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ON (m,n)-DISTRIBUTIVITY OF PSEUDO BOOLEAN ALGEBRAS

In this paper, it has been proved that in the case of a complete pseudo Boolean algebra, the idea of complete meet distributivity and join distributivity are equivalent. We show also that if a complete pseudo Boolean algebra A is (n^m,m)-meet distributive, then A is (m,n)-join distributive. This theorem is a generalization of the Tarski theorem for Boolean algebras (see [4]). The notation and terminology of this paper is consistent with [3] and [4], see also [1].

1. (m,n)-meet and (m,n)-join distributivity

Definition 1.1. A pseudo Boolean algebra A is said to be (m,n)-meet distributive iff for every (m,n)-indexed family $(a_{st})_{s\in S, t\in T}$ of elements of A, where $\bar{\bar{S}}\leq m$, $\bar{\bar{T}}\leq n$, such that:

(i) $\bigcup_{t\in T} a_{st}$ exists for all $s\in S$,

(ii) $\bigcap_{s\in S}\bigcup_{t\in T} a_{st}$ exists,

(iii) $\bigcap_{s\in S} a_{s\varphi(s)}$ exists for all $\varphi\in T^S$,

then for all $\varphi\in T^S$, $\bigcup_{\varphi\in T^S}\bigcap_{s\in S} a_{s\varphi(s)}$ also exists and we have

$$\bigcap_{s\in S}\bigcup_{t\in T} a_{st} = \bigcup_{\varphi\in T^S}\bigcap_{s\in S} a_{s\varphi(s)}.$$

A pseudo Boolean algebra A is said to be (m,n)-join distributive iff for every (m,n)-indexed family $(a_{st})_{s\in S, t\in T}$ of elements of A, where $\bar{\bar{S}} \leq m$, $\bar{\bar{T}} \leq n$, such that:

(i)′ $\bigcap_{t\in T} a_{st}$ exists for all $s \in S$,

(ii)′ $\bigcup_{s\in S} \bigcap_{t\in T} a_{st}$ exists,

(iii)′ $\bigcup_{s\in S} a_{s\varphi(s)}$ exists for all $\varphi \in T^S$,

then for all $\varphi \in T^S$, $\bigcap_{\varphi\in T^S} \bigcup_{s\in S} a_{s\varphi(s)}$ also exist and we have $\bigcup_{s\in S} \bigcap_{t\in T} a_{st} = \bigcap_{\varphi\in T^S} \bigcup_{s\in S} a_{s\varphi(s)}$.

Definition 1.2. A pseudo Boolean algebra A is said to be complete meet (join) distributive, if it is (m,n)-meet (join) distributive for arbitrary cardinal numbers m and n.

2. Complete distributive algebras (see [2]).

Definition 2.1. A pseudo Boolean algebra A is said to be complete distributive if it is complete meet and join distributive.

Theorem 2.2. If for arbitrary elements a,b of a pseudo Boolean algebra A, $a \leq b$ or $b \leq a$, then A is complete distributive.

Proof. We will prove that

$$\bigcap_{s\in S} \bigcup_{t\in T} a_{st} = \bigcup_{\varphi\in T^S} \bigcap_{s\in S} a_{s\varphi(s)}$$

for arbitrary cardinal numbers m and n. Let $(a_{st})_{s\in S, t\in T}$ be a set of elements of a chain such that $\bigcup_{t\in T} a_{st}$ exists for all $s \in S$, and also $\bigcap_{s\in S} \bigcup_{t\in T} a_{st} = b$ and $\bigcap_{s\in S} a_{s\varphi(s)}$ for all $\varphi \in T^S$. We will show that $\bigcup_{\varphi\in T^S} \bigcap_{s\in S} a_{s\varphi(s)} = b$.

Let s be given, we have $a_{s\varphi(s)} \leq \bigcup_{t\in T} a_{st}$, hence

$$\bigcap_{s\in S} a_{s\varphi(s)} \leq \bigcap_{s\in S} \bigcup_{t\in T} a_{s}.$$

Then for all $\varphi\in T^S$, $\bigcap_{s\in S} a_{s\varphi(s)} \leq b$. We have

(1) Let $\forall\varphi(a \geq \bigcap_{s\in S} a_{s\varphi(s)})$ and $b \nleq a$. Hence $a < b$,

$$a < \bigcap_{s\in S} \bigcup_{t\in T} a_{st} = b.$$

(2) For all $s\in S$, we have $a < \bigcup_{t\in T} a_{st}$.

