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THE LATTICE OF VARIETIES OF SYMMETRIC IDEMPOTENT ENTROPIC GROUPOIDS

Consider the following set of the axioms for a set G with a binary operation $\cdot$.

(S) $(x\cdot y)\cdot y = x$ (Symmetry)

(I) $x\cdot x = x$ (Idempotence)

(E) $(x\cdot y)\cdot(z\cdot t) = (x\cdot z)\cdot(y\cdot t)$. (Entropicity)

An algebra $(G,\cdot)$ satisfying all these axioms will be called an SIE-groupoid.

Such groupoids were investigated in many papers. For example in [2] more general algebras so called quandles were considered that have been used to characterization of knots, in [4] in connection with symmetric spaces, in [5] and [6] with reference to groups that are generated by involutions and in [7] as an example of so called modes. Entropic groupoids were investigated in [1], where they were called medial.

The aim of this paper is to describe the lattice of all subvarieties of the variety SIE of all SIE-groupoids. The paper is organized as follows. After some elementary properties of SIE-groupoids presented in Section 1, we describe identities in two variables satisfied in SIE-groupoids in

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Section 2. Then we give a standard form for words in the free SIE-groupoid on a finite set. In Section 4 we investigate identities in n variables satisfied in the variety SIE. We prove that every such identity is equivalent to some identity on two variables. The results of Sections 3 and 4 enable us to describe the lattice of all subvarieties of SIE in Section 5.

1. Preliminaries

We give here some basic properties of SIE-groupoids. In the sequel we write xy for x·y. We also use the following convention

(1.1) $$x_1 \ldots x_n := (x_1 \ldots x_{n-1}) \cdot x_n$$

for every $n \geqslant 2$ and

$$xy \cdot zt := (x \cdot y) \cdot (z \cdot t).$$

The product (1.1) is called left associated.

D e f i n i t i o n 1.2. A groupoid (G,•) is called left distributive if the identity

(i) $$x(yz) = xy \cdot xz$$

holds in (G,•). (G,•) is right distributive if

(ii) $$(xy)z = xz \cdot yz$$

holds in (G,•). Finally (G,•) is distributive in the case it is both left and right distributive.

P r o p o s i t i o n 1.3. [7] Every SIE-groupoid (G,•) is distributive and satisfies the partial associative law

$$xy \cdot x = x \cdot yx,$$

Note that in general SIE-groupoids are not associative.

The axiom of symmetry for SIE-groupoids implies that for every $a \in G$ the mapping $S_a : G \to G$, $S_a(b) = ba$, is a bijection and has the property $S_a^{-1} = S_a$. By the axiom of idempotency every mapping S_a has a fixed point, namely a, since $S_a(a) = aa = a$. By the right distributivity each mapping S_a is a homomorphism. The entropic law means that $f : G\times G \to G$, $f(a,b) = ab$, is a homomorphism too.

Proposition 1.4. The following identities are satisfied in every SIE-groupoid for each natural number n:

(i) $$xyzt = xtzy,$$

(ii) $$x_1y_1\dots x_ny_n = x_1y_{\sigma(1)}\dots x_ny_{\sigma(n)}$$

for each permutation σ of the set $\{1,\dots,n\}$,

(iii) $$x\cdot y_1\dots y_n = x_1y_n\dots y_2y_1y_2\dots y_n.$$

Remark. By (iii) each SIE-groupoid word can be represented in a left associated form.

Proof. (i). By the distributive, entropic and symmetry laws

$$xyzt = xyt\cdot zt = (xt\cdot yt)\cdot zt = (xt\cdot z)(yt\cdot t) = xtzy.$$

(ii). It follows from (i) because every permutation is a composition of transpositions.

(iii). The proof goes by induction on n. The case $n = 1$ is obvious. For $n = 2$, $xy_2y_1y_2 = x\cdot y_1y_2$ holds by symmetry and distributivity. Now assume that (iii) holds for every natural number less than or equal to n. Then by the first part of the proof and the induction hypothesis

$$x\cdot y_1\dots y_{n+1} = x\cdot(y_1\dots y_n)y_{n+1} = xy_{n+1}(y_1\dots y_n)y_{n+1} =$$
$$= xy_{n+1}y_n\dots y_2y_1y_2\dots y_ny_{n+1}. \qquad \square$$

The operation of an SIE-groupoid may be geometrically interpreted as a reflection on the real line R (see [7]). For $G = R$ and $xy := 2y-x$, $(R,\cdot)$ is an SIE-groupoid and xy is the reflection of x in y.

