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Andrzej Kurpiel

A MONADIC APPROACH TO THE MODAL THEORY

In the present paper the concepts of monads and morphisms of monads are applied to the investigation of some relations between modals, modes and semilattices. The key to our theory is the observation that the construction of a modal of non--empty subalgebras of a mode (A,F) is very similar to the construction of a compact space of closed subsets of a compact space. The latter produces a Vietoris monad on the category of compact spaces [3], which is a restriction of a powerset monad to closed subsets. We introduce several monads on modes similar to the Vietoris monad, among them the monad of non--empty subalgebras and monad of sinks. The main aim of this note is to describe modals, complete modals and complete distributive F-lattices as Eilenberg-Moore categories of monads on modes. In particular, we obtain a category of complete Heyting algebras as category of algebras for a monad of sinks on category of meet semilattices.

1. Preliminaries

A mode (A,F) is an algebra that is both idempotent (meaning that for each element x of A, the singleton $\{x\}$ is a subalgebra of (A,F)) and entropic (meaning each n-ary operation

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$f : A^n \to A$ in F is a homomorphism). A <u>modal</u> (A,F,+) is an algebra for which the reduct (A,F) is a mode, (A,+) is a join-semilattice, and the operations F distributive over +. Such structure yields a partial order $\leq$ on the set A on setting $x \leq y$ iff $x + y = y$. A join semilattice (A,+) for which every non-empty subset has a supremum (sup) is called a <u>complete</u> join semilattice. An algebra structure (A,F) on a set is said to be <u>completely distributive</u> over a <u>complete</u> join semilattice structure (A,+) on A iff for each n-ary operation f in F, $1 \leq j \leq n$, and non-empty subset X of A,

$$f(x_1,\ldots,x_{j-1},\sup X,x_{j+1},\ldots,x_n) =$$

$$= \sup\{f(x_1,\ldots,x_{j-1},x,x_{j+1},\ldots,x_n);\ x \in X\}.$$

A modal (A,F,+) is called a <u>complete modal</u> iff (A,+) is a complete join semilattice and (A,F) is completely distributive over (A,+). A modal morphism $h : (A,F,+) \to (C,F,+)$ is said to be <u>complete</u> if h preserves suprema of non-empty subsets.

A monad $\underline{T} = (T,\varrho,\mu)$ on a category $\underline{A}$ consists of a functor $T : \underline{A} \to \underline{A}$ and two natural transformations $\varrho : Id_A \to T$ and $\mu : TT \to T$ such that

$$\mu \cdot \varrho_T = id_T = \mu \cdot T\varrho \quad \text{and} \quad \mu \cdot \mu_T = \mu \cdot T\mu.$$

ϱ is called the <u>unit</u> and μ the <u>counit</u> of T. An <u>algebra</u> (A,a) for the monad $\underline{T}$, or briefly T-algebra, consists of an object A and a morphism $a : TA \to A$ such that $a \cdot \varrho_A = id_A$ and $a \cdot Ta = a \cdot \mu_A$. The morphism a is called the structure of the T-algebra (A,a). A morphism $h : (A,a) \to (B,b)$ of T-algebras is a morphism $h : A \to B$ of $\underline{A}$ such that $h \cdot a = b \cdot Th$. We denote by $\underline{A}^T$ the category of algebras for this monad, and by $U^T : \underline{A}^T \to \underline{A}$ the forgetful functor, with $U^T(A,a) = A$ for T-algebra (A,a).

