorem 1 of [17] it follows that $\langle\delta;c\rangle$ is constructed from $\langle\gamma;b\rangle$ by using an appropriate element $d \in G$. Choosing the element d in some special way we can often observe that $\langle\delta;c\rangle$ is already a C′-system, where C′ is essentially stronger than C. According to the terminology given above we say that the condition C′ is an essential improvement of C. Theorems resolving this type of questions will be referred to as improvement theorems. Now we give such theorems for the condition H and some of its strengthenings. Note that all systems under consideration are n-systems.

D e f i n i t i o n 1. Let $\langle\delta;c\rangle \in H(G,\cdot)$. The system $\langle\delta;c\rangle$ is said to be an L_i-system over $(G,\cdot)$ if there exists an element $a \in G$ such that:

1^o $\delta(a) = a$,

2^o $c = a^{in+1}$.

Putting in Corollary 1 of [17] the element b^{-i} in place of d we get

P r o p o s i t i o n 1. If $\langle\gamma;b\rangle \in H(G,\cdot)$, then the system $\langle\delta;c\rangle$ given by

$$\delta(x) = b^i \cdot \gamma(x) \cdot b^{-i}, \tag{1}$$

$$c = b^{i(n+1)+1} \tag{2}$$

is an L_i-system over the group $(G,\circ) = Ret^{1,2}_{b^{-i}}(G,\cdot)$ $(c = a^{\langle in+1\rangle}$, where $a = b^{i+1})$ and

$$der^n_{\gamma;b}(G,\cdot) = der^n_{\delta;c}(G,\circ). \tag{3}$$

The condition L_0 is just the condition H. For every integer i we have $\underline{D}(L_i) = \underline{D}(H)$. Thus, any condition L_i is an improvement of H. All conditions L_i are stronger than H; so they are (n,1)-regular (for the definition of the regularity of a system see [15]). For certain i and j the conditions L_i and L_j can be uncomparable. The problem of the comparability of L_i and L_j will be investigated in the next section. Now, we are going to show only the way how for any n-L_i-creating system of a given (n+1)-group (G,f) one can find an n-L_j-creating system of the same (n+1)-group (G,f). Namely, putting in Corollary 1 of [17] the element a^{i-j} in place of d we obtain

P r o p o s i t i o n 2. Let j be an integer. If $\langle\gamma;b\rangle \in L_i(G,\cdot)$ (i.e., $b = a^{in+1}$ and $\gamma(a) = a$), then the system $\langle\delta;c\rangle$ given by

$$\delta(x) = a^{j-i}\cdot\gamma(x)\cdot a^{i-j}, \tag{4}$$

$$c = a^{jn+j-i+1} \tag{5}$$

is an L_j-system over the group $(G,\circ) = \mathrm{Ret}^{1,2}_{a^{i-j}}(G,\cdot)$ ($c=\tilde{c}^{\langle jn+1\rangle}$, where $\tilde{c} = a^{j-i+1}$) and (3) holds.

The problem of the (n,1)-nonrestrictivity of the condition L_i is closely related to the Hosszú theorem. By the well-known Hosszú theorem (cf. [8], [18], [4]) an (n+1)-groupoid (G,f) is an (n+1)-group if and only if (G,f) is H-derived from some (binary) group. Note that this theorem remains true when we substitute certain other conditions for the condition H. Thus, from the definition of the condition G (cf. [15], [6], [16]) we infer immediately the following theorem:

Let a condition C be stronger than G and weaker than H. Then an (n+1)-groupoid (G,f) is an (n+1)-group if and only if (G,f) is C-derived from a group.

The Hosszú theorem is equivalent to the fact that any such condition C (i.e., between G and H) is (n,1)-nonrestrictive (cf. [15], [16]). Let us recall the following stronger version of Hosszú theorem (cf. Corollary 4 of [4]):

An (n+1)-groupoid (G,f) is an (n+1)-group if and only if for any $k|n$ (G,f) is H-derived from a (k+1)-group.

Thus, any condition C which is stronger than G and weaker than H is (n,k)-nonrestrictive for $k|n$. Then the question arises whether the Hosszú theorem remains true when we substitute some condition essentially stronger than H for the condition H. In other words, is any condition essentially stronger than H an (n,k)-restrictive condition? In this paper we assume that $k = 1$ and in this case the answer is positive. Taking into account the Hosszú theorem and Proposition 1 we get a corollary, which is a stronger version of the original

Hosszú theorem. To formulate this corollary we use the notation of our papers (which differs considerably from the original Hosszú's notation).