Example 1.5. More general the set R^n with $(i_1,\dots,i_n)\cdot(j_1,\dots,j_n) := (2j_1-i_1,\dots,2j_n-i_n)$ is an SIE-groupoid for every $n\in N$ and $(Z^n,\cdot)$ where Z is the set of all integers, is a subgroupoid of $(R^n,\cdot)$.

Example 1.6. Let $(G,\cdot,e)$ be a nilpotent group of class at most 2. (This means that for all $a,b \in G$, $[a,b] := a^{-1}b^{-1}ab$ is in the center of $(G,\cdot,e)$). Let $x \circ y := yx^{-1}y$. Then $(G,\circ)$ is an SIE-groupoid.

2. Identities in two variables

The aim of this section is to describe identities in two variables satisfied in the variety SIE.

The free SIE-groupoid $F(x,y)$ on two generators x and y was described by Lindner, Mendelsohn in [3] and by Romanowska, Smith in [7]. It is isomorphic to the SIE-groupoid $(Z,\cdot)$, where $xy := 2y-x$. In [3] Lindner, Mendelsohn defined an infinite class of words in the free groupoid on two generators x and y in the following way

$$w_0(x,y) = x, \quad w_1(x,y) = y \quad \text{and}$$

$$w_i(x,y) = w_{i-2}(x,y)w_{i-1}(x,y) \quad \text{for} \quad i \geq 2.$$

Multiplying the last equation on the right by $w_{i-1}(x,y)$ and using the axiom of symmetry one obtains

$$w_i(x,y)w_{i-1}(x,y) = w_{i-2}(x,y).$$

We use the last equation to extend the definition of $w_i(x,y)$ to the case of negative indices. In what follows we abbreviate $w_i(x,y)$ to w_i if no confusion can arise.

Now let $(Z,\cdot)$ be the SIE-groupoid from Example 1.4. It was proved in [3] that each element of $F(x,y)$ may be expressed as w_i for some integer i and that the mapping $h:Z \longrightarrow F(x,y)$, $h(i) := w_i$ is an isomorphism. In this way a standard form of words in $F(x,y)$ is given. In [7] Romanowska, Smith described words in $F(x,y)$ in a different way. They proved the following proposition [7, Proposition 4.14].

Proposition 2.1. In $F(x,y)$ each element may be expressed in the standard form $a_1(a_2...(a_{n-1}a_n))...)$ with $a_i \neq a_{i+1}$ and $a_i \in \{x,y\}$, $i = 1,...,n-1$.

C o r o l l a r y 2.2. Let $i \geq 1$. Then in an SIE-groupoid

$$w_i(x,y) = a_{i-1} \ldots a_2 a_1 y,$$

where $a_1 = x$, $a_k \in \{x,y\}$, $a_k \neq a_{k+1}$ for $1 \leq k \leq i-1$ and

$$w_{-i}(x,y) = a_i \ldots a_1 x,$$

where $a_1 = y$, $a_k \in \{x,y\}$, $a_k \neq a_{k+1}$ for $1 \leq k \leq i-1$.

P r o o f . The proof goes by induction on i. The cases $i = 1,2$ are obvious. Let $w_{i-2} = a_{i-3} \ldots a_1 y$, $w_{i-1} = a_{i-2} \ldots a_1 y$ and $a_1 = x$, $a_k \in \{x,y\}$, $a_k \neq a_{k+1}$ for $1 \leq k \leq i-3$.

Then by distributivity and idempotency

$$w_i = w_{i-2} w_{i-1} = a_{i-3} \ldots a_1 y \cdot a_{i-2} \ldots a_1 y = (a_{i-3} \ldots a_1 \cdot a_{i-2} \ldots a_1) y =$$

$$= (a_{i-3} \cdot a_{i-2} a_{i-3}) a_{i-4} \ldots a_1 y,$$

and since by Proposition 1.3,

$$a_{i-3} \cdot a_{i-2} a_{i-3} = a_{i-3} a_{i-2} a_{i-3},$$

it follows that

$$w_i = a_{i-3} a_{i-2} a_{i-3} \ldots a_1 y.$$

Moreover $a_{i-3} \neq a_{i-2}$. Let us define $a_{i-1} := a_{i-3}$. Then $w_i = a_{i-1} a_{i-2} a_{i-3} \ldots a_1 y$, where $a_1 = x$, $a_k \in \{x,y\}$, $a_k \neq a_{k+1}$, for $1 \leq k \leq i-2$.

The second equality is proved similarly; it is also a consequence of the first one and Lemma 2.3(i) below.

L e m m a 2.3 [3]. Let r and s be integers. Then the identities

(i) $$w_r w_s = w_{-r+2s},$$

(ii) $$w_{-r}(x,y) = w_{r+1}(y,x)$$

hold in SIE. □

The next two obvious consequences of Lemma 2.3 will be used several times.