If two monads $\underline{T}$ over $\underline{A}$ and $\underline{T}'$ over $\underline{B}$ are given, then we define a morphism $(G,\pi) : \underline{T}' \to \underline{T}$ of monads as a pair consisting of a functor $G : \underline{A} \to \underline{B}$ and a natural transformation $\pi : T'G \to GT$ such that

$$\pi\cdot\eta'_G = G\eta \quad \text{and} \quad \pi\cdot\mu'_G = G\mu\cdot\pi_T\cdot T'\pi.$$

Putting $H(A,a) = (GA, Ga\cdot\pi_A)$ for T-algebra (A,a), $H(h) = Gh$ for a morphism of T-algebras, defines a functor $H:\underline{A}^T \longrightarrow \underline{B}^{T'}$ such that $U^{T'}H = G\,U^T$. Conversely, every functor $H:A^T \longrightarrow B^{Z'}$ such that $U^{T'}H = G\,U^T$ for some $G:\underline{A} \longrightarrow \underline{B}$ is of this form.
We denote by P the covariant powerset functor on the category Set of sets and mappings, with PA = set of all non-empty subsets of A, for a set A, and Ph(X) the direct image of X by h, for $h:A \longrightarrow B$ and $X \subseteq A$. Singletons and set unions define natural transformations $s:Id_{Set} \longrightarrow P$ and $u:PP \longrightarrow P$. This define a monad $\underline{P} = (P,s,u)$ on sets which we call the powerset monad.

A monad $\underline{P}_f$ of finite non-empty subsets is obtained by restricting the data of the powerset monad to finite subsets.
We denote by cJSL the category of P-algebras; this is the category of complete join semilattices. An algebra structure $a:PA \longrightarrow A$ on a set A is a sup map with $a(X) = \sup X$ for non-empty subset X of A. A morphism of P-algebras is a map which preserves suprema of non-empty subsets. The category of P_f-algebras is the category JSL of join-semilattices and semilattice homomorphisms.

For all unexplained terms concerning category theory the reader is referred to [1].

2. Some properties of modals

Let $\underline{M}$ be a variety of modes of type F satisfying a given set of linear identities. Here and in what follows we will identify classes of algebras with their corresponding full subcategories of a category of all algebras of type F and F-homomorphisms. We denote by $\underline{Ml}$ and $\underline{cMl}$ classes (categories) of all modals (complete modals) whose mode reducts are in $\underline{M}$.

If (A,F) is a mode, let SA ($S_f A$) denote the set of all non-empty (finitely generated) subalgebras of (A,F). For n-ary operation f of F, denote by $f(B_1,B_2,\dots,B_n)$ the complex f-product of elements $B_1,B_2,\dots,B_n$ of SA. By entropicity,

$f(B_1,\dots,B_n)$ is itself a non-empty subalgebra of (A,F). If every subalgebra B_i is finitely generated by a finite set X_i (i.e. $B_i = cl_A(X_i)$) then

$$f(B_1,\dots,B_n) = cl_A\, f(X_1,\dots,X_n).$$

This makes SA and S_fA into an algebra (SA,F) and (S_fA,F).

Proposition 2.1 (cf. Proposition 313 and 335 of [2]). Let $(A,F)\in \underline{M}$. Then

(i) $(S_fA,F,+)\in \underline{Ml}$, where $B_1 + B_2 = cl_A(X_1 \cup X_2)$ for $B_i = cl_A(X_i)$ and finite subsets X_i $(i = 1,2)$.

(ii) $(SA,F,+)\in \underline{cMl}$, where for every non-empty subset Z of SA, $\sup Z = cl_A(\cup Z)$.

Proposition 2.2. If $(A,F,+)\in \underline{Ml}$ and $t(x_1,\dots,x_n)$ is a term of type F, then for every $a_1,\dots,a_n$ from A, $t(a_1,\dots,a_n)\leqq a_1 + \dots + a_n$.

Proof. It suffices to apply sum-superiority Lemma 318 of [2] and induction on degree of t.

Proposition 2.3. If $B = cl_A(X)$ is a subalgebra of a modal $(A,F,+)$, generated by a non-empty subset X of A, and $a\in A$ then B is upper bounded by a iff X is upper bounded by a.

Proof. The "only if" part is obvious. Conversely, if $b\in B$, then $b = t(x_1,\dots,x_n)$ for some term t and $x_1,\dots,x_n\in X$. Then by 2.2 $b\leqq x_1 + \dots + x_n\leqq a$.

This proposition gives us a motivation to consider the sup maps $\sup : SA \longrightarrow A$ and $\sup : S_fA \longrightarrow A$ for a complete modal and modal respectively.