C o r o l l a r y 1. Let (G,f) be an $(n+1)$-groupoid. Then the following conditions are equivalent:

1^o (G,f) is an $(n+1)$-group;

2^o there exists an integer i such that (G,f) is L_i-derived from a group;

3^o for every integer i, (G,f) is L_i-derived from a group.

C o r o l l a r y 2. For any integer i the condition L_i is $(n,1)$-nonrestrictive.

Thus, the conditions L_i are the strongest from the conditions known so far for which the Hosszú theorem holds (all the time we consider only the case $k = 1$).

For a nonrestrictive condition C any of its improvement C' is also nonrestrictive and, conversely, any nonrestrictive condition C' which is stronger than C is an improvement of C. In Section 5 we will show that there is no $(n,1)$-nonrestrictive condition which is stronger than all conditions L_i.

Corollaries 2 and 3 of [17] together with Propositions 1 and 2 enable us to give further improvement theorems (formulated below as corollaries).

C o r o l l a r y 3. If $\langle b\rangle \in E(G,\cdot)$, then the system $\langle c\rangle$, where c is given by (2), is an EL_i-system over the group $(G,\circ) = \mathrm{Ret}^{1,2}_{b^{-i}}(G,\cdot)$ ($c = a^{\langle in+1\rangle}$, where $a = b^{i+1}$) and

$$\mathrm{der}^n_b(G,\cdot) = \mathrm{der}^n_c(G,\circ). \tag{6}$$

C o r o l l a r y 4. If $\langle b\rangle \in EL_i(G,\cdot)$ (i.e., $b = a^{in+1}$, where $a \in \mathrm{Cent}(G,\cdot)$), then the system $\langle c\rangle$, where c is given by (5), is an EL_j-system over the group $(G,\circ) = \mathrm{Ret}^{1,2}_{a^{i-j}}(G,\cdot)$ ($c = \tilde{c}^{\langle jn+1\rangle}$, where $\tilde{c} = a^{j-i+1}$) and (6) holds.

C o r o l l a r y 5. If $\langle b\rangle \in B(G,\cdot)$ (i.e., $b = \hat{b}^n$, where $\hat{b} \in \mathrm{Cent}(G,\cdot)$), then the system $\langle c\rangle$, where c is given by (2), is a BL_i-system over the group $(G,\circ) = \mathrm{Ret}^{1,2}_{b^{-i}}(G,\cdot)$ ($c = a^{\langle in+1\rangle}$, where $a = b^{i+1}$, and $c = \hat{c}^{\langle n\rangle}$, where $\hat{c} = \hat{b}^{2in+1}$) and (6) holds.

Corollary 6. If $\langle b\rangle \in BL_i(G,\cdot)$ (i.e., $b = \hat{b}^n$ and $b = a^{in+1}$, where $\hat{b}, a \in Cent(G,\cdot)$), then the system $\langle c\rangle$, where c is given by (5), is a BL_j-system over the group $(G,\circ) = Ret^{1,2}_{a^{i-j}}(G,\cdot)$ ($c = \tilde{c}^{\langle jn+1\rangle}$, where $\tilde{c} = a^{j-i+1}$, and $c = \hat{c}^{\langle n\rangle}$, where $c = a^{2(j-i)}\cdot\hat{b}$) and (6) holds.

It should be emphasized that the condition BL_i (which was mentioned in Corollary 5) is not a good improvement of B. Namely, putting in Corollary 4 of [17] the element b in place of d we get the following corollary (which was proved in another way in [16]).

Corollary 7. If $\langle b\rangle \in B(G,\cdot)$ (i.e., $b = \hat{b}^n$, where $\hat{b} \in Cent(G,\cdot)$), then the system $\langle c\rangle$, where $c = \hat{b}^{-1}$, is a PE-system over the group $(G,\circ) = Ret^{1,2}_{\hat{b}}(G,\cdot)$ (i.e., c is the neutral element of $(G,\circ)$) and (6) holds.

4. L_i-systems

In this section we resume the relations between the conditions L_i and L_j for $i \neq j$.