C o r o l l a r y 2.4. For every integer r the identities

(i) $$w_{-r} = w_r w_0,$$

(ii) $$w_{2r} = w_0 w_r$$

hold is SIE.

D e f i n i t i o n 2.5. [3]. Let n be an integer. The identity of the form $w_n(x,y) = x = w_0(x,y)$ is called an n-cyclic identity. An SIE-groupoid satisfying an n-cyclic identity is called an n-cyclic SIE-groupoid.

Denote by V_n the variety of all n-cyclic SIE-groupoids. By Corollary 2.4(i) and idempotency, $w_n = w_0$ if and only if $w_{-n} = w_0$. Hence V_n is equal to V_{-n}.

P r o p o s i t i o n 2.6. Every identity in two variables satisfied in SIE is equivalent to an n-cyclic identity for some n.

P r o o f . Every identity in two variables has the form u = v where u,v where $u,v \in F(x,y)$. This means that there exist integers r,s such that $u = w_r(x,y)$, $v = w_s(x,y)$. Hence the identity u = v has the form $w_r(x,y) = w_s(x,y)$. We consider two cases:

C a s e 1. r or s is even. We may assume that s is even. (The case r is even is analogous). We multiply the identity $w_r = w_s$ on the right by $w_{s/2}$ and obtain (by Lemma 2.3(i)) $w_r w_{s/2} = w_s w_{s/2} = w_0$ and $w_r w_{s/2} = w_{s-r}$. This shows that the identity $w_{s-r} = w_0$ is a consequence of $w_r = w_s$.

To prove the converse implication, multiply the identity $w_{s-r} = w_0$ on the right by $w_{s/2}$. Using Lemma 2.3(i) and Corollary 2.4(ii) we obtain $w_r = w_s$.

C a s e 2. r and s are odd. Since by Lemma 2.3(ii) $w_s(x,y) = w_{-s+1}(y,x)$ and $w_r(x,y) = w_{-r+1}(y,x)$ then by the previous part of the proof the latter is equivalent to $w_{r-s}(y,x) = w_0(y,s) = y$ and consequently equivalent to $w_{r-s}(x,y) = x$.

Let GCD(n,k) denote the greatest common divisor of the integers n and k.

P r o p o s i t i o n 2.7. For all positive integers n and k,

$$V_n \cap V_k = V_{GCD(n,k)}.$$

P r o o f . First note that by the proof of Proposition 2.6, if $w_n = w_0$ is satisfied in a variety of SIE-groupoids then $w_{n+i} = w_i$ and hence $w_{in} = w_0$ is satisfied as well for every i in Z.

Let r, s be natural numbers, k = GCD(n,k)r and n = = GCD(n,k)s. Now assume that an SIE-groupoid G satisfies the identity $w_{GCD(n,k)} = w_0$ i.e. G is in $V_{GCD(n,k)}$. Then G satisfies the identities $w_n = w_{GCD(n,k)s} = w_0$ and $w_k = w_{GCD(n,k)r} = w_0$, i.e. G is in $V_n \cap V_k$.

Conversely let an SIE-groupoid G satisfy identities $w_k = w_0$ and $w_n = w_0$. Since GCD(n,k) = an+bk for some integers a,b, $w_{GCD(n,k)} = w_{an+bk} = w_{an} = w_0$. This means that G is in $V_{GCD(n,k)}$.

3. The standard form of words in the free SIE-groupoids

The purpose of this section is to describe the standard form of words in the free SIE-groupoid $F(x_0,\dots,x_n)$ on n+1 generators $x_0,\dots,x_n$. Let Q_n be the set of all sequences $(k_1,\dots,k_n)$ in Z^n such that at most one k_i is odd. It is easy to see that $(Q_n,\cdot)$ is a subgroupoid of $(Z^n,\cdot)$ from Example 1.5.

T h e o r e m 3.1. (Joyce [2]). The free SIE-groupoid $F(x_0,\dots,x_n)$ is isomorphic to the SIE-groupoid $(Q_n,\cdot)$. The elements $e_0 = (0,\dots,0)$, $e_1 = (1,0,\dots,0),\dots,$ $e_n = (0,\dots,1)$ are free generators of $(Q_n,\cdot)$. □

Note that in [2] SIE-groupoids were called involutary abelian quandles and were used to characterize knots.

Now we define an infinite class of groupoid words in variables $x_0,\dots,x_n$ that coincides with that given in Section 2 in the case n = 1.

