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ON SOME GROUPOID MODES

In [P1] J.Płonka has shown that there are exactly four types of algebras having exactly n n-ary operations depending on all its variables for each $n = 1,2,\ldots$ One of these types form so-called 2-cyclic groupoids satisfying the following axioms

(L) $(x\cdot y)\cdot z = (x\cdot z)\cdot y$ (Left normal law),

(I) $x\cdot x = x$ (Idempotent law),

(R) $x\cdot(y\cdot z) = x\cdot y$ (Reduction law),

(C_2) $(x\cdot y)\cdot y = x$ (2-cyclic law).

More general, k-cyclic groupoids satisfy the axioms (L), (I), (R) and

(C_k) $(\ldots((x\cdot y)\cdot y)\ldots)\cdot y = x$ (k-cyclic law).

K-cyclic groupoids were investigated in [P2].

In this paper we study more general LIR-groupoids that satisfy only the axioms (L), (I) and (R). In Section 1 we show that they are modes i.e. they are idempotent and entropic as defined in [RS]. So our investigations belong to the recently quickly developed theory of groupoids modes. (See e.g. [JK] and references there, [LM], [P1], [P2], [R] and [RS]). In Section 2 we prove that a groupoid is a LIR-groupoid if and only if it can be constructed by means of a construction we describe there, that generalizes a similar construction given by Płonka [P1] for k-cyclic groupoids. Section 3 is devoted to free LIR-groupoids and Section 4 to properties of identities satisfied in LIR-groupoids. Finally, in Section 4 we describe the lattice of all varieties of LIR-groupoids.

1. Preliminaries

In the sequel we write xy for $x\cdot y$ and xy^i for $(\ldots((x\cdot y)\cdot y)\ldots)\cdot y$ where y is repeated i times. Sometimes we write also xy^0 for x.

The variety of LIR-groupoids is denoted by LIR. The subvariety of LIR that satisfies the identity

$$(g_{i,i+p}) \qquad xy^i = xy^{i+p},$$

where $i = 0,1,\ldots$ and $p = 1,2,\ldots$ is denoted by $\underline{G}_{i,i+p}$. For basic algebraic notions we refer the reader to [C] and [RS]. The notation is similar to this used in these books.

Recall that a groupoid mode is a groupoid satisfying the idempotent law and

$$(E) \qquad (xy)(zt) = (xz)(yt) \qquad \text{(Entropic law)}.$$

(See [RS] and [JK] and references there. In [JK] groupoid modes are called idempotent medial groupoids). Note that together the idempotent and entropic laws imply distributive laws

$$(D_r) \qquad (xy)z = (xz)(yz),$$

$$(D_l) \qquad x(yz) = (xy)(xz).$$

In [R] groupoids modes were investigated that additionally satisfy the symmetric law

$$(S) \qquad x\cdot xy = y.$$

(See [RS] and [LM] as well). Instead of the symmetric law the following identity plays an important role in the theory of LIR-groupoids

$$(A) \qquad x\cdot xy = x.$$

1.1. P r o p o s i t i o n . The following sets of identities are equivalent

(i) (L), (I) and (R),
(ii) (E), (I) and (R),
(iii) (E), (I) and (A).

P r o o f . (i) $\Rightarrow$ (ii). It follows by (L) and (R) that

$$xy\cdot zt = xy\cdot z = xz\cdot y = xz\cdot yt,$$

whence the entropic law holds.

(ii) $\Rightarrow$ (iii). It follows by (R) and (I) that

$$x\cdot xy = x\cdot x = x,$$

whence (A) holds.

(iii) $\Rightarrow$ (i). It follows by (A) and (E) that

$$x\cdot yz = (x\cdot xy)(yz) = (xy)(xy\cdot z) = xy,$$

whence (R) holds. Now by the distributive law and (R)

$$xy\cdot z = xz\cdot yz = xz\cdot y,$$

whence the left normal law is satisfied.

By Proposition 1.1, each LIR-groupoid is a mode.

2. The structure of LIR-groupoids

2.1. D e f i n i t i o n . Let I be a non-empty set and for each i in I, let a non-empty set G_i be given. For each pair (i,j) in I^2, let $h_{i,j}:G_i \to G_i$ be a mapping satisfying

(i) $h_{i,i}$ is the identity mapping on G_i,

(ii) $h_{i,j}h_{i,k} = h_{i,k}h_{i,j}$.

