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W. Bartol, D. Niwiński, L. Rudak

# THE SEMILATTICE OF INNER EXTENSIONS OF A PARTIAL ALGEBRA

One of the specific aspects of partial algebras is the fact that each of them induces a non-trivial - unless the algebra be total - set of algebras obtained by extending the operations of the original partial algebra on its carrier. This set admits a natural semi-lattice structure. We give here a complete characterization of such semilattices.

A type of algebras is a pair $\langle F,n\rangle$ consisting of a set $F$ and an arity mapping $n:F \longrightarrow N$ where $N$ is the set of non-negative integers. Then a partial algebra of type $\langle F,n\rangle$ is a pair $\underline{A} = \langle A,(f^{\underline{A}} : f \in F)\rangle$ where for each $f \in F$, $f^{\underline{A}}$ is a partial $n(f)$-ary operation on the set $A$, i.e. $f^{\underline{A}}:\mathrm{dom}(f^{\underline{A}}) \longrightarrow A$ with $\mathrm{dom}(f^{\underline{A}}) \subset A^{n(f)}$ being the domain of $f^{\underline{A}}$. Quite often we shall not mention explicitly the type of an algebra, whenever it can be deduced from the context. An algebra $\underline{A}$ is total when for each $f \in F$, $\mathrm{dom}(f^{\underline{A}}) = A^{n(f)}$ and $\underline{A}$ is discrete when for each $f \in F$, $\mathrm{dom}(f^{\underline{A}}) = \emptyset$.

If $\underline{A} = \langle A,(f^{\underline{A}} : f \in F)\rangle$ and $\underline{B} = \langle B,(f^{\underline{B}} : f \in F)\rangle$ are partial algebras, then $\underline{B}$ is an inner extension of $\underline{A}$ iff $A = B$ and for all $f \in F$, $f^{\underline{A}} \subset f^{\underline{B}}$. An inner extension of $\underline{A}$ is said to be an inner completion iff it is total.

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In [2] J.Schmidt mentions the ordered set of all partial-algebraic structures of the given type on a set A. Such a structure is also briefly mentioned in [1], where Burmeister states that it has the structure of a conditionally complete lower semilattice (where conditional completeness refers to upper bounds). Using the terminology introduced above, the set considered by Schmidt and Burmeister is seen to be the semilattice of all inner extensions of a discrete algebra of a given type. This construction can be easily generalised for an arbitrary partial algebra $\underline{A}$: let Ext($\underline{A}$) be the set of all inner extensions of a partial algebra $\underline{A}$ and for any $\underline{B},\underline{C} \in$ Ext($\underline{A}$) let

$$\underline{B} \underset{e}{\subset} \underline{C} \quad \text{iff} \quad \underline{C} \text{ is an inner extension of } \underline{B}$$

(iff $f^{\underline{B}} \subset f^{\underline{C}}$ for all $f \in F$).

P r o p o s i t i o n . $\langle \mathrm{Ext}(\underline{A}), \underset{e}{\subset} \rangle$ is a complete lower semilattice.

P r o o f . It is easily seen that the g.l.b. $\bigwedge \underline{B}_i$ of a family $\{\underline{B}_i : i \in I\}$ of inner extensions of $\underline{A}$ is defined by

$$f^{\wedge \underline{B}_i} = \bigcap \{ f^{\underline{B}_i} : i \in I \}, \quad \text{for each} \quad f \in F.$$

Let $\cap$ be the binary g.l.b. operation in Ext($\underline{A}$) corresponding to $\underset{e}{\subset}$. Thus each partial algebra $\underline{A}$ induces a semilattice $\langle \mathrm{Ext}(\underline{A}), \cap \rangle$ of its inner extensions. The following theorem characterizes those associated semilattices.

T h e o r e m 1. A lower semilattice $\langle P, \wedge \rangle$ is isomorphic to $\langle \mathrm{Ext}(\underline{A}), \cap \rangle$ for some partial algebra $\underline{A}$ iff

(i) each element of P is contained in a maximal element,

(ii) there is a cardinal $\underline{m}$ such that each maximal principal ideal in $\langle P, \wedge \rangle$ is isomorphic to $2^{\underline{m}}$ (ordered by inclusion),

(iii) there exists a partition $\{A_i : i \in \underline{m}\}$ of the set of all atoms of $\langle P, \wedge \rangle$ such that card($A_i$) = card($A_j$) for all $i,j \in \underline{m}$ and for any $\mathcal{C} \subset \bigcup \{A_i : i \in \underline{m}\}$, sup($\mathcal{C}$) exists in $\langle P, \wedge \rangle$ iff card($\mathcal{C} \cap A_i$) $\leqslant 1$ for each $i \in \underline{m}$.