D e f i n i t i o n 3.2. For every element $(2r_1,\dots,r_j,\dots,2r_n)$ of Q_n we define

$$w(0,\dots,0;x_0,\dots,x_n) := x_0 = w_0,$$

$$w(2r_1,\dots,r_j,\dots,2r_n;x_0,\dots,x_n) :=$$

$$= \begin{cases} w_{r_j}w_0w_{r_1}w_0\dots w_{r_{j-1}}w_0w_{r_{j+1}}\dots w_0w_{r_n} & \text{if } r_j \text{ is odd} \\ w_0w_{r_1}w_0\dots w_0w_{r_j/2}w_0\dots w_0w_{r_n} & \text{otherwise,} \end{cases}$$

where $w_{r_j} := w_{r_j}(x_0,x_j)$ for every $1 \leq j \leq n$. □

In the sequel we write briefly $w(2r_1,\dots,r_j,\dots,2r_n)$ if no confusion can arise and we abbreviate $w(2r_1,\dots,r_j,\dots,2r_n)$ to $w_{\underline{r}}$ where $\underline{r} = (2r_1,\dots,r_j,\dots,2r_n)$ is in Q_n.

Note that, if r is in Z, then $w(r;x,y) = w_r(x,y)$.

R e m a r k . By Corollary 2.4(ii) and Proposition 1.4(ii)

$$w_0w_{r_1}w_0\dots w_0w_{r_n} = w_{2r_1}w_0\dots w_0w_{r_n} =$$

$$= w_{2r_j}w_0w_{r_1}w_0\dots w_0w_{r_{j-1}}w_0w_{r_{j+1}}w_0\dots w_{r_n}$$

for every $1 \leq j \leq n$.

The following is the main result of this Section.

T h e o r e m 3.3. In the free SIE-groupoid $F(x_0,x_1,\dots,x_n)$ on the generators $x_0,x_1,\dots,x_n$ each further element may be expressed in the standard form of $w_{\underline{r}}$ for some $\underline{r}$ in Q_n.

The proof of Theorem 3.3 is divided into several lemmas.

L e m m a 3.4. Let $\underline{r} = (2r_1,\dots,r_i,\dots,2r_n)$ and $\underline{k} = (2k_1,\dots,k_i,\dots,2k_n)$ be elements of Q_n then

$$w_{\underline{r}}w_{\underline{k}} = w_{2\underline{k}-\underline{r}}$$

holds in SIE.

P r o o f . By entropic and distributive laws

$$w_{\underline{r}}w_{\underline{k}} = w_{r_i}w_o w_{r_1} \cdots w_{r_{i-1}} w_o w_{r_{i+1}} w_o \cdots w_{r_n} \cdot$$

$$\cdot w_{k_i} w_o w_{k_1} \cdots w_{k_{i-1}} w_o w_{k_{i+1}} w_o \cdots w_{k_n} =$$

$$\left(w_{r_i} w_{k_i}\right) w_o \left(w_{r_1} w_{k_1}\right) w_o \cdots \left(w_{r_{i-1}} w_{k_{i-1}}\right) w_o \left(w_{r_{i+1}} w_{k_{i+1}}\right) w_o \cdots \left(w_{r_n} w_{k_n}\right).$$

Using Lemma 2.3(i) we obtain

$$w_{\underline{r}} w_{\underline{k}} = w_{2k_i - r_i} w_o w_{2k_1 - r_1} \cdots w_{2k_{i-1} - r_{i-1}} w_o w_{2k_{i+1} - r_{i+1}} \cdots w_{2k_n - r_n} =$$

$$= w_{2\underline{k} - \underline{r}}.$$

L e m m a 3.5. For all integers i_1, j_2, i_2, $w_{i_1} w_{j_2} w_{i_2} = w_{i_1} w_o w_{i_2 - j_2}$ holds in SIE.

P r o o f . In the case $i_2 = 0$ the proof is obvious because by the distributive law and Corollary 2.4(i),

$$w_{i_1} w_{j_2} w_o = w_{i_1} w_o w_{j_2} w_o = w_{i_1} w_o w_{-j_2}.$$

Now let $i_2 = 1$ and $j_2 \geq 0$. In this case the proof goes by induction on j_2. If $j_2 = 0$ then the equality is obvious. Let $j_2 = 1$. Then $w_{i_1} w_1 w_1 = w_{i_1} = w_{i_1} w_o w_o$ as required. Now assume that the identity holds for all positive integers less than j_2. Since $w_{j_2} = w_{j_2 - 2} w_{j_2 - 1}$, it follows by distributivity and the induction hypothesis that

$$w_{i_1} w_{j_2} w_1 = w_{i_1} \left(w_{j_2 - 2} w_{j_2 - 1}\right) w_1 = w_{i_1} w_{j_2 - 2} w_1 \cdot w_{i_1} w_{j_2 - 1} w_1 =$$

$$= w_{i_1} w_o w_{3 - j_2} \cdot w_{i_1} w_o w_{2 - j_2} = w_{i_1} w_o w_{1 - j_2} \quad \text{as required.}$$