Corollary 2.4.

(i) If $(A,F,+)\in \underline{Ml}$ then for every $B\in S_fA$ with $B = cl_A(X)$ and X-finite, sup B exists in A and sup B = sup X. Moreover, the sup map $\sup : S_fA \longrightarrow A$ is a modal homomorphism.

(ii) If $(A,F,+)\in \underline{cMl}$, then for every $B\in SA$ with $B = cl_A(X)$, sup B = sup X, and $\sup : SA \longrightarrow A$ is a complete modal homomorphism.

P r o o f . We prove only the first part of Corollary. Let us observe that sup : $S_f A \longrightarrow A$ is well defined, by 2.3.. Let f be an n-ary operation and $B_i = cl_A(X_i) \in S_f A$ for a finite subsets X_i $(i = 1,\dots,n)$. Then, using Proposition 2.3

$$\sup f(B_1,\dots,B_n) = \sup cl_A(f(X_1,\dots,X_n)) = \sup(X_1,\dots,X_n) =$$
$$= \sup\{f(x_1,\dots,x_n);\ x_i \in X_i \quad \text{for} \quad i = 1,\dots,n\} =$$
$$= f(\sup X_1, \sup X_2,\dots,\sup X_n) = f(\sup B_1,\dots,\sup B_n).$$

3. Monads of subalgebras over modes

We introduce a modal monad $\underline{S}_f$ and a complete modal monad $\underline{S}$ on the category $\underline{M}$ of modes. We show that subalgebras generated by subsets provide a morphism of monads, from the powerset monad $\underline{P}$ to the modal monad $\underline{S}$, and we show that the categories of S-algebras and cMl are isomorphic.

P r o p o s i t i o n 3.1. If $(A,F) \in \underline{M}$, then putting $\varrho_A(x) = \{x\}$ for $x \in A$ defines a homomorphism $\varrho_A : (A,F) \longrightarrow (SA,F)$, and set unions of subalgebras define a homomorphism $\mu_A : (SSA,F) \longrightarrow (SA,F)$. Moreover, $\mu_A(C) = cl_A(\cup Z)$ if $C = cl_{SA}\{cl_A(X);\ X \in Z\}$ for some subset Z of PA.

P r o o f . Since (A,F) is an idempotent algebra, it follows that ϱ_A is well-defined homomorphism. If K is a subalgebra of SA, then for every n-ary operation f of F and $x_1,\dots,x_n \in \cup K$ there are subalgebras $B_1,\dots,B_n \in K$ of A such that $x_i \in B_i$ for $i = 1,\dots,n$, and then $f(x_1,\dots,x_n) \in f(B_1,\dots,B_n) \in K$. Thus $\cup K$ is a subalgebra of (A,F) and $\mu_A : SSA \longrightarrow SA$ is well-defined map. Directly from Corollary 2.4(ii), for a complete modal (SA,F,+), we obtain that μ_A is a complete modal homomorphism.

If $h:(A,F) \longrightarrow (C,F)$ is a homomorphism of modes, then the direct image Ph maps subalgebras into subalgebras. We define Sh as the restriction of Ph to subalgebras, and we note that

$$Sh(f(B_1,\dots,B_n)) = f(Sh(B_1),\dots,Sh(B_n))$$

for every n-ary operation f of F and $B_1,\dots,B_n \in SA$. This shows that Sh is a complete modal homomorphism. Thus, we obtain a functor $S:\underline{M} \to \underline{M}$.

T h e o r e m 3.2. The functor S and the maps ϱ_A and μ_A define a monad S on the category $\underline{M}$, and subalgebras generated by subsets in modes define a morphism $(U,cl):\underline{P} \to \underline{S}$ of monads, where $U:\underline{M} \to$ Set is a forgetful functor with $U(A,F) = A$ for a mode (A,F).