Proposition 3. A condition L_j is stronger than a condition L_i if and only if there exists an integer t such that $j = (ni+1)t+i$. Moreover, for every $t \neq 0$ the condition L_j is then essentially stronger than L_i provided $(i+j)n \neq -2$.

Proof. Let $j = (ni+1)t+i$ for some integer t. Consider an L_j-system $\langle \gamma;b\rangle$ over $(G,\cdot)$. Thus there exists an element $a \in G$ such that $\gamma(a) = a$ and $b = a^{jn+1}$. Observe that $a^{jn+1} = a^{(tn+1)(in+1)} = (a^{tn+1})^{in+1}$. So $\langle \gamma;b\rangle \in L_i(G,\cdot)$.

Conversely, let for some integers i, j the condition L_j be stronger than L_i. Consider the additive group of integers $(Z,+)$. The system $\langle b\rangle$, where $b = jn+1$, is an L_j-system over $(Z,+)$ (in fact, $\langle b\rangle$ is even an EL_j-system). Thus, by assumption, $\langle b\rangle \in L_i(Z,+)$, i.e., there exists an element $x \in Z$ such that

(7) $$jn+1 = (in+1)x.$$

From (7) we infer that $j = (in+1)t+i$ for some $t \in Z$.

Suppose that for some $j = (in+1)t+i$, with $j \neq i$ and $(i+j)n \neq -2$, the conditions L_j and L_i are equivalent. From (7) we get

(8) $$in+1 \mid jn+1.$$

On the other hand, the condition L_i is stronger than L_j (they are equivalent, by assumption), whence

(9) $$jn+1 \mid in+1.$$

From (8) and (9) we infer that $in+1 = jn+1$ (consequently, $i = j$) or $in+1 = -(jn+1)$ (consequently, $(i+j)n = -2$), which is impossible. This completes the proof of Proposition 3.

The case $(i+j)n = -2$ requires a separate treatment.

P r o p o s i t i o n 4. For every integer i the condition L_i is equivalent to the condition L_{-i-2} provided $n = 1$, and it is equivalent to L_{-i-1} provided $n = 2$.

From Proposition 4 we conclude that for $n = 1$ it is sufficient to consider only conditions L_i with $i \geqslant -1$, and for $n = 2$ only L_i with $i \geqslant 0$.

Taking into account Corollary 2, Propositions 3 and 4 we see that for $i \neq -2, -1, 0$ all conditions L_i are $(n,1)$-nonrestrictive and essentially stronger than H. So we get the solution of Problem 1 of [15].

The main purpose of [16] was to give some characterizations of n-C-identity elements for various conditions C. Now we are going to give such a characterization for the condition L_i. We start with two easy lemmas.

L e m m a 1. Given an $(n+1)$-group $(G,f) = \mathrm{der}^n_{\delta;o}(G,\circ)$, where $n > 1$, let a be an element of G such that $\delta(a) = a$. Then

(10) $$\mathrm{co}\bar{a} = a^{\langle 1-n \rangle}.$$

P r o o f . Indeed,

$$a = f(\bar{a}, \overset{(n)}{a}) = \bar{a} \circ a^{\langle n \rangle} \circ c,$$

which implies (10).

L e m m a 2. Given a 2-group $(G,f) = \mathrm{der}^1_{\delta;c}(G,\circ)$, let a be an element of G such that $\delta(a) = a$ and let $\hat{a}$ be the inverse of a in (G,f). Then

$$\hat{a} \circ c = c^{\langle -1 \rangle} \circ a^{\langle -1 \rangle}. \tag{11}$$

T h e o r e m 1. An element e is an n-L_i-identity element in an $(n+1)$-group (G,f) if and only if there exists an element $a \in G$ such that $e = a^{[i]}$. Moreover, $(G,f) = \mathrm{der}^n_{\delta;c}(G,\circ)$, where $\langle \delta;c \rangle$ is an n-L_i-system over $(G,\circ)$ and e is the neutral element of $(G,\circ)$, if and only if

$$(G,\circ) = \mathrm{ret}^{n,2}_{a_1^{2n-1}}(G,f), \tag{12}$$

$$\delta(x) = f_{(2)}(a^{[i]}, x, a_1^{2n-1}), \tag{13}$$

$$c = a^{[i(n+1)]}, \tag{14}$$

where the $(2n-1)$-ad $\langle a_1^{2n-1} \rangle$ is inverse to the 1-ad $\langle a^{[i]} \rangle$ in (G,f). Furthermore, $c = \tilde{c}^{\langle in+1 \rangle}$, where $\tilde{c} = a^{[i+1]}$.