(3) $\forall_{s\in S} \exists_{\varphi_a(s)}$ such that $a < a_{s\varphi_a(s)}$ let $\varphi_a(s) = t$.

Let s be given. If $\forall_t(a_{st} \leq a)$, then $\bigcup_{t\in T} a_{st} \leq a$. This is a contradiction with (2). Hence from (3) we get $a \leq \bigcap_{s\in S} a_{s\varphi_a(s)}$.

We also have

(4) $a = \bigcap_{s\in S} a_{s\varphi_a(s)}$.

(5) If $a \leq a' < b$, then $a = a'$.

Using (3) to a', we obtain $a' = \bigcap_{s\in S} a_{s\varphi_a(s)}$. From (1) $a \geq \bigcap_{s\in S} a_{s\varphi_a(s)} = a'$, hence $a = a'$.

Let s be given. We have $a < a_{s\varphi_a(s)}$, but from (5) we see that it is impossible to be $a < a_{s\varphi_a(s)} < b$ and so $b \leq a_{s\varphi_a(s)}$ for all $s\in S$. Thus $b \leq \bigcap_{s\in S} a_{s\varphi_a(s)} = a$, but this is a contradiction with the hypothesis that $a < b$.

By the same method we can prove that

$$\bigcup_{s\in S} \bigcap_{t\in S} a_{st} = \bigcap_{\varphi\in T^S} \bigcup_{s\in S} a_{s\varphi(s)}.$$

This ends the proof.

Theorem 2.3. For every complete pseudo Boolean algebra A, if A is a complete meet distributive, then A is a complete join distributive.

P r o o f . Let

$$\bigcap_{s\in S}\bigcup_{t\in T} a_{st} = \bigcup_{\varphi\in T^S}\bigcap_{s\in S} a_{s\varphi(s)}$$

for all S and T. We will prove that

$$\bigcup_{s\in S}\bigcap_{t\in T} a_{st} = \bigcap_{\varphi\in T^S}\bigcup_{s\in S} a_{s\varphi(s)}$$

for any cardinal numbers m and n, $\overline{\overline{S}} = m$, $\overline{\overline{T}} = n$.

For all $s \in S$ and $\varphi \in T^S$ we denote $a_{s\varphi(s)} = b_{\varphi s}$. We have

$$\bigcap_{\varphi\in T^S}\bigcup_{s\in S} a_{s\varphi(s)} = \bigcap_{\varphi\in T^S}\bigcup_{s\in S} b_{\varphi s} = \bigcup_{\psi\in S^{(T^S)}}\bigcap_{\varphi\in T^S} b_{\varphi\psi(\varphi)}.$$

Moreover

(∗) For all $\psi \in S^{(T^S)}$, the family $\{b_{\varphi\psi(\varphi)}\}$ contains a subfamily $\{a_{st}\}_{t\in T}$ for a certain $s \in S$.

Let for $s \in S$, ψ_s be a constant function from T^S in to S such that $\psi_s(\varphi) = s$ for all $\varphi \in T^S$.

P r o o f of (∗): Assume that it is not true. This means that there exists $\varphi_0 \in T^S$ and $\psi_0 \in S^{(T^S)}$, such that $a_{s\varphi_0(s)} \notin \{b_{\varphi\psi_0(\varphi)}\}_{\varphi\in T^S}$ for all $s \in S$. Take $s_0 = \psi_0(\varphi_0)$ then $a_{s_0\varphi_0(s_0)} = b_{\varphi_0 s_0\psi(\varphi_0)}$, this is a contradiction. Next we continue the proof of Theorem 2.3. We have

$$(1) \qquad \{b_{\varphi\psi_s(\varphi)}\}_{\varphi\in T^S} = \{b_{\varphi s}\} = \{a_{s\varphi(s)}\} = \{a_{st}\}_{t\in T},$$

$$\forall_{\psi\in S^{(T^S)}}\ \exists_{s\in S}\ \{b_{\varphi\psi(\varphi)}\}_{\varphi\in T^S} \supset \{a_{st}\}_{t\in T}.$$