P r o o f . Necessity of (i)-(iii).

The three conditions (i)-(iii) are preserved under semilattice isomorphisms, thus it is enough to show that for any partial algebra A, $\langle Ext(\underline{A}), \cap \rangle$ satisfies (i)-(iii). Let $\underline{A} = \langle A, (f^{\underline{A}} : f \in F) \rangle$ be a partial algebra and let $\underline{Ext}(\underline{A}) = \langle Ext(\underline{A}), \cap \rangle$. The maximal elements in $\underline{Ext}(\underline{A})$ are exactly the inner completions of $\underline{A}$. Clearly every inner extension of $\underline{A}$ is contained in some inner completion of $\underline{A}$, which proves (i).

A principal ideal in $\underline{Ext}(\underline{A})$ is maximal iff it is generated by an inner completion of $\underline{A}$. If $(\underline{A}']$ is the maximal principal ideal generated by the inner completion $\underline{A}'$, then each $\underline{B} \in (\underline{A}']$ is uniquely determined by the domains of operations in $\underline{B}$, since all the operations in $\underline{B}$ must be compatible with those in $\underline{A}'$. Thus for any $\underline{B}, \underline{C} \in (\underline{A}']$, $\underline{B} \subseteq_e \underline{C}$ iff for each $f \in F$, $dom(f^{\underline{B}}) \subset dom(f^{\underline{C}})$. This implies that $\langle (\underline{A}'], \subseteq_e \rangle \cong \langle 2^{\underline{m}}, \subset \rangle$ where

$$\underline{m} = card(\bigcup \{(A^{n(f)} - dom(f^{\underline{A}})) \times \{f\} : f \in F\}).$$

The atoms in $\underline{Ext}(\underline{A})$ are the one-point inner extensions of $\underline{A}$, i.e.

$\underline{B}$ is an atom iff $card(\bigcup \{(dom(f^{\underline{B}}) - dom(f^{\underline{A}})) \times \{f\} : f \in F\}) = 1$.

Let $D = \bigcup \{(A^{n(f)} - dom(f^{\underline{A}})) \times \{f\} : f \in F\}$. For each $(d,f) \in D$ define:

$A(d,f) = \{\underline{B} \in Ext(\underline{A}) : \underline{B}$ is an atom and $d \in dom(f^{\underline{B}}) - dom(f^{\underline{A}})\}$.

Then $card(A(d,f)) = card(A)$ (the bijection is given by assigning to each $c \in A$ the atom $\underline{B}$ with $f^{(\underline{B})}(d) = c$) and $card(\{A(d,f) : (d,f) \in D\}) = card(D) = \underline{m}$, since for any $(d,f), (e,g) \in D$, $(d,f) \neq (e,g)$ implies $A(d,f) \cap A(e,g) = \emptyset$. Moreover, each atom $\underline{B}$ is in $A(d,f)$ for some $(d,f) \in D$. Thus $\{A(d,f) : (d,f) \in D\}$ is a partition of the set of all atoms of $Ext(\underline{A})$ into $\underline{m}$ equipotent blocks.

Assume $\mathcal{C} \subset \bigcup \{A(d,f) : (d,f) \in D\}$ and suppose $\underline{B}_1, \underline{B}_2 \in \mathcal{C} \cap A(d,f)$, (with $\underline{B}_1 \neq \underline{B}_2$) for some $(d,f) \in D$. Then $d \in dom(f^{\underline{B}_i}) - dom(f^{\underline{A}})$ for i=1,2 and $f^{\underline{B}_1}(d) \neq f^{\underline{B}_2}(d)$, which implies there is no

