DEMONSTRATO MATHEMATICA
Vol. XX No 1-2 1987

Radosław Godowski

PARTIAL GREECHIE DIAGRAMS FOR MODULAR ORTHOLATTICES

1. Introduction

We shall develope a new diagram technique which permits an easy reference to certain kinds of modular ortholattices. Those diagrams are very helpful in modular ortholattice theory because order-graph presentation as well as Greechie diagrams tends to get rather complicated already for relatively simply algebras. The idea of this new technique is more or less implicitely included in G.Bruns's paper [1].

In this paper such new diagrams are introduced under the name "partial Greechie diagrams" (abbreviated pGd). Using these diagrams we shows some fact from orthomodular lattice theory, namely we prove that the equality $TSFSS \wedge MOL = [MO\omega]$ holds in the lattice of all subvarieties of the variety of orthomodular lattices.

2. Basic definitions and properties

As in [3], an orthomodular lattice is considered as an universal algebra $(L;\vee,\wedge,',0,1)$ such that a reduct $(L;\vee,\wedge,0,1)$ is a bounded lattice and the following hold:

(om1) $(a')' = a$

(om2) $(a\vee b)' = a'\wedge b'$

(om3) $a\vee a' = 1$

(om4) $a\vee b = a\vee[a'\wedge(a\vee b)]$

We write $a\perp b$ if $a\leq b'$; $a\not\perp b$ if not $a\perp b$.

A modular ortholattice is an orthomodular lattice which is modular. The variety of all modular ortholattices is denoted by MOL. A commutator of elements $a,b\in L$ is defined as follows:

$$C(a,b) =: (a \vee b) \wedge (a \vee b') \wedge (a' \vee b) \wedge (a' \vee b').$$

By $MO\omega$ we denote a modular ortholattice of length 2 consisting 0;1 and a countable set of atoms $\{a_1, a_1', a_2, a_2', \dots\}$. The variety $[MO\omega]$ generated by $MO\omega$ can be characterized by the equations characterizing MOL and the equation $C(x, C(y,z)) = 0$.

A Greechie diagram associated with an orthomodular lattice L of length 3 consist of a set P of points and a set Q of lines, where $Q \subseteq \mathcal{P}(P)$. The points are in one-to-one correspondence with the atoms of L; the lines are in one-to-one correspondence with the blocks (maximal Boolean subalgebras) of L such that every line $l = \{p_t : t \in T\}$ correspondes to a block generated by the set of atoms corresponding to the set of points $\{p_t : t \in T\}$ (for details see [3]).

An example of Greechie diagram is presented on Figure 1, left side. This is the diagram G of the product $MO2 \times 2$, where MO2 is the six-element modular ortholattice and 2 is a two-element Boolean algebra. The Hasse diagram of this algebra is presented on the right.

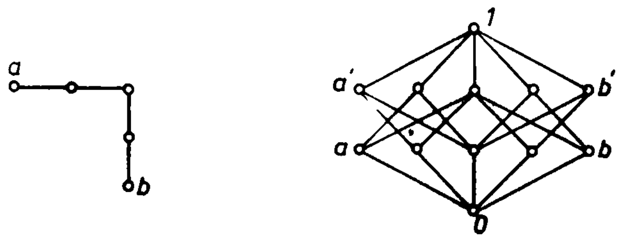

Figure 1

Observe that we can describe every point of G. This description is given in Figure 2.

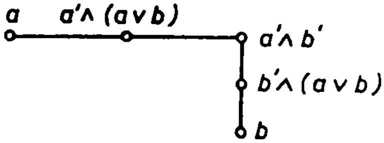

Figure 2

A state on a modular ortholattice L is a real map $m: L \to \langle 0;1 \rangle$ such that the following conditions hold:

(s1) $m(1) = 1$

(s2) if $a \perp b$ then $m(a \vee b) = m(a) + m(b)$

A state is two-valued if $m(L) = \{0;1\}$. A set $\{m_t : t \in T\}$ of two-valued states on L is said to be full if for any $a, b \in L$:

$$a \not\leq b \Longrightarrow \exists t \in T \quad m_t(a) = m_t(b) = 1.$$

By TSFSS we denote the class of orthomodular lattices with a full set of two-valued states. As it is shown in [2] the class TSFSS forms a variety.

3. Partial Greechie diagrams

D e f i n i t i o n 1. A partial Greechie diagram (abbreviated pGd) G for a modular ortholattice L consist of a set P of points and a set Q of lines such that the following conditions hold:

(pG1) Every line is a three-element subset of P.

(pG2) Every point lies on (belongs to) some line.