Then

$$\forall_{\psi\in S^{(T^S)}}\ \exists_{s\in S}\ \bigcap_{\varphi\in T^S} b_{\varphi\psi(\varphi)} \leq \bigcap_{t\in T} a_{st}.$$

Thus

(2) $$\bigcup_{\psi\in S^{(T^S)}} \bigcap_{\varphi\in T^S} b_{\varphi\psi(\varphi)} \leqslant \bigcup_{s\in S} \bigcap_{t\in T} a_{st}$$

by (1) we obtain

$$\forall_{s\in S} \exists_{\psi_s} \{b_{\varphi\psi_s(\varphi)}\}_{\varphi\in T^S} = \{a_{st}\}_{t\in T} \bigcap_{t\in T} a_{st} \leqslant \bigcup_{\psi\in S^{(T^S)}} \bigcap_{\varphi\in T^S} b_{\varphi\psi(\varphi)}$$

from here we have

(3) $$\bigcup_{s\in S} \bigcap_{t\in T} a_{st} \leqslant \bigcup_{\psi\in S^{(T^S)}} \bigcap_{\varphi\in T^S} b_{\varphi\psi(\varphi)}.$$

From (2) and (3) we obtain

$$\bigcup_{s\in S} \bigcap_{t\in T} a_{st} = \bigcup_{\psi\in S^{(T^S)}} \bigcap_{\varphi\in T^S} b_{\varphi\psi(\varphi)}$$

thus

$$\bigcap_{\varphi\in T^S} \bigcup_{s\in S} a_{s\varphi(s)} = \bigcup_{s\in S} \bigcap_{t\in T} a_{st}.$$

T h e o r e m 2.4. If a complete pseudo Boolean algebra A is (n^m,m)-meet distributive, then A is (m,n)-join distributive, where m and n are infinite cardinal numbers.

P r o o f . Let A be (n^m,m)-meet distributive and complete. Since A is (n^m,m)-meet distributive, if $(a_{st})_{s\in S,\, t\in T}$ is a family of element of A which satisfies the conditions:

(i) $\bar{\bar{S}} \leqslant n^m$, $\bar{\bar{T}} \leqslant m$,

(ii) $\bigcup_{t\in T} a_{st}$ exists for all $s\in S$, and $\bigcap_{s\in S} \bigcup_{t\in T} a_{st}$ also exists,

(iii) $\bigcap_{s\in S} a_{s\varphi(s)}$ exists for all $\varphi\in T^S$,

then $\bigcup_{\varphi\in T^S} \bigcap_{s\in S} a_{s\varphi(s)}$ also exists and

$$\bigcap_{s\in S} \bigcup_{t\in T} a_{st} = \bigcup_{\varphi\in T^S} \bigcap_{s\in S} a_{s\varphi(s)}.$$

Suppose that $\overline{\overline{U}} \leq m$, $\overline{\overline{V}} \leq n$. We get from the completeness of A that if $(a_i)_{i\in I} \subseteq A$ then $\bigcap_{i\in I} a_i$ and $\bigcup_{i\in I} a_i$ exists for all I.

Let $(b_{uv})_{u\in U, v\in V} \subseteq A$, we will prove that

(**) $$\bigcup_{u\in U} \bigcap_{u\in V} b_{uv} = \bigcap_{f\in V^u} \bigcup_{u\in U} b_{uf(u)}.$$

Let

(1) $$a_{fu} = b_{uf(u)}$$

then

(2) $$\bigcap_{f\in V^U} \bigcup_{u\in U} b_{uf(u)} = \bigcap_{f\in V^U} \bigcup_{u\in U} a_{fu} = \bigcup_{\varphi\in U^{(V^U)}} \bigcap_{f\in F^U} a_{f\varphi(f)}$$

for all $u \in U$. Define the mapping $\varphi_u : V^U \longrightarrow U$ as follows: $\varphi_u(f) = u$ for all $f \in V^U$. We have

$$\forall_{\varphi' \in U^{(V^U)}} \left[\bigcap_{f\in V^U} a_{f\varphi'(f)} \leq \bigcup_{\varphi\in U^{(V^U)}} \bigcap_{f\in V^U} a_{f\varphi(f)} \right],$$

therefore from the definition of φ_u

$$\forall_{u\in U} \left[\bigcap_{f\in V^U} a_{f\varphi_u(f)} \leq \bigcup_{\varphi\in U^{(V^U)}} \bigcap_{f\in V^u} a_{f\varphi(f)} \right].$$