$\underline{B} \in \mathrm{Ext}(\underline{A})$ with $\underline{B}_1 \subseteq_e \underline{B}$ and $\underline{B}_2 \subseteq_e \underline{B}$. Thus $\mathcal{C}$ has no upper bound in $\underline{\mathrm{Ext}}(\underline{A})$. On the other hand, if $\mathrm{card}(\mathcal{C} \cap A(d,f)) \leqslant 1$ for each $(d,f) \in D$, then the algebra $\underline{B} = \langle A, (f^{\underline{B}} : f \in F)\rangle$ such that $f^{\underline{B}} = \bigcup \{ f^{\underline{C}} : \underline{C} \in \mathcal{C} \}$ for all $f \in F$ is a well-defined inner extension of $\underline{A}$ and $\underline{B} = \sup(\mathcal{C})$ in $\underline{\mathrm{Ext}}(\underline{A})$.

Sufficiency of (i)-(iii).

Let $\underline{P} = \langle P, \wedge \rangle$ be a semilattice satisfying (i)-(iii) and let $\underline{m}$ be the cardinal of condition (ii). Let $\{A_i : i \in \underline{m}\}$ be the partition of the set of atoms of $\underline{P}$ assumed in (iii). For any $p \in P$, define $C_p$ as the set of all atoms contained in p. We prove first some simple consequences of (i)-(iii).

L e m m a 1. For every element p of P, $p = \sup(C_p)$.

P r o o f . Observe that by (i) and (ii) each $p \in P$ belongs to some maximal principal ideal isomorphic to $\langle 2^{\underline{m}}, \subset \rangle$, which proves the lemma.

L e m m a 2. For any $p, q \in P$, $\sup(C_p) = \sup(C_q)$ iff $C_p = C_q$.

P r o o f . Obviously $C_p = C_q$ implies $\sup(C_p) = \sup(C_q)$. Assume now $p = \sup(C_p) = \sup(C_q) = q$. Let $I_p$ be any maximal principal ideal with $p \in I_p$ (see (i)) and let h be an isomorphism of $\langle I_p, \wedge \rangle$ onto $\langle 2^{\underline{m}}, \subset \rangle$ (which exists by (ii)). Since both $C_p \subset I_p$ and $C_q \subset I_p$, we have in $2^{\underline{m}}$ that $h(p) = \sup(h(C_p)) = \sup(h(C_q))$, which clearly implies $C_p = C_q$.

L e m m a 3. For each $p \in P$, $C_p \cap A_i$ has at most one element for any $i \in \underline{m}$.

P r o o f . $\sup(C_p)$ exists by Lemma 1, so apply condition (iii).

Now, let F be any set with $\mathrm{card}(F) = \underline{m}$ and take $n : F \to N$ such that $n(f) = 1$ for all $f \in F$. From now on we shall represent the partition of the set of atoms of $\underline{P}$ as $\{A_f : f \in F\}$. Moreover, let A be a set such that $\mathrm{card}(A) = \mathrm{card}(A_f)$ for all $f \in F$ (see condition (iii)) and let $p_f : A \to A_f$ be a bijection for every $f \in F$. We consider an algebra $\underline{A}$ of type $\langle F, n \rangle$ such that:

$$\underline{A} = \langle A, (f^{\underline{A}} : f \in F)\rangle \text{ and for any } f \in F,\ \mathrm{dom}(f^{\underline{A}}) = A - \{u\}$$

for some fixed $u \in A$.

For any $p \in P$ let $\underline{B}_p$ be an inner extension of $\underline{A}$ such that for any $f \in F$:

(*) $\quad u \in \mathrm{dom}(f^{\underline{B}_p})$ and $f^{\underline{B}_p}(u) = v$ iff $p_f(v) \in C_p$.

It follows in particular that $u \in \mathrm{dom}(f^{\underline{B}_p})$ iff $A_f \cap C_p \neq \emptyset$. Define $H: P \longrightarrow \mathrm{Ext}(\underline{A})$ so that $H(p) = \underline{B}_p$. We claim that H is a semilattice isomorphism from $\underline{P}$ onto $\underline{\mathrm{Ext}}(\underline{A})$.