P r o o f . The unary term operations in $(n+1)$-groups are described by different formulas for $n > 1$ and for $n = 1$ (cf. [16]). For this reason this proof differs in details for $n > 1$ and for $n = 1$.

Let $n > 1$. Assume that $e = a^{[i]}$, where $a \in G$. By Proposition 6 of [15], any element of (G,f) (so, in particular, the element a) is an n-H-identity element. Thus there exists a group $(G,\cdot)$ with a as the neutral element and there exists an H-system $\langle \gamma;b \rangle$ over $(G,\cdot)$ such that $(G,f) = \mathrm{der}^n_{\gamma;b}(G,\cdot)$. Using once more Proposition 6 of [15] we get

$$(G,\cdot) = \overline{\mathrm{Ret}}^n_a(G,f), \tag{15}$$

(16) $$\gamma(x) = f(a,x,\overset{(n-2)}{a},\bar{a}),$$

(17) $$b = a^{[1]}.$$

Define a system $\langle\delta;c\rangle$ as in Proposition 1, i.e., let δ and c be given by (1) and (2). Then $\langle\delta;c\rangle$ is already an L_i-system over $(G,\circ) = \mathrm{Ret}^{1,2}_{b^{-i}}(G,\cdot)$ ($c = \tilde{c}^{\langle in+1\rangle}$, where $\tilde{c} = b^{i+1}$) and $(G,f) = \mathrm{der}^{n}_{\delta;c}(G,\circ)$. We prove that

(18) $$b^{i} = a^{[i]}.$$

Indeed, for $i > 0$

$$b^{i} = f_{(i-1)}(\underbrace{b,\overset{(n-2)}{a},\bar{a},b}_{i-1}) = f_{(2i-1)}(\underbrace{\overset{(n+1)}{a},\overset{(n-2)}{a},\bar{a},}_{i-1}\overset{(n+1)}{a}) = a^{[i]}.$$

For $i < 0$ the proof of (18) is lightly more complicated. Using Lemma 4 of [16] we get

$$b^{i} = (b^{-1})^{-i} = f_{(-i-1)}(\underbrace{b^{-1},\overset{(n-2)}{a},\bar{a},b^{-1}}_{-i-1}) =$$

$$= f_{(-2i-1)}(\underbrace{a,\overset{(n-2)}{b},\bar{b},a,\overset{(n-2)}{a},\bar{a},}_{-i-1}a,\overset{(n-2)}{b},\bar{b},a) = f_{(-i)}(\underbrace{a,\overset{(n-2)}{b},\bar{b},a}_{-i}) =$$

$$= f_{(\cdot)}(\underbrace{\overset{((n-2)(n+1)+1)}{a},\overset{(n+1)}{\bar{a}}}_{-i},a) = f(\underbrace{\overset{(1-i)}{a},\overset{(-i)}{\bar{a}}}) =$$

$$= f_{(-i)}(\overset{(-in+1+2i)}{a},\overset{(-2i)}{\bar{a}}) = a^{[i]}.$$

Therefore, (18) holds for any integer i (for $i = 0$ equality (18) is evident). It is easy to check that e is the neutral element of $(G,\circ)$. Observe that

$$a^{[i]}\cdot a^{[-i]} = f(\overset{(in+1)}{a}, \overset{(n-2)}{a}, \bar{a}, \overset{(-in+1)}{a}) = a,$$

whence

$$b^{-i} = a^{[-i]}. \tag{19}$$

Let a $(2n-1)$-ad $\langle a_1^{2n-1}\rangle$ be inverse to the 1-ad $\langle a^{[i]}\rangle$ in (G,f). Using (1), (18), (19), (16), and (17) we have

$$\delta(x) = b^{i}\cdot\gamma(x)\cdot b^{-i} = f_{(3)}(a^{[i]}, \overset{(n-2)}{a}, \bar{a}, a, x, \overset{(n-2)}{a}, \bar{a}, \overset{(n-2)}{a}, \bar{a}, a^{[-i]}) =$$

$$= f_{(-i)}(a^{[i]}, x, \overset{(-in-1)}{a}) = f_{(2)}(a^{[i]}, x, a_1^{2n-1}),$$

i.e., (13) holds. Note that

$$x\circ y = x\cdot b^{-i}\cdot y = f_{(-i)}(x, \overset{(-in-1)}{a}, y) = f_{(2)}(x, a_1^{2n-1}, y),$$

which proves that $(G,\circ)$ is given by (12).