Define a groupoid structure on the disjoint union G of G_i, i in I, by

(iii) $a_i\cdot a_j := a_ih_{i,j}$,

where a_i is in G_i and a_j is in G_j. Then evidently, each G_i is a subgroupoid of $(G,\cdot)$ and is a left-zero band. Moreover, the mapping $f:G \to I$, $a_i \mapsto i$ is a homomorphism and $f(G) = I$ is a left-zero band. The groupoid $(G,\cdot)$ is said to be the sum of left-zero bands $(G_i,\cdot)$ over the left-zero band $(I,\cdot)$ by the mappings $h_{i,j}$, or more briefly LZ-LZ-sum of $(G_i,\cdot)$.

A similar construction, though formulated in a different language, was considered in [P1], [P2]. It was show there that each k-cyclic groupoid is an LZ-LZ-sum.

Let us note that by the definition above each LZ-LZ-sum belongs to the Mal'cev product $LZ \circ LZ$ (see [M]), where LZ is the variety of left-zero bands. Now we will show that the class of all such sums is just the variety of LIR-groupoids.

2.2. Theorem. A groupoid $(G,\cdot)$ is a LIR-groupoid if and only if it is a sum of left-zero bands $(G_i,\cdot)$ over a left-zero band $(I,\cdot)$ by some mappings $h_{i,j}$.

Proof. ($\Leftarrow$) It follows by 2.1(i) and (iii) that the idempotent law is satisfied in the LZ-LZ-sum $(G,\cdot)$ of $(G_i,\cdot)$. Now let a_i be in G_i, a_j in G_j and a_k in G_k. Then $a_i\cdot b_jc_k = a_i\cdot b_jh_{j,k} = a_ih_{i,j} = a_ib_j$ implying that the reduction law is satisfied in $(G,\cdot)$. Since by 2.1, $a_ib_j\cdot c_k =$ $= a_ih_{i,j}\cdot c_k = a_ih_{i,j}h_{i,k} = a_ih_{i,k}h_{i,j} = a_ih_{i,k}\cdot b_j = a_ic_k\cdot b_j$ it follows that the left normal law holds in $(G,\cdot)$.

($\Rightarrow$) Let us define a relation R in $(G,\cdot)$ by the formula $a\,R\,b$ if and only if $xa = xb$ for each x in G. Obviously R is an equivalence relation.

Now let us denote G/R by I and let $\{G_i \mid i \in I\}$ be the set of all R-classes of $(G,\cdot)$. Let a_i be in G_i and b_j and b'_j in G_j. Then obviously $a_ib_j = a_ib'_j$. Define $h_{i,j}: G_i \to G_i$, $a_ih_{i,j} := a_ib_j$ for $i \neq j$ and $h_{i,i}: G_i \to G_i$, $a_ih_{i,i} = a_i$. Then the left normal law implies that $a_ih_{i,j}h_{i,k} = a_ib_j\cdot c_k =$ $= a_ic_k\cdot b_j = a_ih_{i,k}h_{i,j}$, where c_k is in G_k.

It follows that the groupoid $(G,\cdot)$ is the LZ-LZ-sum of subgroupoids $(G_i,\cdot)$ over the left-zero band $(I,\cdot)$ by the mappings $h_{i,j}$. □

Remark. The Theorem 2.2 can be generalised to obtain a characterisation of all groupoids satisfying the identities (I) and (R). Let us drop the condition (ii) in the Definition 2.1 of LZ-LZ-sum. Let us call the sum obtained in this way a general LZ-LZ-sum. Then the following may be proved in the similar way as Theorem 2.2.

2.2′. Theorem. A groupoid $(G,\cdot)$ is an IR-groupoid if and only if it is a general LZ-LZ-sum. □

Note that the variety of IR-groupoids may be equivalently characterised as the class of groupoids $(G,\cdot)$ in $LZ \circ LZ$ having the following property: if θ is a congruence relation on $(G,\cdot)$ such that $(G,\cdot)/\theta$ and all θ-classes are in LZ, and if for a, b, c in G, $b\,\theta\,c$, then $a\cdot b = a\cdot c$.

3. Free LIR-groupoids

3.1. Lemma. The identity

$$x_1 x_2^{j_2} \dots x_r^{j_r} = x_1 x_{2\pi}^{j_{2\pi}} \dots x_{r\pi}^{j_{r\pi}}$$

holds in LIR for each permutation π of the set $\{2,\dots,r\}$.