Now let $i_2 = 1$ and $j_2 < 0$. By Corollary 2.4(i) $w_{j_2} = w_{-j_2} w_o$. Hence by the distributive law and the previous part of the proof we obtain

$$w_{i_1}w_{j_2}w_1 = w_{i_1}\left(w_{-j_2}w_o\right)w_1 = w_{i_1}w_{-j_2}w_1\cdot w_{i_1}w_ow_1 =$$

$$= w_{i_1}w_ow_{1+j_2}\cdot w_{i_1}w_ow_1 = w_{i_1}w_ow_{1-j_2}.$$

Now consider the case $i_2 > 0$ and assume that the identity holds for all positive integers less than i_2. Then since $w_{i_2} = w_{i_2-2}w_{i_2-1}$, it follows that

$$w_{i_1}w_{j_2}w_{i_2} = w_{i_1}w_{j_2}\cdot w_{i_2-2}w_{i_1-1} = w_{i_1}w_{j_2}w_{i_2-2}\cdot w_{i_1}w_{j_2}w_{i_1-1} =$$

$$= w_{i_1}w_ow_{i_2-2-j_2}\cdot w_{i_1}w_ow_{i_2-1-j_2} = w_{i_1}w_o\cdot w_{i_2-2-j_2}w_{i_2-1-j_2} =$$

$$= w_{i_1}w_ow_{i_2-j_2},$$

what completes the proof in the case $i_2 \geq 0$.
If $i_2 < 0$, then the proof is similar to the case $i_2 = 1$, $j_2 < 0$.

L e m m a 3.6. Let $\underline{r} = (r_1, 2r_2)$ and $\underline{k} = (2k_1, k_2)$ be element of Q_2. Then $w_{\underline{r}}w_{\underline{k}} = w_{2\underline{k}-\underline{r}}$ holds in SIE.

P r o o f . By Definition 3.2 we have $w_{\underline{r}} = w_{r_1}w_ow_2$, $w_{\underline{k}} = w_{k_2}w_ow_{k_1}$, and by Propositions 1.4(iii),(ii) and the axiom of symmetry

$$w_{\underline{r}}w_{\underline{k}} = w_{r_1}w_ow_{r_2}\cdot w_{k_2}w_ow_{k_1} = w_{r_1}w_ow_{r_2}w_{k_1}w_ow_{k_2}w_ow_{k_1} =$$

$$= w_{r_1}w_{k_1}w_{r_2}w_{k_1}w_ow_{k_2}w_ow_o = w_{r_1}w_{k_1}w_ow_{k_1}w_{r_2}w_{k_2}.$$

Since by Lemma 2.3(i),

$$w_{r_1}w_{k_1}w_ow_{k_1} = w_{2k_1-r_1}w_ow_{k_1} = w_{4k_1-r_1}$$

we have $w_{\underline{r}}w_{\underline{k}} = w_{4k_1-r_1}w_{r_2}w_{k_2}$ and by Lemma 3.5

$$w_{\underline{r}}w_{\underline{k}} = w_{4k_1-r_1}w_ow_{k_2-r_2} = w(4k_1-r_1, 2k_2-2r_2) = w_{2\underline{k}-\underline{r}}$$

what completes the proof.

L e m m a 3.7. Let $\underline{r} = (2r_1,\dots,r_i,\dots,2r_n)$ and $\underline{k} = (2k_1,\dots,k_j,\dots,2k_n)$ be elements of Q_n. Then $w_{\underline{r}}w_{\underline{k}} = w_{2\underline{k}-\underline{r}}$ holds in SIE.

P r o o f . It remains to prove the Lemma in the case $n>2$ and $i<j$. In this case Proposition 1.4(ii) implies that

$$w_{\underline{r}} = w_{r_i}w_ow_{r_1}\dots w_{r_{i-1}}w_ow_{r_{i+1}}\dots w_ow_{r_j}w_o\dots w_{r_n} =$$

$$= w_{r_i}w_ow_{r_j}w_ow_{r_1}\dots w_{r_{i-1}}w_ow_{r_{i+1}}\dots w_ow_{r_{j-1}}w_ow_{r_{j+1}}\dots w_{r_n}$$

and analogously

$$w_{\underline{k}} = w_{k_j}w_ow_{k_i}w_ow_{k_1}\dots w_ow_{k_{i-1}}w_ow_{k_{k+1}}\dots w_{k_{j-1}}w_ow_{k_{j+1}}\dots w_{k_n}.$$