Taking into account (17), (2), and (18) we obtain

$$c = b^{i(n+1)+1} = (a^{[i]})^{n+1}\cdot a^{[1]} =$$

$$= f_{((i+1)(n+1)+1)}(\underbrace{\overset{(in+1)}{a}, \overset{(n-2)}{a}, \bar{a},}_{n+1} \overset{(n+1)}{a}) = a^{[i(n+1)+1]}.$$

By Proposition 1 we have $c = \tilde{c}^{\langle in+1\rangle}$, where $\tilde{c} = b^{i+1} = b^{i}\cdot b =$ $= f_{(i+2)}(\overset{(in+1)}{a}, \overset{(n-2)}{a}, \bar{a}, \overset{(n+1)}{a}) = a^{[i+1]}$.

Conversely, let e be an n-L_i-identity element in (G,f). Then there exists a group $(G,\circ)$ with e as the neutral element, and there exists $\langle\delta;c\rangle \in L_i(G,\circ)$ such that $(G,f) =$ $= \mathrm{der}^{n}_{\delta;c}(G,\circ)$. Since $\langle\delta;c\rangle$ is an L_i-system, the element c is of the form $c = \tilde{c}^{\langle in+1\rangle}$ with $\delta(\tilde{c}) = \tilde{c}$. Let $a = \tilde{c}^{\langle -i\rangle}$. Then $a^{\langle n\rangle}\circ c = a^{\langle n\rangle}\circ\tilde{c}^{\langle in+1\rangle} = \tilde{c}$.

Note that $\delta(a) = a$, $a \circ c = c \circ a$, and $(G,f) = \mathrm{der}^{n}_{\delta;c}(G,\circ)$. Thus for $i > 0$ we have

$$a^{[i]} = f_{(i)}(\overset{(in+1)}{a}) = a^{\langle in+1\rangle} \circ c^{\langle i\rangle} = a \circ \bar{c}^{\langle i\rangle} = e.$$

The case $i < 0$ is more complicated. Using Lemma 1 we obtain

$$a^{[i]} = f_{(-i)}(\overset{(-in+1+2i)}{a}, \overset{(-2i)}{\bar{a}}) = a^{\langle -in+1+2i\rangle} \circ \bar{a}^{\langle -2i\rangle} \circ c^{\langle -i\rangle} =$$

$$= a^{\langle -in+1+2i\rangle} \circ \bar{a}^{\langle -2i\rangle} \circ c^{\langle -2i\rangle} \circ c^{\langle i\rangle} =$$

$$= a^{\langle -in+1+2i\rangle} \circ a^{\langle -2i(n-1)\rangle} \circ c^{\langle i\rangle} =$$

$$= a^{\langle in+1\rangle} \circ c^{\langle i\rangle} = a \circ c^{\langle i\rangle} = e.$$

Therefore, $e = a^{[i]}$ for any integer i.

Now we pass to the proof of Theorem 1 for $n = 1$. By Proposition 7 of [15], $(G,f) = \mathrm{der}^{1}_{\gamma;b}(G,\cdot)$, where a is the neutral element of $(G,\cdot)$, and $\langle\gamma;b\rangle \in H(G,\cdot)$. Moreover,

$$(20) \qquad (G,\cdot) = \mathrm{Ret}^{1,2}_{d}(G,f),$$

$$(21) \qquad \gamma(x) = f_{(2)}(a,x,d),$$

$$(22) \qquad b = f(\overset{(2)}{a}),$$

where d is the inverse of a in (G,f). Define a system $\langle\delta;c\rangle$ as in Proposition 1, i.e., let δ and c be given by (1) and (2). Then $\langle\delta;c\rangle$ is already an L_1-system over $(G,\circ) = \mathrm{Ret}^{1,2}_{b^{-1}}(G,\cdot)$ and $(G,f) = \mathrm{der}^{1}_{\delta;c}(G,\circ)$. The rest of the proof is similar to that for the case $n > 1$ (now we use Lemma 4 of [16] and Lemma 2).