Injectivity of H:

Assume $p, q \in P$ and $p \neq q$. Then by lemmas 1 and 2, $C_p \neq C_q$. Let e.g. $a \in C_p - C_q$. Then there is exactly one $g \in F$ with $a \in A_g$ and by the definition of $\underline{B}_p$, $u \in \mathrm{dom}(g^{\underline{B}_p})$ and $g^{\underline{B}_p}(u) = p_g^{-1}(a)$. On the other hand, if $u \in \mathrm{dom}(g^{\underline{B}_q})$, then $g^{\underline{B}_q}(u) \neq p_g^{-1}(a)$, since $a \notin C_q$. Thus $\underline{B}_p \neq \underline{B}_q$.

Surjectivity of H:

Let $\underline{B} \in \mathrm{Ext}(\underline{A})$ and define $F_{\underline{B}} = \{f \in F : u \in \mathrm{dom}(f^{\underline{B}})\}$. Let

$$p = \sup\{p_f(f^{\underline{B}}(u)) : f \in F_{\underline{B}}\} .$$

Observe that $p$ is well-defined, since for any $f, g \in F_{\underline{B}}$ with $f \neq g$, $p_f(f^{\underline{B}}(u))$ and $p_g(g^{\underline{B}}(u))$ belong to disjoint sets $A_f$ and $A_g$, correspondingly, and thus by (iii) the sup exists in $\underline{P}$. We shall prove that $\underline{B} = \underline{B}_p = H(p)$. Since both $\underline{B}$ and $\underline{B}_p$ are inner extensions of $\underline{A}$, it is enough to prove that

$$u \in \mathrm{dom}(f^{\underline{B}}) \text{ and } f^{\underline{B}}(u) = v \text{ iff } u \in \mathrm{dom}(f^{\underline{B}_p}) \text{ and } f^{\underline{B}_p}(u) = v$$

(iff $p_f(v) \in C_p$, by (*)).

But by Lemma 2, $C_p = \{p_f(f^{\underline{B}}(u)) : f \in F_{\underline{B}}\}$, thus by injectivity of $p_f$ for any $f \in F$, $p_f(v) \in C_p$ iff $f \in F_{\underline{B}}$ and $v = f^{\underline{B}}(u)$ iff $u \in \mathrm{dom}(f^{\underline{B}})$ and $f^{\underline{B}}(u) = v$. Thus $\underline{B} = \underline{B}_p$.

Monotonicity of H and $H^{-1}$:

Let $\leq_P$ be the ordering induced in P by $\wedge$ and assume $p \leq_P q$. Then $C_p \subset C_q$ and consequently if $u \in \mathrm{dom}(f^{\underline{B}_p})$ and

$f^{\underline{B}_p}(u) = v$ then $p_f(v) \in C_p$ which implies $p_f(v) \in C_q$; hence, by $(*)$, $u \in dom(f^{\underline{B}_q})$ and $f^{\underline{B}_q}(u) = v$. This proves $\underline{B}_p \subseteq_e \underline{B}_q$ in $\mathbf{Ext}(\underline{A})$.

On the other hand, assume $\underline{B}_p \subseteq_e \underline{B}_q$ and let $a \in C_p$. Since $a \in A_f$ for some (unique) $f \in F$, we have $a = p_f(p_f^{-1}(a)) \in C_p$, which implies $u \in dom(f^{\underline{B}_p})$ and $f^{\underline{B}_p}(u) = p_f^{-1}(a)$. By assumption, also $u \in dom(f^{\underline{B}_q})$ and $f^{\underline{B}_q}(u) = p_f^{-1}(a)$, consequently $a \in C_q$ (by definition of $\underline{B}_q$). Hence $C_p \subset C_q$ which proves $p \leqslant_P q$. Thus H is a semilattice isomorphism. This proves

$$\langle P, \wedge \rangle \cong \langle Ext(\underline{A}), \cap \rangle .$$

Theorem 1 suggests that the structure of $\mathbf{Ext}(\underline{A})$ depends only on two cardinal numbers related with the algebra $\underline{A}$: the cardinality of its universe and the cardinality of the disjoint union of the complements of domains of operations in $\underline{A}$. This leads to a representation of semilattices of extensions by simpler structures.

Let X,Y be any sets. Define Fun(X,Y) as the set of all functions on subsets of X with values in Y (or equivalently, of all partial functions from X into Y). This set is ordered by inclusion and the latter is a lower semilattice ordering in Fun(X,Y), since for any $f,g \in Fun(X,Y)$, the g.l.b. of $\{f,g\}$ is the function $f \cap g$. Let $\underline{Fun}(X,Y) = \langle Fun(X,Y), \cap \rangle$.