Proof. It follows by (L). □

3.2. Theorem. In the free LIR-groupoid $F(X)$ on a set X each element may be expressed in the standard form

$$x_1 x_2^{k_2} \dots x_s^{k_s}$$

where x_j is in X for each $j = 1,\dots,s$ and $x_i \neq x_j$ for $i \neq j$.

Proof. Let $w(x_1,\dots,x_s)$ be a groupoid word with variables in the set X. The proof goes by induction on s. If $s = 2$, then $w(x_1,x_2)$ equals x_1x_2 or x_2x_1 and it is already in the standard form. Now suppose the theorem holds for all $s < n$ and consider a word $w(x_1,\dots,x_n)$. Then

$$w(x_1,\dots,x_n) = w_1(y_1,\dots,y_p)w_2(z_1,\dots,z_q)$$

where both p and q are less than n and $\{y_1,\dots,y_p\}$, $\{z_1,\dots,z_q\} \subseteq \{x_1,\dots,x_n\}$. By induction we may assume that $w_1(y_1,\dots,y_p) = y_1 y_2^{i_2} \dots y_p^{i_p}$ and $w_1(z_1,\dots,z_q) = z_1 z_2^{j_2} \dots z_q^{j_q}$. Hence by (R)

$$w(x_1,\dots,x_n) = (y_1 y_2^{i_2} \dots y_p^{i_p})(z_1 z_2^{j_2} \dots z_q^{j_q}) = (y_1 y_2^{i_2} \dots y_p^{i_p})z_1.$$

In the case $z_1 = y_m$ for some $m = 1,\ldots,p$, (L) implies that

$$w(x_1,\ldots,x_n) = y_1 y_2^{i_2}\ldots y_m^{i_m+1}\ldots y_p^{i_p}.$$

It follows that $w(x_1,\ldots,x_n)$ m y be expressed in the standard form. □

3.3. Corollary. In the free LIR-groupoid $F(x_1,\ldots,x_n)$ on the generators $x_1,\ldots,x_n$ each further element may be expressed in the standard form

$$x_{i_1} x_{i_2}^{k_2}\ldots x_{i_s}^{k_s},$$

where x_{i_j} is in $\{x_1,\ldots,x_n\}$ for $j = 1,\ldots,s$, $x_{i_p} \neq x_{i_q}$ for $p \neq q$ and $i_2 < \ldots < i_s$. □

3.4. Corollary. In the free $\underline{G}_{i,j}$-groupoid $F_{i,j}(x_1,\ldots,x_n)$ on the generators $x_1,\ldots,x_n$ each further element may be expressed in the standard form

$$x_{i_1} x_{i_2}^{k_2}\ldots x_{i_s}^{k_s}$$

where x_{i_p} is in $\{x_1,\ldots,x_n\}$ for $p = 1,\ldots,s$, $x_{i_p} \neq x_{i_q}$ for $p \neq q$, $i_1 < \ldots < i_s$ and $k_p < j$ for all $p = 2,\ldots,s$. □

3.5. Corollary. Let $A_{i,j}(n)$ be the set of the words $x_p x_1^{k_1}\ldots x_{p-1}^{k_{p-1}} x_{p+1}^{k_{p+1}}\ldots x_n^{k_n}$, where $p = 1,\ldots,n$, $0 \leqq k_s < j$. Let us denote such word by $x_p x_1^{k_1}\ldots \hat{x}_p\ldots x_n^{k_n}$ and define

$$x_p x_1^{k_1}\ldots \hat{x}_p\ldots x_n^{k_n} \cdot x_r x_1^{l_1}\ldots \hat{x}_r\ldots x_n^{l_n} :=$$

$$\begin{cases} x_p x_1^{k_1}\ldots \hat{x}_p\ldots x_n^{k_n} & \text{if } p = r \\ x_p x_1^{k_1}\ldots \hat{x}_p\ldots x_r^{k_r+1}\ldots x_n^{k_n} & \text{if } k_r+1 < j \text{ and } r \neq p \\ x_p x_1^{k_1}\ldots \hat{x}_p\ldots x_r^{i}\ldots x_n^{k_n} & \text{if } k_r+1 = j \text{ and } r \neq p. \end{cases}$$

Then $(A_{i,j}(n),\cdot)$ is the free $\underline{G}_{i,j}$-groupoid on the generators $x_1,\dots,x_n$. □

4. Identities in LIR-groupoids

4.1. Lemma. If the identity

$$x_1 x_2^{j_2}\dots x_r^{j_r} = y_1 y_2^{k_2}\dots y_s^{k_s} \tag{4.2}$$

where $x_1 \neq y_1$ holds in an LIR-groupoid $(G,\cdot)$, then $(G,\cdot)$ is trivial.

