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DEFINITION OF PARTIALITY

We call definition of partiality a condition which is satisfied in a class of partial algebras if the class contains no nontrivial total algebras. In other words one can say that the condition of partiality is a condition, which cannot be satisfied by a nontrivial total algebra.

The aim of this paper is to formulate the definition of partiality for a variety (i.e. an equationally definable class) of partial algebras. We introduce this definition in the form of a Mal'cev condition.

When dealing with equations in partial algebras one must decide what kind of equations will be considered. We have chosen so called weak equations. For strong and existence equations our problem is easily solvable (see end of this paper).

A signature is a pair $\langle F,n\rangle$ with F a set of operation symbols and n a mapping from F into the set of nonnegative integers. $\underline{A} = \langle A,(f^{\underline{A}}:f\in F)\rangle$ is a partial algebra of signature $\langle F,n\rangle$ if A is a nonempty set and $f^{\underline{A}}$ a partial $n(f)$-ary operation in A. Fix a signature $\langle F,n\rangle$. In the following we deal with partial algebras of signature $\langle F,n\rangle$ only and thus the term "algebra" will always be used in the sense "partial algebra of signature $\langle F,n\rangle$". If for some reason we will need an

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algebra in the usual sense we will use the term "total algebra". We call an algebra discrete if all its operations are empty. If p is a polynomial symbol (see e.g. [3]) $p^{\underline{A}}$ will denote the partial function induced in an algebra $\underline{A}$ by p and $dom(p^{\underline{A}})$ will be its domain. In the case that p is n-ary and $\underline{a} \in dom(p^{\underline{A}})$ where $\underline{a}$ is a n-tuple of elements of A, we will say that p is defined on $\underline{a}$ in the algebra $\underline{A}$.

D e f i n i t i o n . Let p and q be n-ary polynomial symbols and let $\underline{A} = \langle A, (f^{\underline{A}} : f \in F) \rangle$ be an algebra. An equation $p = q$ is a weak equation in an algebra $\underline{A}$ (or is weakly satisfied in $\underline{A}$), in symbols $p \overset{w}{=} q$ $(\underline{A} \models p \overset{w}{=} q)$, if for any n-tuple $\underline{a}$ of elements of A we have: if $\underline{a} \in dom(p^{\underline{A}}) \cap dom(q^{\underline{A}})$ then $p^{\underline{A}}(\underline{a}) = q^{\underline{A}}(\underline{a})$.

Let for a class of algebras K:

$$Eq_w K = \left\{ p \overset{w}{=} q \;:\; \underline{A} \models p \overset{w}{=} q \text{ for all } \underline{A} \in K \right\}.$$

A class of algebras V is a WE-variety if it is the class of all algebras satisfying all weak equations from a set E of equations, that is

$$V = \left\{ \underline{A} \;:\; \underline{A} \models p \overset{w}{=} q \text{ for all } p \overset{w}{=} q \in E \right\}.$$

E x a m p l e . Let $\underline{A} = \langle A, (f^{\underline{A}} : f \in F) \rangle$ be an algebra and p an n-ary polynomial symbol. Let x be a variable with no occurence in p. Then if $\underline{A}$ is nontrivial we have: $\underline{A} \models p \overset{w}{=} x$ iff $dom(p^{\underline{A}}) = \emptyset$. Indeed, assume $\underline{a} \in dom(p^{\underline{A}})$ and let b, c be any elements of A with $b \neq c$. Then we have $p^{\underline{A}}(\underline{a}) = b$ and $p^{\underline{A}}(\underline{a}) = c$, a contradiction. The "if" part is obvious. □

It is easy to see that all weak equations reduce to usual equations in total algebras. It means that if $\underline{A}$ is a total algebra then $\underline{A} \models p \overset{w}{=} q$ iff $\underline{A} \models p = q$. Hence we have:

P r o p o s i t i o n 1. If V is a WE-variety then the class $T(V) = \{\underline{A} : \underline{A} \in V$ and $\underline{A}$ is a total algebra$\}$ is a variety of total algebras.

P r o p o s i t i o n 2. If V is a WE-variety, then all one-elements total algebras are in V. □

It is easy to see that in every weak variety there is a discrete algebra; it follows from the observation that any discrete algebra satisfies all weak equations exept, the following: $x \overset{W}{=} y$ with x and y different variables and any discrete one-element algebra satisfies all weak equations.