Hence by the distributive and entropic laws and by Lemma 2.3(i)

$$w_{\underline{r}}w_{\underline{k}} =$$

$$= \left(w_{r_i}w_ow_{r_j}\cdot w_{k_j}w_ow_{k_i}\right)w_o\left(w_{r_1}w_{k_1}\right)w_o\dots\left(w_{r_{i-1}}w_{k_{i-1}}\right)w_o\left(w_{r_{i+1}}w_{k_{i+1}}\right)\dots$$

$$\left(w_{r_{j-1}}w_{k_{j-1}}\right)w_o\left(w_{r_{j+1}}w_{k_{j+1}}\right)w_e\dots w_o\left(w_{r_n}w_{k_n}\right) =$$

$$= \left(w_{r_i}w_ow_{r_j}\cdot w_{k_j}w_ow_{k_i}\right)w_ow_{2k_1-r_1}w_o\dots w_{2k_{i-1}-r_{i-1}}w_ow_{2k_{i+1}-r_{i+1}}\dots$$

$$w_{2k_{j-1}-r_{j-1}}w_ow_{2k_{j+1}-r_{j+1}}w_o\dots w_{2k_n-r_n}.$$

Since in the case $n = 2$

$$w_{r_i}w_ow_{r_j}\cdot w_{k_j}w_ow_{k_i} = w_{4k_i-r_i}w_ow_{k_j-r_j}$$

then using Lemma 3.6 and Proposition 1.4(ii) we conclude that

$$w_{\underline{r}}w_{\underline{k}} = w_{2\underline{k}-\underline{r}}.$$

This completes the proof Lemma 3.7.

P r o o f of Theorem 3.3. The proof goes by induction on the length of expression of the element. The shortest expression of such an element is x_ix_j, where $i \neq j$ and x_i, x_j are in the set $\{x_0,\dots,x_n\}$. By previous Section $x_ix_j = w_1(x_0,x_i)w_1(x_0,x_j) = w_{-1}(x_0,x_i)w_0w_1(x_0x_j)$ and this is already in the standard form. By induction, a longer element not in standard form may be expressed as $w_{\underline{r}}w_{\underline{k}}$, a product of standard forms. Then by Lemma 3.7 $w_{\underline{r}}w_{\underline{k}} = w_{2\underline{k}-\underline{r}}$ is in the required standard form.

Theorem 3.8. Let

$$W := \{w(r_1,\dots,r_n;x_0,x_1,\dots,x_n):(r_1,\dots,r_n)\in Q_n\}.$$

For $\underline{k},\underline{r}\in Q_n$, define $w_{\underline{r}}w_{\underline{k}} := w_{2\underline{k}-\underline{r}}$. Then $(W,\cdot)$ is an SIE-groupoid and the mapping $h:Q_n \to W$, $h(\underline{r}) = w_{\underline{r}}$ is an isomorphism.

Proof. Obviously the mapping h is surjective and by Lemma 3.7 it is a homomorphism. Now we prove that it is injective. Let $\underline{r} = (2r_1,\dots,r_i,\dots,2r_n)$ and $\underline{k} = (2k_1,\dots,k_j,\dots,2k_n)$ be elements of Q_n. Assume that $w_{\underline{r}} = w_{\underline{k}}$. Substituting x_0 for all variables x_s different from x_1 in this identity we get one of the following identities:

(a) $w_{2r_1}(x_0,x_1) = w_{2k_1}(x_0,x_1)$ in the case $i\neq 1$, $j\neq 1$,

(b) $w_{r_1}(x_0,x_1) = w_{2k_1}(x_0,x_1)$ in the case $i=1$, $j\neq 1$,

(c) $w_{2r_1}(x_0,x_1) = w_{k_1}(x_0,x_1)$ in the case $i\neq 1$, $j=1$,

(d) $w_{r_1}(x_0,x_1) = w_{k_1}(x_0,x_1)$ in the case $i=j=1$.

By [3] the identity $w_p(x_0,x) = w_q(x_0,x)$ is satisfied in the free SIE-groupoid $F(x_0,x)$ iff $p = q$. It follows that the first coordinate of $\underline{r}$ is equal to the first coordinate of $\underline{k}$. In general substituting x_0 for all variables x_s different from x_p in this identity we get that the p-th coordinate of $\underline{r}$ is equal to the p-th coordinate of $\underline{k}$ for $1\leq p\leq n$, hence $\underline{r} = \underline{k}$.