P r o o f . The first part follows immediately from the corresponding facts for the powerset monad $\underline{P}$ on sets, by restriction to subalgebras. If $h:(A,F) \to (C,F)$ is a mode homomorphism, then $Sh(cl_A(X)) = cl_C(Ph(X))$, thus cl_A is natural in (A,F). In our case, the two conditions for a morphism of monads become

$$cl_A \cdot s_A = U\varrho_A \text{ and } cl_A \cdot u_A = U\mu_A \cdot cl_{SA} \cdot S\, cl_A \quad \text{for a mode } (A,F).$$

The first condition is clearly valid. If Z is a non-empty subset of PA and $B \in SA$, then by Corollary 2.4 we have $cl_A(\cup Z) \subseteq B$ iff $\cup Z \subseteq B$ iff $cl_A(X) \subseteq B$ for every $X \in Z$ iff $C \subseteq B$ for every $C \in cl_{SA}\{cl_A(X);\ X \in Z\}$ iff $\bigcup cl_{SA}\{cl_A(X);\ x \in Z\} \subseteq B$ and the second condition is also satisfied.

R e m a r k 3.3. The modal monad $\underline{S}_f$ is obtained by restriction of Proposition 3.1 and Theorem 3.2 to the finitely generated subalgebras. In this case, the pair (U,cl) forms a morphism of monads from $\underline{P}_f$ to $\underline{S}_f$. Note that the monad $\underline{S}_f$ is induced by a pair of adjoint functors with $U^{\dagger}:\underline{Ml} \to \underline{M}$ - the forgetful functor - as a right adjoint (cf. Theorem 351 of [2]).

C o r o l l a r y 3.4. Every S-algebra (S_f-algebra) ((A,F),k) is a complete join-semilattice (join-semilattice) with $\sup X = k\, cl_A(X)$ for non-empty (finite) subset X of A. Morphisms of S-algebras (S_f-algebras) preserve these suprema.

P r o o f . The morphism (U,cl) of monads induces a funtor $H:\underline{M}^S \to \mathrm{Set}^P = \mathrm{cJSL}$ (cf. Preliminaries).

T h e o r e m 3.5. There is an isomorphism of categories $\underline{M}^S$ ($\underline{M}^S f$) and $\underline{cMl}$ ($\underline{Ml}$) which preserves underlying mode structure and underlying complete join-semilattice (join-semilattice) structure.

P r o o f . We define a functor $R:\underline{M}^S \longrightarrow \underline{cMl}$ as follows: $R((A,F),k) = (A,F,+)$ cf. 3.4 on objects and $R(h) = h$ on morphisms. It is easy to see that R is well-defined and injective on objects. We show that R is surjective on objects. Let $(A,F,+) \in \underline{cMl}$, then the sup map sup : $SA \longrightarrow A$ is a mode homomorphism, by 2.4(ii) and it preserves all existing suprema. Now consider two conditions $\sup\cdot \varrho_A = id_A$ and $\sup \cdot S\sup = \sup\cdot \mu_A$.

The first is obvious. Let $C \in SSA$ with $C = cl_{SA}\{cl_A(x);\ X\in Z\}$ for some non-empty subset Z of PA. Using 2.4(ii) and 3.2 we have

$$\sup \mu_A(C) = \sup cl_A(\cup Z) = \sup \cup Z = \sup \{\sup X;\ X\in Z\} =$$
$$= \sup\{\sup cl_A(X);\ X\in Z\} = \sup cl_{SA}\{\sup cl_A(X);\ X\in Z\} =$$
$$= \sup S\sup cl_{SA}\{cl_A(X);\ X\in Z\} = \sup S\sup C.$$

As every $C\in SSA$ is of the form $cl_{SA}\{cl_A(X);\ X\in Z\}$ for some Z (for example Z = C), we see that ((A,F),sup) is an S-algebra. From 3.4 it follows that $R((A,F),\sup) = (A,F,+)$.

We show now that R is full and faithful. The functor preserves underlying maps, thus it is faithful. Now consider a morphism $h:R((A,F),k) \longrightarrow ((C,F),l)$ in $\underline{cMl}$ for S-algebras $((A,F),k)$ and $((C,F),l)$. Using the fact that h is a complete modal homomorphism and 3.4, we have $hk(B) = h \sup B = \sup Ph(B) = \sup Sh(B) = l\ Sh(B)$ for every $B \in SA$. Thus R is full.