In the example above we have shown that it is possible to express the fact that a polynomial function is never defined, except possibly for the trivial one-element algebras using a weak equation. We can consider an equation of the form $p \overset{W}{=} x$ with x a variable not in p, as a definition of partiality, but it is not enough. There are varieties containing no nontrivial total algebras - such that no equation of the above form is satisfied in any algebra in this variety.

To obtain a full characterization of WE-varieties with no nontrivial total algebras we need the following two lemmas.

Lemma 1. Let $E = Eq_w V$ for some WE-variety V. Then an equation $p \overset{W}{=} q$ is provable from E using Birkhoff's rules (see [1]) iff it is provable from E using transitivity as the only rule.

Proof. Let $e_1,\dots,e_n$ be a proof of $p \overset{W}{=} q$ from $E = Eq_w V$ (with V a WE-variety), i.e. $e_1 \in E$, $e_n \equiv p \overset{W}{=} q$ and for $i = 1,2,3,\dots,n$ $e_i \in E$ or is corollary of Birkhoff's rule with premises in $\{e_1,\dots,e_{i-1}\}$. We show how to change this proof to get one with transitivity as the only rule.

Let e_k be the first equation in the proof of $p \overset{W}{=} q$, such that $e_k \notin E$ and is obtained from previous equations by Birkhoff's rules without transitivity (if there is no such e_k then either $p \overset{W}{=} q \in E$ or $e_1,\dots,e_n$ is the required proof; in both cases we do nothing). If e_k is obtained by substituting a term $r(x)$ for a variable y throughout an equation e_i, for some $i < k$, we have $e_i \equiv p'(y) \overset{W}{=} q'(y)$ and there are terms $s_1,\dots,s_m$ such that

$$p'(y) \overset{W}{=} s_1(y), s_1(y) \overset{W}{=} s_2(y),\dots, s_m(y) \overset{W}{=} q'(y) \in E.$$

Hence

$$p'(r(x)) \overset{W}{=} s_1(r(x)), s_1(r(x)) \overset{W}{=} s_2(r(x)),\dots, s_m(r(x)) \overset{W}{=} q'(r(x)) \in E$$

since E is closed under substitution. We include this sequence of equations into the proof of $p \overset{w}{=} q$ as a proof of e_k by transitivity.

For other Birkhoff's rules an analogous argument will do since E is closed under all Birkhoff's rules but transitivity (see [4]).

Following the described procedure we get the required proof of $p \overset{w}{=} q$ which completes the proof of the lemma since the "if" part is obvious. □

L e m m a 2. Let V be a WE-variety which contains no nontrivial total algebras. Then there is a sequence of n binary polynomial symbols $p_1, p_2, \ldots, p_n$ for some natural number n such that the following equations are in $Eq_w V$:

$$(*) \qquad x \overset{w}{=} p_1(x,y), p_1(x,y) \overset{w}{=} p_2(x,y), \ldots, p_n(x,y) \overset{w}{=} y$$

where x and y are different variables.

P r o o f . Let x and y be different variables. Let $\underline{A}$ be a total algebra in V. By assumption it is trivial so there is a sequence of equations in the set $Eq_w V$ from which $x = y$ can be deduced using all Birkhoff's rules for total algebras. By Lemma 1 we can choose this sequence to be as follows:

$$(**) \qquad x = p'_1, p'_1 = p'_2, \ldots, p'_n = y \in Eq_w V$$

Now we will substitute y for all variables except x in all equations (**) to obtain

$$x = p_1(x,y), p_1(x,y) = p_2(x,y), \ldots, p_n(x,y) = y.$$

Then $p_1, p_2, \ldots, p_n$ is required sequence. □

Now we are ready to state the main theorem: definition of partiality in the form of Mal'cev condition.

T h e o r e m . A WE-variety V contains no nontrivial total algebras iff there are two polynomial symbols p (unary) and q (binary) such that

$$p(x) \overset{w}{=} x, q(x,y) \overset{w}{=} y, p(x) \overset{w}{=} q(x,y) \in Eq_w V$$

where x and y are different variables.

P r o o f . Let x and y be different variables. The "if" part is obvious since if $\underline{A}$ is a total algebra in V then we have $\underline{A} \models x = p(x)$, $\underline{A} \models p(x) = q(x,y)$ and $\underline{A} \models q(x,y) = y$. Then by transitivity $\underline{A} \models x = y$. So $\underline{A}$ is trivial.