4. Identities in n+1 variables

The goal of this section is to describe identities in n+1 variables satisfied in SIE. Every such identity has the form $w_{\underline{r}} = w_{\underline{k}}$ for some elements $\underline{r}$ and $\underline{k}$ of Q_n. We shall prove that any such identity is equivalent to one identity in two variables. First we prove that the identity $w_{\underline{r}} = w_{\underline{k}}$ is equivalent to $w_{\underline{p}} = w_0 = x_0$ for some $\underline{p}$ in Q_n.

P r o p o s i t i o n 4.1. Let $\underline{r}$ and $\underline{k}$ be in Q_n. Then the identity $w_{\underline{r}} = w_{\underline{k}}$ is equivalent to the identity $w_{\underline{p}} = w_0$ for some $\underline{p}$ in Q_n.

P r o o f . Let $\underline{k} = (k_1,\dots,k_n)$ and $\underline{r} = (r_1,\dots,r_n)$ be elements of Q_n. Since

$$w(k_1,\dots,k_i,\dots,k_n;x_0,x_1,\dots,x_i,\dots,x_n) =$$

$$= w(k_i,k_2,\dots,k_{i-1},k_1,k_{i+1},\dots,k_n;$$

$$x_0,x_i,x_2,\dots,x_{i-1},x_1,x_{i+1},\dots,x_n)$$

we can assume that all k_i, $i = 2,\dots,n$, are even and $k_1 \neq 0$. (In the case all k_i equals zero it is nothing to prove). Recall that $w_{\underline{k}} = w_{k_1}w_0w_{k_2/2}\dots w_0w_{k_n/2}$. It is easy to see that the identity $w_{\underline{r}} = w_{\underline{k}}$ is equivalent to

$$w_{\underline{r}}w_{k_n/2}w_0\dots w_{k_2/2}w_0 = w_{k_1}.$$

(Multiply the identity $w_{\underline{r}} = w_{\underline{k}}$ on the right first by $w_{k_n/2}$, then by w_0, then by $w_{k_{n-1}/2}$ etc., and finally by $w_{k_2/2}$ and by w_0 and use the axiom of symmetry).

Moreover by Lemma 3.7 and because w_{k_i} is equal to $w(0,\dots,k_i,\dots,0;x_0,\dots,x_n)$ the left side of the last identity is equal to $w(r_1,r_2-k_2,\dots,r_n-k_n;x_0,\dots,x_n)$.

Now we consider two cases:

C a s e 1. k_1 is even. By Corollary 2.4(ii), $w_{k_1} = w_0w_{k_1/2}$. Hence multiplying the last identity on the right by $w_{k_1/2}$, that equals $w(k_1/2,0,\dots,0;x_0,\dots,x_n)$, using Lemma 3.7 and the axiom of symmetry we conclude that the identity

$w(r_1,r_2-k_2,\ldots,r_n-k_n;x_0,\ldots,x_n) = w_0w_{k_1/2}$ is equivalent to $w(r_1-k_1,\ldots,r_n-k_n;x_0,\ldots,x_n) = w_0$.

Case 2. k_1 is odd. Then by Lemma 2.3(ii) $w_{k_1}(x_0,x_1) = w_{-k_1+1}(x_1,x_0)$ and $-k_1+1$ is even. By Theorem 3.3

$$w(r_1,r_2-k_2,\ldots,r_n-k_n;x_0,\ldots,x_n) = w(p_1,\ldots,p_n;x_1,x_0,\ldots,x_n)$$

for some $(p_1,\ldots,p_n)$ in Q_n hence we have case 1. This completes the proof.

It is obvious that without loss of generality we can consider only the identities

$$w(k_1,2k_2,\ldots,2k_n;x_0,\ldots,x_n) = x_0, \tag{4.2}$$

where k_i are integers and $i = 1,\ldots,n$. Substitute in this identity x_0 for all $x_i \neq x_1$ and x for x_1. Since $w_p(x_0,x_0) = x_0$ for every integers p and by the axiom of symmetry we obtain $w_{k_1}(x_0,x) = x_0$.

Similarly for every $j \in \{2,\ldots,n\}$ we can substitute x_0 for all $x_i \neq x_j$ and x for x_j in (4.2). In this way we obtain $w_0w_{k_j}(x_0,x) = x_0$ that is equivalent to $w_{2k_j}(x_0,x) = x_0$. As consequences of the identity (4.2) we get n identities in two variables

$$\begin{aligned} w_{k_1}(x_0,x) &= x_0, \\ w_{2k_j}(x_0,x) &= x_0 \end{aligned} \tag{4.3}$$

where $2 \leq j \leq n$.