(pG3) Every two different lines are either disjoint or neighbouring, i.e. having an one-point intersection.

(pG4) Triangles and quadrangles do not occur in G.

(pG5) The points of G are in one-to-one correspondence with some non-zero elements of L. Moreover, every two neighbouring lines form a Greechie diagrams for subalgebras of L.

Observe that the conditions (pG1)-(pG4) mean that G is a Greechie diagram for some orthomodular (in general not modular) lattice of length 3.

E x a m p l e 1. The diagram G_3 presented on Figure 3 is a pGd for a modular ortholattice L.

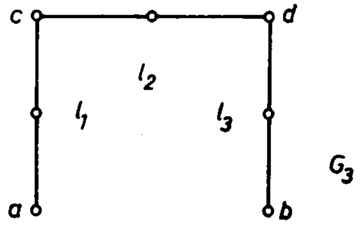

Figure 3

If consist of a seven-element set of points and a three-element set of lines $\{l_1, l_2, l_3\}$. It follows from the axiom (pG5) that the L_{12} and L_{23} presented on its Hasse diagrams in Figure 4 are subalgebras of L.

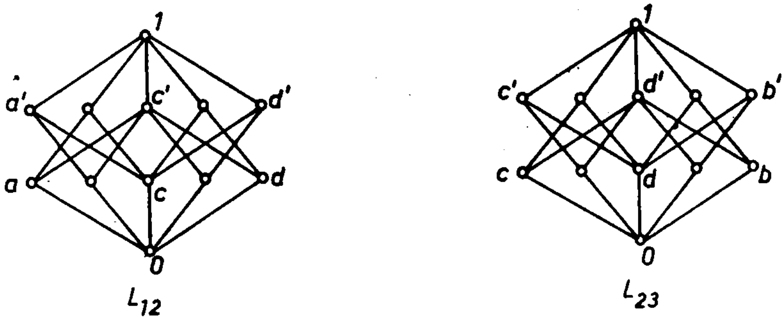

Figure 4

Observe that $C(a,C(b,c)) = C(a,b \vee c) = C(a,d') = a \vee d \neq 0$. Thus $L \notin [MO\omega]$. As a Corollary of G.Bruns's result in [1] we obtain the following:

P r o p o s i t i o n . If L is a modular ortholattice not contained in $[MO\omega]$ then G_3 is a pGd for L.

4. The pentagon Theorem

In this chapter we prove that if L is a modular ortholattice, $L \notin [MO\omega]$, then pentagon is a pGd for L.

P r o p o s i t i o n 1. Let G_3 (Figure 3) be a pGd for a modular ortholattice L. Then $a' \wedge b' \neq 0$ and G_4 (Figure 5) is a pGd for L too.

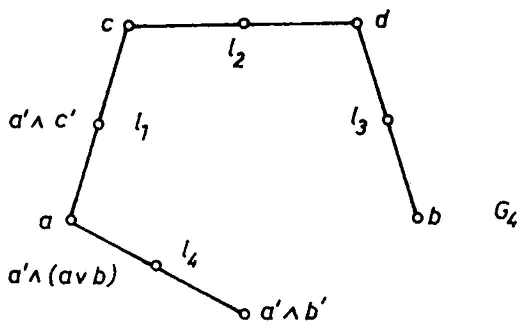

Figure 5

P r o o f . First we prove that $a' \wedge b' \neq 0$. Observe that $c \vee b' = 1$ and $c < a'$ (Figure 4). Now from the modularity law: $c \vee (b' \wedge a') = (c \vee b') \wedge a'$. Hence $c \vee (b' \wedge a') = a'$. Thus $c \vee (b' \wedge a') \neq c$. Therefore $a' \wedge b' \neq 0$. Now we have to prove that G_4 is a pGd for L. Because $c \vee [(a \vee b) \wedge a'] = [c \vee (a \vee b)] \wedge a' = a'$ then $a' \wedge (a \vee b) \neq 0$ and it is enough to prove that $\{l_1, l_4\}$ forms a Greechie diagram for some subalgebra of L, i.e. that the following hold:

1) $c \vee [a' \wedge (a \vee b)] = c \vee (a' \wedge b') = (a' \wedge c') \vee [a' \wedge (a \vee b)] =$
$= (a' \wedge c') \vee (a' \wedge b') = a'$

2) $c \wedge (a \vee b) = (a' \wedge c') \wedge (a \vee b) = c \wedge [a \vee (a' \wedge b')] =$
$= (a' \wedge c') \wedge [a \vee (a' \wedge b')] = 0$

3) $(a' \wedge b') \wedge c' = [a' \wedge (a \vee b)] \wedge c' = (a' \wedge b') \wedge (a \vee c) =$
$= [a' \wedge (a \vee b)] \wedge (a \vee c) = 0.$

The proof of any above identity is not difficult. We shows for example that (1a) and (1b) hold.