Proof. Multiplying the identity (4.2) on the left by x_1 and using (R) one gets $x_1 = x_1 y_1$ what together with (4.2) implies that $x_1 = y_1$. □

Let GCD(i,j) denote the greatest common divisor of the natural numbers i and j.

4.3. Lemma. The following hold in the variety LIR.

(i) The identity

$$x\, x_1^{j_1}\dots x_m^{j_m} y_{m+1}^{j_{m+1}}\dots y_r^{j_r} = x\, x_1^{j_1}\dots x_m^{j_m} z_{m+1}^{k_{m+1}}\dots z_s^{k_s} \tag{4.4}$$

where the set $\{x_1,\dots,x_m\}$ may be empty, is equivalent to the identity

$$xy^k = x$$

where $k = \mathrm{GCD}(j_{m+1},\dots,j_r,k_{m+1},\dots,k_s)$.

(ii) The identity

$$x\, x_1^{j_1}\dots x_r^{j_r} = x\, x_1^{j_1}\dots x_r^{j_r} y_{r+1}^{j_{r+1}}\dots y_s^{j_s} \tag{4.5}$$

is equivalent to the identity

$$xy^k = x$$

where $k = \mathrm{GCD}(j_{r+1},\dots,j_s)$.

(iii) The identity

$$x\, x_1^{j_1}\dots x_r^{j_r} = x\, x_1^{k_1}\dots x_r^{k_r} \tag{4.6}$$

is equivalent to the set of identities

$$x\, x_p^{j_p} = x\, x_p^{k_p}$$

where $p = 1,\dots,r$.

P r o o f . (i) Substituting x for all variables different from y_p in (4.4) one gets

$$xy_p^{j_p} = x \tag{4.7}$$

for all $p = m+1,\dots,r$. In a similar way (4.4) implies

$$xz_q^{k_q} = x \tag{4.8}$$

for all $q = m+1,\dots,s$. It is easy to see that in fact (4.7) and (4.8) are equivalent to (4.4). On the other hand, by Płonka [P2] the set of identities consisting of the identities (4.7) and (4.8) is equivalent to the unique identity

$$xy^k = x$$

with $k = \mathrm{GCD}(j_{m+1},\dots,j_r,k_{m+1},\dots,k_s)$.

The proof of (ii) and (iii) is similar to this of (i). □

4.9. C o r o l l a r y . Let the identity

$$x\, x_1^{j_1}\dots x_r^{j_r} = x\, y_1^{k_1}\dots y_s^{k_s},$$

where the set $\{x_1,\dots,x_r\}$ may be empty, that is not a consequence of the axioms of LIR, hold in an LIR-groupoid $(G,\cdot)$. Then $(G,\cdot)$ is in a variety $\underline{G}_{i,j}$ for some $i = 0,1,\dots$ and $j > i$. □

4.10. L e m m a . If the identity

$$x\, y^{i+r} = x\, y^i \tag{4.11}$$

holds in a LIR-groupoid $(G,\cdot)$, then so does the identity

$$x\, y^{i+kr} = x\, y^i. \tag{4.12}$$

P r o o f . The identity (4.11) implies that $xy^{i+2r} = xy^{i+r}y^r = xy^iy^r = xy^{i+r} = xy^i$. Now if the identity (4.12) holds in $(G,\cdot)$, then $xy^{i+kr+r} = xy^{i+kr}y^r = xy^iy^r = xy^{i+r} = xy^i$. It follows by induction that (4.12) holds in $(G,\cdot)$. □

4.13. C o r o l l a r y . Let i be a non-negative integer and p and r natural numbers. If $i \leqq j$ and $p \mid r$, then $\underline{G}_{i,i+p} \subseteq \underline{G}_{j,j+r}$.

P r o o f . If $i = j$ it follows by Lemma 4.10. Assume that $i < j$. Multiplying the identity

$$x\,y^i = x\,y^{i+p}$$

by y^{j-i} on the right one gets the identity

$$x\,y^j = x\,y^{j+p}.$$

Now by Lemma 4.10, if the last identity is satisfied in a LIR-groupoid $(G,\cdot)$, then so does the identity

$$x\,y^j = x\,y^{j+r}. \quad \square$$

4.14. L e m m a . Let r and s be natural numbers and let $r < s$. Then the identities

(4.15) $$x\,y^{i+r} = x\,y^i$$

and

(4.16) $$x\,y^{i+s} = x\,y^i$$

are satisfied in a LIR-groupoid $(G,\cdot)$ if and only if the identity

$$x\,y^{i+GCD(r,s)} = x\,y^i$$

is satisfied in $(G,\cdot)$ as well.