Conversely, suppose an SIE-groupoid $(G,\cdot)$ satisfies each of the identities (4.3). Therefore by Corollary 2.4(ii) and the idempotency

$$w(k_1,2k_2,\ldots,2k_n;x_0,\ldots,x_n) =$$

$$= w_{k_1}(x_0,x_1)w_0w_{k_2}(x_0,x_2)\ldots w_{k_n}(x_0,x_n) =$$

$$= w_0 w_{k_2}(x_0,x_2)\dots w_{k_n}(x_0,x_n) =$$

$$= w_{2k_2}(x_0,x_2)w_0\dots w_{k_n}(x_0,x_n) = x_0$$

what means that $(G,\cdot)$ satisfies the identity (4.2).

In this way we have proved the following.

P r o p o s i t i o n 4.4. Let $(k_1,\dots,k_n)$ be in Q_n. Then every identity in $n+1$ variables $w(k_1,\dots,k_n;x_0,\dots,x_n) = x_0$ is equivalent in an SIE-groupoid to the set of n identities in two variables $w_{k_i}(x_0,x) = x_0$, $1 \leq i \leq n$.

P r o p o s i t i o n 4.5. The set of identities in two variables $w_{k_i}(x_0,x) = x_0$, $1 \leq i \leq n$ is equivalent to the unique identity in two variables

$$w_k(x_0,x) = x.$$

P r o o f . By Proposition 2.7 we have $k = GCD(k_1,\dots,k_n)$.

5. The main theorem

Using the results from the previous sections we here prove the main theorem which characterizes the lattice of all subvarieties of the variety SIE. Denote this lattice by L(SIE).

Let L be a lattice. We will denote by L^+ the lattice $L \cup \{\infty\}$ in which the element ∞ is greater than all elements of L. Let N denote the lattice of all natural numbers with respect to the partial order $\leq_N$ defined by $k \leq_N n$ if and only if k divides n. Finally let $V_{\underline{r}}$ be the variety of SIE-groupoids satisfying the identity $w_{\underline{r}} = w_0$, where $\underline{r} = (r_1,\dots,r_n)$ is in Q_n.

By Proposition 4.4, $V_{\underline{r}} = V_{r_1} \cap \dots \cap V_{r_n}$ and consequently, by Proposition 4.5 we can easily get the following statement.

C o r o l l a r y 5.1. For every $\underline{r} = (r_1,\dots,r_n)$ in Q_n, $V_{\underline{r}} = V_{GCD(r_1,\dots,r_n)}$.

T h e o r e m 5.2. Let V be a variety of SIE-groupoids. Then V = SIE or $V = V_n$ for some natural number n.

P r o o f . Let V be a nontrivial subvariety of the variety SIE. By Propositions 4.4 and 4.5 the set of all identities satisfied in V is equivalent to a set I of identities in two variables having the form $w_k = w_0$.

Let n be a minimal natural number such that the identity $w_n = w_0$ is in I. By Proposition 2.7, if the identity $w_k = w_0$ is satisfied in V then the identity $w_{GCD(n,k)} = w_0$ is satisfied in V, as well. Obviously $n = GCD(n,k)$. This means that n divides all natural numbers k such that the identity $w_k = w_0$ is in I. On the other hand as it was mentioned in the proof of Proposition 2.7 identity $w_n = w_0$ implies $w_{in} = w_0$ for every i in Z. Consequently $I = \{w_{in} = w_0 : i \in Z\}$ and each identity of I is a consequence of the identity $w_n = w_0$. It follows that $V = V_n$. □

R e m a r k . By [3], $V_n = V_k$ if and only if $n = k$.

Finally we describe the lattice L(SIE).

T h e o r e m 5.3. The lattice L(SIE) of all varieties of SIE-groupoids is isomorphic to N^+. The variety corresponding to ∞ is SIE. If n is a natural number then the variety corresponding to n is just the variety of SIE-groupoids satisfying the identity $w_n(x,y) = x$.

P r o o f . Let n, k be natural numbers and let LCM(n,k) denote the least common multiple of n and k.

By Theorem 5.2 and Proposition 2.7 it is enough to prove that the join $V_n \vee V_k$ of V_n and V_k is equal to $V_{LCM(n,k)}$. First note that by Proposition 2.7 and by above Remark $V_n \subseteq V_k$ if and only if n divides k.

By Theorem 5.2, $V_n \vee V_k = V_m$ for some natural number m. Since $V_n \subseteq V_m$ and $V_k \subseteq V_m$ then n divides m and k divides m. Hence LCM(n,k) divides m. Otherwise $V_n \subseteq V_{LCM(n,k)}$ and $V_k \subseteq V_{LCM(n,k)}$ whence $V_n \vee V_k = V_m \subseteq V_{LCM(r,k)}$ and m divides LCM(n,k). It follows that $m = LCM(n,k)$. □

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