In the similar way one can prove that the categories $\underline{M}^S f$ and $\underline{Ml}$ are isomorphic.

We note some consequences of Theorem 3.5:

C o r o l l a r y 3.6. Let $(A,F)\in \underline{M}$ and $(A,+) \in cJSL((A,+)\in JSL)$. Then $(A,F,+)\in \underline{cMl}$ ($(A,F,+)\in \underline{Ml}$) if and only if the following two conditions holds:

(i) $\sup X = \sup cl_A(X)$ for every non-empty (finite) subset of A.

(ii) the suprema of non-empty (finitely generated) subalgebras define a mode homomorphism $\sup:(SA,F) \longrightarrow (A,F)$ $(\sup:(S_f A,F) \longrightarrow (A,F))$.

4. Monads of sinks over modes

We show that sinks in modes define a monad $\underline{D}$ on modes and we introduce the category of D-algebras.

All modes (A,F) in this section are plural (i.e. F is non-empty and an arity of every f of F is at least two).

Definition 4.1. A subset S of A is said to be a sink if for every n-ary operation f and every $x_1,\ldots,x_n \in A$, the following holds: if $x_i \in S$ for some $i = 1,\ldots,n$ then $f(x_1,\ldots,x_n) \in S$. Directly from the defining property of sinks and Proposition 366 of [2] we have

Proposition 4.2. If (A,F) is a mode, then

(i) the union and intersection of sinks of (A,F) is also a sink,

(ii) the set of all non-empty sinks DA of (A,F) is a complete submodal of $(SA,F,+)$ with $f(S_1,\ldots,S_n) = S_1 \cap \ldots \cap S_n$ for every n-ary operation f and $S_1,\ldots,S_n \in DA$. Moreover, $(DA,F,\cup)$ is a complete distributive F-lattice i.e. (DA,F) is an F-semilattice and both partial orders $\leqq_F$ and $\leqq_+$ coincides (cf. [2]).

For a given element a of (A,F) let $[a]$ denote the principal sink generated by a. One sees easily that the map $p_A: A \longrightarrow DA$ which assigns to every $a \in A$ the principal sink $[a]$ is a mode homomorphism. Moreover, every sink S in (A,F) has a unique form

$$S = \bigcup\{[a];\ a \in S\}.$$

Proposition 4.3. For every mode homomorphism $h:(A,F) \longrightarrow (C,F)$, the map $Dh:(DA,F,\cup) \longrightarrow (DC,F,\cup)$ with $Dh(S) = \bigcup\{[h(a)];\ a \in S\}$ for $S \in DA$ is a complete modal homomorphism which preserves finite infima.

P r o o f . Follows from the Proposition 368 of [2].

Let q_A:DDA ⟶ DA denote the set-union map for a given mode (A,F).

T h e o r e m 4.4. The functor D:$\underline{M}$ ⟶ $\underline{M}$, defined above, and the maps q_A and p_A define a monad $\underline{D}$ = (D,p,q) on $\underline{M}$, and sinks generated by subalgebras define a morphism $(Id_{\underline{M}},\varphi):\underline{S} \longrightarrow \underline{D}$ of monads.

P r o o f . The first part of theorem follows easily from 4.3 and the definition of q. Note that q_A is a complete modal homomorphism which preserves finite intersections. For every mode (A,F) from $\underline{M}$, we define φ_A:SA ⟶ DA as follows: $\varphi_A(B) = \bigcup\{[b];\ b \in B\}$ for every subalgebra $B \in SA$. It is easy to see that this map is natural in (A,F) and that both conditions of monad morphism are satisfied.

Thus we obtain a functor $H:\underline{M}^D \longrightarrow \underline{M}^S$, given by H((A,F),k) = = ((A,F),k•φ_A) for an D-algebra ((A,F),k), so every D-algebra is a complete modal such that

$$k(\bigcup\{[b];\ b\in B\}) = \sup B$$

for every subalgebra $B \in SA$. Moreover, k preserves all non-empty suprema and finally k(S) = sup S for every sink $S \in DA$.