L e m m a 4. If card(X) = card(Z) and card(Y) = = card(W) then

$$\underline{Fun}(X,Y) \cong \underline{Fun}(Z,W).$$

The proof may be left as a simple exercise in set theory.

L e m m a 5. For any partial algebra $\underline{A}$ there exist cardinal numbers $\underline{m},\underline{n}$ such that $\mathbf{Ext}(\underline{A}) \cong \underline{Fun}(\underline{m},\underline{n})$.

P r o o f . Let $\underline{A} = \langle A, (f^{\underline{A}} : f \in F) \rangle$ be an arbitrary partial algebra (of type $\langle F,n \rangle$). Let $\underline{n}$ = card(A) and $\underline{m}$ = card(D) with

$$D = \bigcup\{(A^{n(f)} - \mathrm{dom}(f^{\underline{A}}))\times\{f\} : f\in F\}.$$

We shall prove that $\underline{\mathrm{Ext}}(\underline{A}) \cong \underline{\mathrm{Fun}}(\underline{m},\underline{n})$.

By lemma 4, it is sufficient to prove $\underline{\mathrm{Ext}}(\underline{A}) \cong \underline{\mathrm{Fun}}(D,A)$. For any $\underline{B}\in \mathrm{Ext}(\underline{A})$ let $D_{\underline{B}} = \bigcup\{(\mathrm{dom}(f^{\underline{B}}) - \mathrm{dom}(f^{\underline{A}}))\times\{f\} : f\in F\}$. Thus $D_{\underline{B}}$ represents, intuitively speaking, the field onto which the structure of $\underline{B}$ has been extended with respect to that of $\underline{A}$; clearly $D_{\underline{B}}\subset D$. Define $G : \mathrm{Ext}(\underline{A}) \longrightarrow \mathrm{Fun}(D,A)$ so that

$$\text{for } \underline{B}\in \mathrm{Ext}(\underline{A}), \mathrm{dom}(G(\underline{B})) = D_{\underline{B}} \text{ and}$$

$$\text{for } (d,f)\in D_{\underline{B}},\ G(\underline{B})(d,f) = f^{\underline{B}}(d).$$

The injectivity of G follows easily from its definition: if $\underline{B} \neq \underline{C}$ then either $D_{\underline{B}} \neq D_{\underline{C}}$ or there exists a $(d,f)\in D_{\underline{B}} = D_{\underline{C}}$ such that $f^{\underline{B}}(d) \neq f^{\underline{C}}(d)$.

To show that G is surjective, take any $t\in \mathrm{Fun}(D,A)$ and let $\mathrm{dom}(t) = T\subset D$. Define an inner extension $\underline{C}$ of $\underline{A}$ so that for any $f\in F$ and $d\in A^{n(f)} - \mathrm{dom}(f^{\underline{A}})$,

$$d\in \mathrm{dom}(f^{\underline{C}}) \text{ and } f^{\underline{C}}(d) = a \text{ iff } (d,f)\in T \text{ and } t(d,f) = a.$$

An easy verification proves that $G(\underline{C}) = t$.

It remains to prove that G and $G^{-1}$ are monotone. But if $\underline{B},\underline{C}\in \mathrm{Ext}(\underline{A})$ and $\underline{B} \subseteq_e \underline{C}$, then $D_{\underline{B}}\subset D_{\underline{C}}$ and for any $(d,f)\in D_{\underline{B}}$, $f^{\underline{B}}(d) = f^{\underline{C}}(d)$. Thus $G(\underline{B})\subset G(\underline{C})$. Reversing this argument we prove the monotonicity of $G^{-1}$.

The following lemma yields the inverse of lemma 5.

L e m m a 6. For any cardinal numbers $\underline{m}$, $\underline{n}$ there exists a partial algebra $\underline{A}$ such that $\underline{\mathrm{Ext}}(\underline{A}) \cong \underline{\mathrm{Fun}}(\underline{m},\underline{n})$.