To prove the "only if" part let $p_1(x,y),\dots,p_n(x,y)$ be the sequence of binary polynomial symbols which exists by Lemma 2. Let m be the maximal number such that the variable y does not occur in p_i for $i < m+1$ (if there is no such number i.e. if y occurs in p_1 then take $m = 0$ and consider the polynomial symbol $p_0 \equiv x$). We claim that

$$p(x) \equiv p_m(p_{m-1}(\dots p_1(x)\dots)) \quad \text{and}$$

$$q(x,y) \equiv p_{m+1}(x, p_{m+2}(x, \dots p_n(x,y)\dots))$$

are the required polynomial symbols.

For the proof assume that $\underline{A} \in V$ and $a \in A$ is such that $a \in \mathrm{dom}(p^{\underline{A}})$. Then we have

$$a \in \mathrm{dom}(p_1^{\underline{A}}), p_1^{\underline{A}}(a) \in \mathrm{dom}(p_2^{\underline{A}}), \dots, p_{m-1}^{\underline{A}}(\dots p_1^{\underline{A}}(a)\dots) \in \mathrm{dom}(p_m^{\underline{A}}).$$

Using equations (*) from Lemma 2 we infer that $p_1^{\underline{A}}(a) = a$, thus $a \in \mathrm{dom}(p_2^{\underline{A}})$, and $p_2^{\underline{A}}(a) = p_1^{\underline{A}}(a) = a$ again by Lemma 2. In the same way we have $a \in \mathrm{dom}(p_3^{\underline{A}})$ and $p_3^{\underline{A}}(a) = a$. Following this procedure we finally obtain $a \in \mathrm{dom}(p_m^{\underline{A}})$ and $p_m^{\underline{A}}(a) = a$. Since a is arbitrary we have: if $a \in \mathrm{dom}(p^{\underline{A}})$ then $p^{\underline{A}}(a) = a$ by construction of p. So $\underline{A} \models p(x) \overset{W}{=} x$.

On the other hand if $\langle a,b \rangle \in \mathrm{dom}(q^{\underline{A}})$ we have: $\langle a,b \rangle \in \mathrm{dom}(p_n^{\underline{A}}(x,y))$ and by equations (*) from Lemma 2 $p_n^{\underline{A}}(a,b) = b$. Further $\langle a,b \rangle = \langle a, p_n^{\underline{A}}(a,b) \rangle \in \mathrm{dom}(p_{n-1}^{\underline{A}})$ and $p_{n-1}^{\underline{A}}(a,b) = p_n^{\underline{A}}(a,b) = b$ again by Lemma 2. As before after n-m steps we obtain $\langle a,b \rangle \in \mathrm{dom}(p_{m+1}^{\underline{A}})$ and $p_{m+1}^{\underline{A}}(a,b) = b$. So we have proved that if $\langle a,b \rangle \in \mathrm{dom}(q^{\underline{A}})$ then $q^{\underline{A}}(a,b) = b$. So $\underline{A} \models q(x,y) \overset{W}{=} y$.

Moreover if $a, b \in A$ are such that $a \in \mathrm{dom}(p^{\underline{A}})$ and $\langle a,b\rangle \in \mathrm{dom}(q^{\underline{A}})$ we can prove that $a \in \mathrm{dom}(p_m^{\underline{A}})$ and $\langle a,b\rangle \in \mathrm{dom}(p_{m+1}^{\underline{A}})$ using a similar argument. So we get

$$p^{\underline{A}}(a) = p_m^{\underline{A}}(a) = p_{m+1}^{\underline{A}}(a,b) = q^{\underline{A}}(a,b)$$

by equations (*) from Lemma 2 and construction of p and q. Since a, b are arbitrary we have: $\underline{A} \models p(x) \overset{w}{=} q(x,y)$ which completes the proof. □

R e m a r k s . For an equation $p = q$ we say that $p = q$ is a strong in an algebra $\underline{A}$ if it is a weak equation and $\mathrm{dom}(p^{\underline{A}}) = \mathrm{dom}(q^{\underline{A}})$ (see [5]). We say that an equation $p = q$ is existence equation if both $p^{\underline{A}}$ and $q^{\underline{A}}$ are everywhere defined and equal (see [2]). It is easily seen that for strong and existence equations both Proposition 1 and 2 hold.

In all SE- and E-varieties (classes defined by strong and existence equations respectively) which contain at least one nontrivial algebra there is a nontrivial total one. The situation with discrete algebra is somewhat more complicated. There is such an algebra in an E-variety iff it is the variety of all algebras of the same type. For SE-varieties we have: there is no discrete algebra in a variety V iff there is a strong equation of the form $p = x$ (where p is not a variable) which holds in all algebras in V.

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