Both the conditions H and L_1 are $(n,1)$-nonrestrictive. However, the former has a certain property which does not belong to the latter. Namely, every element of an arbitrary $(n+1)$-group is an n-H-identity element, whereas one can easily give the examples of $(n+1)$-groups where there exist elements which are not n-L_i-identity elements.

Putting together Theorem 1 of [16] with the above Theorem 1 we get

C o r o l l a r y 8. An element e is an n-EL_i-identity element in an $(n+1)$-group (G,f) if and only if e is a central element of (G,f) and it is of the form $e = a^{[i]}$ for some $a \in G$.

Note that the condition PH is essentially stronger than the condition L_i for every integer i. Putting together Corollary 10 of [16] with Theorem 1 and taking into account the following Lemma 3 we can obtain a new criterion for a given $(n+1)$-group to be PH-derived from a group.

L e m m a 3. If an $(n+1)$-group (G,f) has an element a such that

$$a^{[in+1]} = a \tag{23}$$

for some integer i, then (G,f) has an idempotent element (this may be, e.g. $a^{[i]}$).

P r o p o s i t i o n 5. An $(n+1)$-group (G,f) is PH-derived from a group if and only if there exists an element $a \in G$ such that

$$a^{[u]} = a, \tag{24}$$

where $GCD(u,n) = 1$ (i.e., $lu = in+1$ for some integers l, i). Moreover, $(G,f) = der^n_\delta(G,\circ)$, where $\langle\delta\rangle \in PH(G,\circ)$ and e is the neutral element of $(G,\circ)$, if and only if

$$(G,\circ) = Ret^n_e(G,f), \tag{25}$$

$$\delta(x) = f(e,x,\overset{(n-1)}{e}), \tag{26}$$

$$e = a^{[i]}. \tag{27}$$

P r o o f . Let $a^{[u]} = a$ and $lu = in+1$. Then $a^{[in+1]} = a^{[lu]} = a$; hence, by Lemma 3, the element $e = a^{[i]}$ is idempotent in (G,f). By Corollary 10 of [16], $(G,f) = der^n_\delta(G,\circ)$, where the group $(G,\circ)$ and δ are given by (25) and (26), which completes the proof.

In a similar way, using Proposition 4 of [16] and Proposition 5 we get

T h e o r e m 2. An $(n+1)$-group (G,f) is PE-derived from a group if and only if there exists an element $a \in G$ such that for any $x \in G$ and some integer u with $GCD(u,n) = 1$ we have

$$(28) \qquad \langle \overset{(u)}{\bar{a}}, x\rangle \underset{f}{=} \langle x, \overset{(u)}{a}\rangle .$$

In this case, moreover, $(G,f) = der^n_e(G,\circ)$, where e is the neutral element of $(G,\circ)$, if and only if (25) and (27) hold (where $lu = in+1$ for some integers l, i).

P r o o f . Assume that (23) holds for some $a \in G$ and u with $GCD(u,n) = 1$ (i.e., $lu = in+1$). Equality (28) is obviously equivalent to

$$(29) \quad f_{(u+2|u|)}(\overset{((u+|u|)(n-1))}{a}, \overset{(|u|)}{\bar{a}}, x, \overset{(|u|\cdot(n-1)+u)}{a}, \overset{(|u|)}{\bar{a}}) = x.$$

Putting in (29) the element a in place of x we get (24). Then, by Proposition 5, $(G,f) = der^n_\delta(G,\circ)$, where $(G,\circ)$ and δ are described by (25), (26), and, furthermore, the element $e = a^{[i]}$ is an idempotent of (G,f). Hence, using (28) we have

$$\langle x,e\rangle \underset{f}{=} \langle \overset{(n)}{e}, x, e\rangle \underset{f}{=} \langle \overset{(lun)}{a}, x, \overset{(lu)}{a}\rangle \underset{f}{=}$$

$$\underset{f}{=} \langle \overset{(lun)}{a}, \overset{(lu)}{\bar{a}}, x\rangle \underset{f}{=} \langle \overset{(lu)}{a}, x\rangle \underset{f}{=} \langle e,x\rangle,$$

which proves that e is a central element of (G,f). Thus δ is an identity mapping of $(G,\circ)$. Consequently, (G,f) is PE-derived from $(G,\circ)$, and so the proof of Theorem 2 is complete.