P r o o f . If $\underline{n} = 0$, then any total algebra $\underline{A}$ will do, since in that case $\mathrm{Ext}(\underline{A})$ is a singleton, too. Assume now $\underline{n} \neq 0$ and let u be any fixed element of $\underline{n}$. Define an algebra $\underline{A} = \langle \underline{n}, (f^{\underline{A}} : f\in \underline{m})\rangle$ so that for all $f\in \underline{m}$, $\mathrm{dom}(f^{\underline{A}}) = \underline{n} - \{u\}$ and $f^{\underline{A}}(a) = a$ for any $a\in \mathrm{dom}(f^{\underline{A}})$. For $\underline{B}\in \mathrm{Ext}(\underline{A})$, let $F_{\underline{B}} = \{f\in \underline{m} : u\in \mathrm{dom}(f^{\underline{B}})\}$. Now to any inner extension $\underline{B}$ of $\underline{A}$ assign a function $t_{\underline{B}} : F_{\underline{B}} \longrightarrow \underline{n}$ such that $t_{\underline{B}}(f) = f^{\underline{B}}(u)$ for

$f \in F_{\underline{B}}$. The correspondence $\underline{B} \longmapsto t_{\underline{B}}$ is an isomorphism of $\underline{Ext}(\underline{A})$ onto $\underline{Fun}(\underline{m},\underline{n})$.

The following lemma completes the result of lemma 4.

L e m m a 7. For any non-empty sets X,Y,Z,W $\underline{Fun}(X,Y) \cong \underline{Fun}(Z,W)$ iff $card(X) = card(Z)$ and $card(Y) = card(W)$.

From Lemmas 5, 6, 7 we deduce immediately the following.

T h e o r e m 2. Let $\underline{A} = \langle A,(f^{\underline{A}} : f \in F)\rangle$ and $\underline{B} = \langle B,(f^{\underline{B}} : f \in F')\rangle$ be two partial algebras of type $\langle F,n\rangle$ and $\langle F',n'\rangle$, correspondingly. If $\underline{n}_{\underline{A}} = card(A)$ and $\underline{n}_{\underline{B}} = card(B)$ and

$$\underline{m}_{\underline{A}} = card\Big(\bigcup\big\{(A^{n(f)} - dom(f^{\underline{A}}))\times\{f\} : f \in F\big\}\Big) \text{ and}$$

$$\underline{m}_{\underline{B}} = card\Big(\bigcup\big\{(B^{n'(f)} - dom(f^{\underline{B}}))\times\{f\} : f \in F'\big\}\Big)$$

then $\underline{Ext}(\underline{A}) \cong \underline{Ext}(\underline{B})$ iff $\underline{n}_{\underline{A}} = \underline{n}_{\underline{B}}$ and $\underline{m}_{\underline{A}} = \underline{m}_{\underline{B}}$ or both $\underline{A}$ and $\underline{B}$ are total.

Given two partial algebras $\underline{A}$ and $\underline{B}$, the existence of an isomorphism between their extension semilattices can be expressed by a Cantor-Bernstein-type condition. Define an upper embedding of a poset $\langle X,\leq_X\rangle$ into a poset $\langle Y,\leq_Y\rangle$ to be an injective order-homomorphism $h: X \longrightarrow Y$ such that

(i) $h^{-1}$ is an order homomorphism on $h(X)$;

(ii) if a is maximal in $\langle X,\leq_X\rangle$, then h(a) is maximal in $\langle Y,\leq_Y\rangle$;

(iii) h(X) is convex in Y i.e. if $b \in Y$ and for some $a_1,a_2 \in X$, $h(a_1) \leq_Y b \leq_Y h(a_2)$, then $b \in h(X)$.

When such an embedding exists, we say that $\langle X,\leq_X\rangle$ is upper-embedabble in $\langle Y,\leq_Y\rangle$.

T h e o r e m 3. Let $\underline{A}$ and $\underline{B}$ be arbitrary partial algebras. Then

$$\underline{Ext}(\underline{A}) \cong \underline{Ext}(\underline{B}) \quad \text{iff}$$

each of the semilattices $\underline{Ext}(\underline{A})$ and $\underline{Ext}(\underline{B})$ is upper-embeddable in the other.

By Lemmas 5 and 6 we can reduce this theorem to the following form.