P r o o f . $(\Leftarrow)$ It follows by Lemma 4.10.
$(\Rightarrow)$ By the Euclidean Algorythm

$$s = rq_1+p_1$$
$$r = p_1q_2+p_2$$
$$p_1 = p_2q_3+p_3$$
$$\vdots$$
$$p_{k-2} = p_{k-1}q_k+p_k$$
$$p_{k-1} = p_kq_{k+1}$$

and p_k = GCD(r,s). Now by Lemma 4.10, the identities (4.15) and (4.16) imply that

(4.17) $$x\,y^{i+p_1} = x\,y^i$$

is satisfied in (G,•), (4.15) and (4.17) imply that

$$x\,y^{i+p_2} = x\,y^i$$

is satisfied in (G,•), and so on. Finally, we get that the identity

$$x\,y^{i+p_k} = x\,y^i$$

is satisfied in (G,•) as well. □

4.18. L e m m a . Let i be a non-negative integer and j, r, s natural numbers. Let $i<j$. The identities

(4.15) $$x\,y^{i+r} = x\,y^i$$

and

(4.19) $$x\,y^{j+s} = x\,y^j$$

are satisfied in a LIR-groupoid (G,•) if and only if the identity

(4.20) $$xy^{i+\mathrm{GCD}(r,s)} = xy^i$$

is satisfied in (G,•).

P r o o f . ($\Leftarrow$) If the identity (4.20) holds in $(G,\cdot)$, then by Lemma 4.10 the identities (4.15) and (4.16) hold in $(G,\cdot)$ as well. Multiplying (4.16) by y^{j-i} on the right one gets the identity (4.19).
($\Rightarrow$) If $s = ar$, then $GCD(r,s) = r$ and (4.20) holds in $(G,\cdot)$.

Now assume that $r \nmid s$. Multiplying (4.15) by y^{j-i} on the right one gets the identity

$$(4.21) \qquad xy^{j+r} = xy^{j}.$$

By Lemma 4.14, it follows that together (4.19) and (4.21) are equivalent to

$$(4.22) \qquad xy^{j+GCD(r,s)} = xy^{j}.$$

Let $j = i+k$ and $k = ar+b$ for a, b integers and $a \geqq 0$, $0 \leqq b < r$. Multiplying (4.22) by y^{r-b} and using (4.15) and Lemma 4.10 one gets (4.20) what completes the proof. □

4.23. P r o p o s i t i o n . In the variety LIR, the set of identities

$$x\,y^{i_k+r_k} = x\,y^{i_k}$$

where $k = 1,\ldots,q$ and $0 \leqq i_1 \leqq i_2,\ldots,i_q$, is equivalent to the identity

$$x\,y^{i_1+GCD(r_1,\ldots,r_q)} = x\,y^{i_1}.$$

P r o o f . It follows by Lemma 4.18. □

4.24. C o r o l l a r y . Let $0 \leqq i_1 \leqq i_2,\ldots,i_q$. Then

$$\underline{G}_{i_1,i_1+r_1} \cap \ldots \cap \underline{G}_{i_q,i_q+r_q} = \underline{G}_{i_1,i_1+GCD(r_1,\ldots,r_q)}. \qquad □$$

5. The lattice of subvarieties of LIR

Corollary 4.9 shows that each non-trivial variety $\underline{K}$ of LIR-groupoids different from LIR is contained in some of $\underline{G}_{i,j}$. Now we show that each $\underline{K}$ equals some of $\underline{G}_{i,j}$.

5.1. P r o p o s i t i o n . If $\underline{K}$ is a variety of LIR-groupoids contained in $\underline{G}_{i,i+j}$, then $\underline{K}$ is trivial or $\underline{K} = \underline{G}_{p,p+q}$ for some $p \leqq i$.