P r o p o s i t i o n 4.5. For every D-algebra ((A,F,k), k:DA ⟶ A preserves finite infima.

P r o o f . A D-algebra structure k of (A,F) must be of the form sup, by remarks above. In order to show that $\sup S_1 \cap S_2 = \inf\{\sup S_1, \sup S_2\}$ for every S_1,S_2 from DA, observe that Dsup([S]) = [sup S] and $q_A([S]) = S$ for every $S \in DA$. Note first that $\sup S_1 \cap S_2$ is a lower bounded of $\sup S_1$ and $\sup S_2$. Now suppose that $z \in A$ is another lower bounded of them, then $\sup S_i = \sup[z] \cup [\sup S_i]$ for i = 1,2. Using the condition sup•Dsup = sup•q_A we have

$$\sup S_1 \cap S_2 = \sup q_A([S_1] \cap [S_2]) =$$

$$= \sup \mathrm{Dsup}\Big[[z] \cup [\sup S_1]\Big] \cap \Big[[z] \cup [\sup S_2]\Big] =$$

$= \sup([z] \cup [\sup S_1]) \cap ([z]\ [\sup S_2]) = \sup \mathrm{Dsup}(\big[[z]\big] \cup [S_1 \cap S_2]) =$

$$= \sup q_A(\big[[z]\big] \cup [S_1 \cap S_2]) =$$

$$= \sup [z] \cup (S_1 \cap S_2) = \sup\{z, \sup S_1 \cap S_2\}$$

and then $z \leqq \sup S_1 \cap S_2$.

T h e o r e m 4.6. For a complete modal $(A,F,+)$, the following are equivalent:

(i) The sup map $\sup : DA \longrightarrow A$ is a D-algebra structure.

(ii) $(A,F,+)$ is a complete distributive F-lattice.

P r o o f . By 4.5, the sup map preserves finite infima for D-algebra $((A,F),\sup)$. Hence for every n-ary operation f and $x_1,\ldots,x_n \in A$ we have

$$f(x_1,\ldots,x_n) = f(\sup[x_1],\ldots,\sup[x_n]) =$$

$$= \sup f([x_1],\ldots,[x_n]) = \sup[x_1] \cap \ldots \cap [x_n] =$$

$$= \inf\{\sup[x_1],\ldots,[\sup[x_n]]\} = \inf\{x_1,\ldots,x_n\}.$$

Thus (i) implies (ii). If (ii) is satisfied, then the map $\sup : DA \longrightarrow A$ is a well-defined complete modal homomorphism. It remains to show that $\sup \cdot p_A = id_A$ and $\sup \cdot \mathrm{Dsup} = \sup \cdot q_A$. The first condition is obvious. If $\underline{S}$ is a sink in (DA,F), then $\mathrm{Dsup}(\underline{S})$ is a sink in (A,F) generated by the elements $\sup S$ for $S \in \underline{S}$ and

$$\sup \mathrm{Dsup}(\underline{S}) = \sup\{\sup S;\ S \in \underline{S}\}.$$

On the other hand, $\sup q_A(\underline{S}) = \sup(\bigcup \underline{S})$. This implies that the second condition is also satisfied.

C o r o l l a r y 4.7. For a category of plural modes $\underline{M}$, the category of D-algebras $\underline{M}^D$ is the same as the category of complete distributive F-lattices which mode reducts are in $\underline{M}$.

P r o o f . This follows immediately from Theorem 4.6 for objects. One sees easily that a mode homomorphism is a morphism of D-algebras iff it is a complete modal homomorphism of corresponding complete distributive F-lattices.

R e m a r k 4.8. Similar to the modal monad $\underline{S}_f$ we can consider the monad $\underline{D}_f$ of finitely generated sinks. In this case, the category of D_f-algebras is isomorphic to the category of distributive F-lattices in the sense of [2].