T h e o r e m 3. For any cardinal numbers $\underline{m},\underline{n},\underline{m}',\underline{n}'$ :

$$\underline{Fun}(\underline{m},\underline{n}) \cong \underline{Fun}(\underline{m}',\underline{n}') \quad \text{iff}$$

each of the semilattices $\underline{Fun}(\underline{m},\underline{n})$, $\underline{Fun}(\underline{m}',\underline{n}')$ is upper-embedabble in the other.

This theorem in turn is a direct consequence of the following.

L e m m a 8. For any cardinals $\underline{m},\underline{n},\underline{m}',\underline{n}'$: if $\underline{m}\leq\underline{m}'$ and $\underline{n}\leq\underline{n}'$, then $\underline{Fun}(\underline{m},\underline{n})$ is upper-embedabble in $\underline{Fun}(\underline{m}',\underline{n}')$. The inverse implication holds, when $m \neq 0 \neq n$.

P r o o f . Let $\underline{m}\leq\underline{m}'$ and $\underline{n}\leq\underline{n}'$. If $\underline{n}' = 0$, then both $\underline{Fun}(\underline{m},\underline{n})$ and $\underline{Fun}(\underline{m}',\underline{n}')$ are trivial one-element semilattices, so we may assume $\underline{n}' \neq 0$. Let also $w_0$ be an arbitrary fixed function from $\underline{m}'-\underline{m}$ into $\underline{n}'$. For any $w\in \mathrm{Fun}(\underline{m},\underline{n})$ define $h(w)\in \mathrm{Fun}(\underline{m}',\underline{n}')$ so that

$$\mathrm{dom}(h(w)) = \mathrm{dom}(w)\cup(\underline{m}'-\underline{m}) \text{ and for any } a\in\mathrm{dom}(h(w)),$$

$$h(w)(a) = \begin{cases} w_0(a) & \text{if } a\in\underline{m}'-\underline{m} \\ w(a) & \text{if } a\in\mathrm{dom}(w). \end{cases}$$

Then the mapping $h:\mathrm{Fun}(\underline{m},\underline{n})\longrightarrow\mathrm{Fun}(\underline{m}',\underline{n}')$ is an upper-embedding of $\underline{Fun}(\underline{m},\underline{n})$ into $\underline{Fun}(\underline{m}',\underline{n}')$.

On the other hand, if $\underline{m}\neq 0\neq\underline{n}$ and $h:\mathrm{Fun}(\underline{m},\underline{n})\longrightarrow\mathrm{Fun}(m,n)$ is an upper embedding, take an arbitrary maximal element $a$ in $\underline{Fun}(\underline{m},\underline{n})$. Then $h(a)$ is maximal in $\underline{Fun}(\underline{m}',\underline{n}')$ and

$$h((a])\subset(h(a)],$$

which by lemma 6 and theorem 1 implies $2^{\underline{m}}\leq 2^{\underline{m}'}$ i.e. $\underline{m}\leq\underline{m}'$.

To prove that also $\underline{n}\leq\underline{n}'$ take any $b\in\underline{m}$ and consider any function $f\in\mathrm{Fun}(\underline{m},\underline{n})$ with $\mathrm{dom}(f) = \underline{m} - \{b\}$. Then $f$ has exactly $\underline{n}$ succesors in $\mathrm{Fun}(\underline{m},\underline{n})$ and these are maximal functions $f_x:\underline{m}\longrightarrow\underline{n}$ with $f\subset f_x$ and $f_x(b) = x$, for all $x\in\underline{n}$. By the definition of an upper embedding, $h(f)$ is covered by $h(f_x)$, $x\in\underline{n}$, which are all different and maximal in $\underline{Fun}(\underline{m}',\underline{n}')$, thus $\mathrm{dom}(h(f)) = \underline{m}' - \{b'\}$ for some $b'\in\underline{m}'$ and $h(f)$ has exactly $\underline{n}'$ succesors in $\underline{Fun}(\underline{m}',\underline{n}')$. Hence $\underline{n}\leq\underline{n}'$.

REFERENCES

[1] P. Burmeister: A Model Theoretic Oriented Approach to Partial Algebras, Akademie-Verlag, Berlin 1986.

[2] J. Schmidt: Allgemeine Algebra, Vorlesung, Math. Inst. Univ. Bonn, Bonn 1966 (typescript).

INSTITUTE OF MATHEMATICS, WARSAW UNIVERSITY, 00-901 WARSZAWA
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