Because $c < a'$ than from modularity $c \vee [a' \wedge (a \vee b)] = [c \vee (a \vee b)] \wedge a'$. But (Figure 4) $a \vee b \vee c = a \vee d' = 1$. Hence $[c \vee (a \vee b)] \wedge a' = 1 \wedge a' = a'$ and (1a) holds. Now $c \wedge (a \vee b) = (c \wedge d') \wedge (a \vee b) = c \wedge [d' \wedge (a \vee b)]$. From the modularity law $d' \wedge (a \vee b) = (d' \wedge a) \vee b$. But $d' \wedge a = 0$ (Figure 4). Thus $d' \wedge (a \vee b) = b$. Hence $c \wedge [d' \wedge (a \vee b)] = c \wedge b = 0$.

T h e o r e m 1. Let G_3 (Figure 3) be a pGd for a modular ortholattice L. Then G_5 (Figure 5) is a partial Greechie diagram for L too.

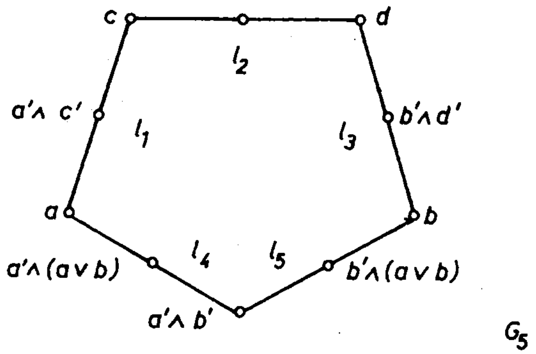

Figure 6

P r o o f . It follows from Proposition 1 that $\{l_1, l_4\}$ and, by symmetry, $\{l_3, l_5\}$ form a Greechie diagram for subalgebras of L. Now we applicate Proposition 1 to diagram $\{l_2, l_3, l_5\}$. Because $b = d' \wedge (a \vee b)$ then $\{l_4, l_5\}$ forms a Greechie diagram for a subalgebra of L and the proof is complete.

5. States on modular ortholattices

It is well known that every modular ortholattice L from the variety $[MO\omega]$ has a full set of two-valued states. Now we prove the converse.

T h e o r e m 2. Let L be a modular ortholattice, $L \notin [MO\omega]$. Then L has no full set of two-valued states.

P r o o f . Let G_5 (Figure 6) be a pGd for L. It follows from the previous chapter that $(a \vee c) \wedge (b \vee d) \neq 0$ and G_6 (Figure 7) is a pGd for L.

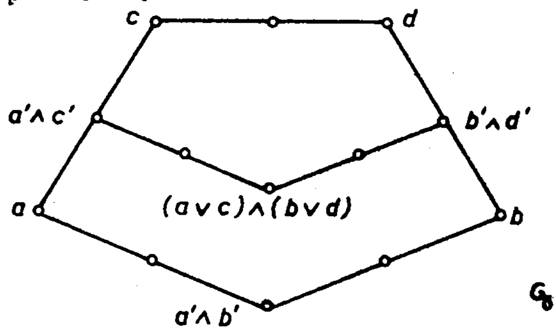

Figure 7

In particulary $a' \wedge b' \not\leq (a \vee c) \wedge (b \vee d)$. Let m be a two-valued state on L such that $m((a \vee c) \wedge (b \vee d)) = 1$. Then $m(a' \wedge c') = m(b' \wedge d') = 0$. If $m(a) = 1$, then $m(a' \wedge b') = 0$. Otherwise $m(a) = m(a' \wedge c') = 0$ and hence $m(c) = 1$. Thus $m(d) = 0$ and hence $m(b) = 1$. Therefore $m(a' \wedge b') = 0$.

Corollary. In the lattice of all subvarieties of the variety of crthomodular lattices the following holds:

$$\mathrm{TSFSS} \wedge \mathrm{MOL} = [\mathrm{MO}\omega].$$

REFERENCES

[1] G. Bruns: Varieties of modular ortholattices. Houston J. Math. 9 (1983) 1-7.

[2] R. Godowski: Varieties of orthomodular lattices with a strongly full set of states, Demonstratio Math. 14 (1981) 725-733.

[3] G. Kalmbach: Orthomodular lattices, Academic Press, London, 1983.

INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF WARSAW, 00-661 WARSZAWA

Received September 30, 1986.