5. Example

In this section we give a direct proof that the category cHA of complete Heyting algebras and maps preserving finite infima and arbitrary suprema is isomorphic to the category MSL^D of D-algebras for a monad of sinks on the category of meet semilattices MSL.

At first we note that if category of plural modes M is defined by a given set of balanced identities (i.e. linear and regular) then the empty sink can be included in DA for a mode (A,F) from $\underline{M}$. Let us note that a partially ordered set $(A,\leqq)$ is said to be a complete Heyting algebra if it is a complete lattice and for every subset X of A and an element a of A the following distributive law holds

$$\inf\{a, \sup X\} = \sup\{\inf\{a,x\};\ x \in X\}.$$

Let $\underline{M}$ = MSL be a category of all semilattices and semilattice homomorphisms. If $(A,\cdot) \in$ MSL, then a subset S of A is a sink iff it is a lower set i.e. $S = {\downarrow}S = \{y \in A;\ y \leqq x$ for some $x \in S\}$. Then the functor part D of the monad $\underline{D} = (D,p,q)$ on MSL assigns to a semilattice $(A,\cdot)$ the semilattice of all lower subsets of $(A,\cdot)$ (the empty set is included). If $h:(A,\cdot) \longrightarrow (C,\cdot)$ is a semilattice homomorphism, then Dh(S) is a lower set $\{y \in C;\ y \leqq hx$ for some $x \in S\}$. The unit p_A assigns to $x \in A$ the principal lower set $[x] = \{y \in A;\ y \leqq x\}$. The counit q_A assigns to a lower subset $\underline{S}$ in $(DA,\cap)$ the set-union of elements of $\underline{S}$.

L e m m a 5.1. Suppose $((A,\cdot),k)$ is a D-algebra for a semilattice $(A,\cdot)$. Then k(S) = sup S for a lower subset S of A. As a consequence $(A,\leqq)$ is a complete Heyting algebra.

P r o o f . Let $a = k(S)$. Suppose $x \leqq z$ for every $x \in S$. Then $S \subseteq [z] = p_A(z)$, hence $a = k(S) \leqq k\ p_A(z) = id_A(z) = z$. Now let $x \in S$, then $p_A(x) = [x] \subset S$, hence $x = k\ p_A(x) \leqq k(S) = a$. Thus, every subset X of A has a supremum with sup X = sup $\downarrow X$, and $(A,\leqq)$ is a complete lattice. The sup map sup : DA $\longrightarrow$ A is a meet semilattice homomorphism i.e. $(A,\cdot)$ is completely distributive over (A,sup). Finally $(A,\leqq)$ is a complete Heyting algebra.

As a consequence a semilattice $(A,\cdot)$ admits at most one D-algebra structure.

L e m m a 5.2. Suppose $(A,\leqq)$ is a complete Heyting algebra, then $k:(DA,\cap) \longrightarrow (A,\cdot)$ defined by $k(S) = \sup \hat{S}$ for $S \in DA$ is a D-algebra structure on $(A,\cdot)$.

P r o o f . (i) Clearly $k \cdot p_A = id_A$.

(ii) In order to show that $k \cdot DK = k \cdot q_A$, let $\underline{S}$ be a lower subset of $(DA,\cap)$, i.e. $\underline{S} \in DDA$, then $k\ q_A(\underline{S}) = k(\cup\ \underline{S}) = \sup\{\sup S;\ S \in \underline{S}\}$. On the other hand, $Dk(\underline{S}) = \{x \in A;\ x \leqq k(S)$ for some $S \in \underline{S}\}$; thus $k\ Dk(\underline{S}) = \sup\{\sup S;\ S \in \underline{S}\}$.

It is immediate from the proofs of 5.1 and 5.2 that a semilattice homomorphism $h:(A,\cdot) \longrightarrow (C,\cdot)$ is a D-morphism iff it preserves suprema of all subsets.

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INSTITUTE OF MATHEMATICS, NICOLAUS COPERNICUS UNIVERSITY,
87-100 TORUŃ, POLAND
Received June 20, 1986.