P r o o f . Let us assume that $\underline{K}$ is non-trivial and $\underline{K} \neq \underline{G}_{i,i+j}$. By Lemma 4.3, the set of identities satisfied in $\underline{K}$, that are not consequences of the axioms of LIR, is equivalent to a set I of identities

$$x\,y^{r_k} = x\,y^{r_k+s_k}$$

where $r_k+s_k < i+j$. Evidently, this set is finite, say has l elements, and some of r_k, say r_1, is less than all other. Hence, by Proposition 4.23 the set I is equivalent to the unique identity

$$xy^{p} = xy^{p+q}$$

where $p = r_1$ and $q = GCD(s_1,\dots,s_l)$. □

5.2. P r o p o s i t i o n . $\underline{G}_{i,i+p} \subseteq \underline{G}_{j,j+r}$ if and only if $i \leq j$ and $p \mid r$.

P r o o f . ($\Leftarrow$) It follows by Corollary 4.13. ($\Rightarrow$) If $\underline{G}_{i,i+p} \subseteq \underline{G}_{j,j+r}$, then by Propoisition 5.1 and Corollary 4.24, $j = i+k$ and $\underline{G}_{i,i+p} \cap \underline{G}_{i+k,i+k+r} = \underline{G}_{i,i+GCD(p,r)} = \underline{G}_{i,i+p}$. Hence $GCD(p,r) = p$ and consequently $p \mid r$. □

Let N_d be the lattice of the natural numbers with the meet of two numbers i and j being their greatest common divisor GCD(i,j), and the join of i and j being their least common multiple LCM(i,j). Let N_o be the chain of the natural numbers with the join of i and j being their maximum and the meet of i and j their minimum. Denote

$$L := N_o \times N_d.$$

Now let $L_{0,1}$ be the lattice obtained from L by adding a new greatest element 1 and a new least element 0.

5.3. T h e o r e m . The lattice L(LIR) of all subvarieties of the variety LIR is isomorphic to the lattice $L_{0,1}$. The isomorphism is given by the mapping

$$h: L(LIR) \longrightarrow L_{0,1}$$

where the image of the trivial variety is 0, the image of the variety LIR is 1 and $\underline{G}_{i,i+p}h = (i,p)$.

P r o o f . It is evident that the mapping h is onto. We show that it is one-to-one. Indeed, if $\underline{K} = \underline{G}_{i,i+p} = \underline{G}_{j,j+r}$, then $\_ \subseteq \_{,i+p} = \underline{G}_{j,j+r}$ and by Propositions 5.1 and 5.2, $\underline{K} = \underline{G}_{k,k+s}$ with $k \leqq i$, $k \leqq j$, $s \mid p$, $s \mid r$. On the other hand $\underline{G}_{i,i+p} = \underline{G}_{j,j+r} \subseteq \underline{K} = \underline{G}_{k,k+s}$, whence $i \leqq k$, $j \leqq k$ and $p \mid s$, $q \mid s$. It follows that $i = j = k$ and $p = r = s$.

Now by Corollary 4.24, for $i \leqq j$

$$(\underline{G}_{i,i+p} \cap \underline{G}_{j,j+r})h = \underline{G}_{i,i+GCD(p,r)}h = (i,GCD(p,r)).$$

On the other hand

$$\underline{G}_{i,i+p}h \wedge \underline{G}_{j,j+r}h = (i,p) \wedge (j,r) = (i,GCD(p,r)).$$

It follows that h is a meet-homomorphism.

We show that

$$\underline{G}_{i,i+p} \vee \underline{G}_{i+k,i+k+r} = \underline{G}_{i+k,i+LCM(p,r)}. \tag{5.4}$$

By Proposition 5.2, it follows that

$$\underline{G}_{i,i+p} \vee \underline{G}_{i+k,i+k+r} \subseteq \underline{G}_{i+k,i+LCM(p,r)}.$$

Now if $\underline{G}_{i,i+p} \subseteq \underline{K}$ and $\underline{G}_{i+k,i+k+r} \subseteq \underline{K}$, then by Propositions 5.1 and 5.2, $\underline{K} = \underline{G}_{s,s+t}$, where $i+k \leqq s$ and $p \mid t$ and $r \mid t$, whence $LCM(p,r) \mid t$. It follows that $\underline{G}_{i+k,i+LCM(p,r)} \subseteq \underline{G}_{s,s+t}$ and consequently (5.4) holds.

The equality (5.4) implies that h is a join-homomorphism. Indeed, for $i \leqq j$

$$(\underline{G}_{i,i+p} \vee \underline{G}_{j,j+r})h = \underline{G}_{j,j+LCM(p,r)}h = (j,LCM(p,r)).$$

On the other hand

$$\underline{G}_{i,i+p}h \vee \underline{G}_{j,j+r}h = (i,p) \vee (j,r) = (j,LCM(p,r)).$$

It follows that h is a join-homomorphism. □

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