Then

(3) $$\bigcup_{u\in U} \bigcap_{f\in V^U} a_{f\varphi_u(f)} \leq \bigcup_{\varphi\in U^{(V^U)}} \bigcap_{v\in V^u} a_{f\varphi(f)}.$$

Since for all $u \in U$

$$(b_{uf(u)})_{f \in V^U} = (b_{uv})_{v \in V},$$

hence from (1)

$$(a_{f\varphi_u(f)})_{f \in V^U} = (b_{uv})_{v \in V} \quad \text{for all} \quad u \in U.$$

Then we have

$$(4) \qquad \bigcap_{f \in V^U} a_{f\varphi_u(f)} = \bigcap_{v \in V} b_{uv} \quad \text{for all} \quad u \in U.$$

From (3) and (4), we have

$$(5) \qquad \bigcup_{u \in U} \bigcap_{v \in V} b_{uv} \leq \bigcup_{\varphi \in U^V} \bigcap_{f \in V^U} a_{f\varphi(f)}.$$

We will prove that

$$(6) \qquad \bigcup_{\varphi \in U^{(V^U)}} \bigcap_{f \in V^U} a_{f\varphi(f)} \leq \bigcup_{u \in U} \bigcap_{v \in V} b_{uv}.$$

First we prove that

$$(7) \qquad \forall_{\varphi \in U^{(V^U)}} \exists_{u \in U} \forall_{v \in V} \exists_{f \in V^U} \left[u = \varphi(f) \text{ and } v = f(\varphi(f)) \right].$$

Let

$$\exists_{\varphi \in U^{(V^U)}} \forall_{u \in U} \exists_{v \in V} \forall_{f \in V^U} \left[u \neq \varphi(f) \text{ or } v \neq f(\varphi(f)) \right].$$

From this for a certain $\varphi_0 \in U^{(V^U)}$, we have

$$\forall_{u \in U} \exists_{v \in V} \forall_{f \in V^U} \left[u \neq \varphi_0(f) \text{ or } v \neq f(\varphi_0(f)) \right].$$

From this we get that there exists a function $f_0 : U \to V$, such that

$$\forall_{f\in V^U}\left[u \neq \varphi_0(f) \text{ or } f_0(u) \neq f(\varphi_0(f))\right].$$

Then $u \neq \varphi_0(f_0)$ or $f_0(u) \neq f_0(\varphi_0(f_0))$ for all $u \in U$. Hence for $u = \varphi_0(f_0)$ we have a contradiction

$$f_0(\varphi_0(f_0)) \neq f_0(\varphi_0(f_0)).$$

From (7) we get

$$\forall_{\varphi\in U^{(V^U)}} \exists_{u\in U} (b_{uv})_{v\in V} \subset (b_{\varphi(f)f(\varphi(f))})_{f\in V^U}.$$

Hence from (1)

$$\forall_{\varphi\in U^{(V^U)}} \exists_{u\in U} (b_{uv})_{v\in V} \subset (a_{f\varphi(f)})_{f\in V^U},$$

then

$$\forall_{\varphi\in U^{(V^U)}} \exists_{u\in U} \left[\bigcap_{f\in V^U} a_{f\varphi(f)} \leq \bigcap_{v\in V} b_{uv}\right].$$

Therefore

$$\forall_{\varphi\in U^{(V^U)}} \left[\bigcap_{v\in V^U} a_{f\varphi(f)} \leq \bigcup_{u\in U} \bigcap_{v\in V} b_{uv}\right]$$

and

$$\bigcup_{\varphi\in U^{(V^U)}} \bigcap_{v\in V^U} a_{f\varphi(f)} \leq \bigcup_{u\in U} \bigcap_{v\in V} b_{uv}.$$

From (6) and (5) using (2) we obtain (**).

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