Observe that for $u = 1$ Theorem 2 takes the form of Proposition 4 of [16], which is closely related to the well-known Dörnte criterion (cf. [3] and [18]).

5. L_∞-systems

In Section 3 we put the question about the existence of an $(n,1)$-nonrestrictive condition which would be stronger than L_i for all integers i. Now we can come back to that problem.

As was mentioned in Section 4, the condition L_i is essentially weaker than PH for every i. We are going to define a new condition, which turns out to be essentially stronger than all conditions L_i and essentially weaker than PH.

Definition 2. Let $\langle\delta;c\rangle \in H(G,\cdot)$. The system $\langle\delta;c\rangle$ is said to be an L_∞-system over $(G,\cdot)$ if for every integer i there exists an element $a \in G$ such that

1^0 $\delta(a) = a$,

2^0 $c = a^{[i]}$.

In other words, a system $\langle\delta;c\rangle$ is an L_∞-system over $(G,\cdot)$ if and only if $\langle\delta;c\rangle \in L_i(G,\cdot)$ for every i. From the definition of the condition L_∞ and by Theorem 1 we get immediately

Corollary 9. An element e is an n-L_∞-identity element in an $(n+1)$-group (G,f) if and only if for any integer i there exists an element $a \in G$ such that $e = a^{[i]}$.

Consider an infinite cyclic $(n+1)$-group (i.e., a free $(n+1)$-group generated by one element). This $(n+1)$-group (G,f) can be described as the set $nZ+1$ (i.e., the set of integers which equal 1 modulo n) with the usual addition (cf. [1], [12]). It is easy to check that for $n > 1$ (G,f) has no L_∞-identity element. So (G,f) is an n-L_∞-primitive $(n+1)$-group (for the definition of an n-C-primitive $(n+1)$-group see [15]). Thus we have

Corollary 10. The condition L_∞ is $(n,1)$-restrictive for $n > 1$.

Since the condition PE is $(1,1)$-nonrestrictive (cf. [15]), any weaker condition is also $(1,1)$-nonrestrictive (and so is L_∞). The case $n = 1$ is then very special. For this reason, further remarks will be done separately for $n > 1$ and $n = 1$.

Let $n > 1$. The condition L_∞ is essentially stronger than the conditions L_i (which are $(n,1)$-nonrestrictive). Therefore, there is no $(n,1)$-nonrestrictive condition stronger than all conditions L_i (observe that such a condition would be stronger than L_∞, but the latter is $(n,1)$-restrictive). On the other hand, it is evident that L_∞ is weaker than PH. As we now prove, this is essentially weaker. Indeed, consider a cyclic $(n+1)$-group of order n (cf. [18]). This $(n+1)$-group can be described as follows (cf. [9], [10], [11]). The $(n+1)$-group $\underline{C}_{n,n+1} = (Z_n,\varphi)$, where $Z_n = \{0,1,\ldots,n-1\}$, $\varphi(1_1^{n+1}) \equiv 1_1 + \ldots + 1_{n+1} + 1 \pmod n$, is a cyclic $(n+1)$-group generated, e.g., by 0. Observe that the equation $e = x^{[i]}$ has a solution for any $e \in Z_n$ and for any integer i. Hence, by Corollary 9 any element $e \in Z_n$ is an n-L_∞-identity element in $\underline{C}_{n,n+1}$, i.e., $\underline{C}_{n,n+1}$ is L_∞-derived from a group. Nevertheless, $\underline{C}_{n,n+1}$ has no idempotent element (cf. [18]), and so it is not PH-derived from a group (cf. Corollary 10 of [16]). Thus the condition PH is just enough stronger than L_∞ to be not even an improvement of the latter (the classes of PH- and L_∞-derived $(n+1)$-groups are different).

This is not the case for $n = 1$. The classes of groups PH- and L_∞-derived from groups are equal (both PH and L_∞ are $(1,1)$-nonrestrictive conditions). But now the condition PH is also essentially stronger than L_∞. Indeed, let $(G,\cdot)$ be the multiplicative group of positive real numbers and let $a \neq 1$ be an element of G. Then $\langle a\rangle$ is an L_∞-system over $(G,\cdot)$, whereas it is not a PH-system over $(G,\cdot)$.

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