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SEMIISOMORPHISM VS. ISOMORPHISM OF INNER EXTENSIONS OF A PARTIAL ALGEBRA

In [1] we introduced the semilattice of inner extensions, associated with a given partial algebra. Such semilattices have been completely characterized and it was shown that the information on the partial algebra which can be deduced from its semilattice of inner extensions is rather unsatisfactory and of a set-theoretic nature only. In this paper we define an order-compatible equivalence relation in a semilattice of inner extensions and we use it to investigate some properties of partial algebras. This relation - called semi-isomorphism - identifies two inner extensions of a partial algebra $\underline{A}$ whenever they are structurally very close ("almost" isomorphic).

We recall ([1]) that for any partial algebra $\underline{A}$ we can define its semilattice of inner extensions $\underline{\mathrm{Ext}}(\underline{A}) = \langle \mathrm{Ext}(\underline{A}), \subseteq_e \rangle$ where $\mathrm{Ext}(\underline{A})$ is the set of all inner extensions of $\underline{A}$ and $\underline{B} \subseteq_e \underline{C}$ means that $\underline{C}$ is an inner extension of $\underline{B}$, for any $\underline{B}, \underline{C} \in \mathrm{Ext}(\underline{A})$. An inner extension of $\underline{A} = \langle A, (f^{\underline{A}} : f \in F) \rangle$ is any partial algebra $\underline{B} = \langle B, (f^{\underline{B}} : f \in F) \rangle$ of the same type such that $A = B$ and for all $f \in F$, $f^{\underline{A}} \subset f^{\underline{B}}$. (We assume here that all algebras considered are of a fixed type $\langle F, n \rangle$, where F is a set of operation symbols and $n: F \longrightarrow N$ is an arity function

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into the set of all non-negative integers). A homomorphism of a partial algebra $\underline{B} = \langle B,(f^{\underline{B}} : f \in F)\rangle$ into a partial algebra $\underline{C} = \langle C,(f^{\underline{C}} : f \in F)\rangle$ is any mapping $h:B \longrightarrow C$ such that for any $f \in F$ and $\underline{a} \in dom(f^{\underline{B}})$ (where $dom(f^{\underline{B}}) \subset B^{n(f)}$ is the domain of the n(f)-ary partial operation $f^{\underline{B}}$ in B) we have $h \cdot \underline{a} \in dom(f^{\underline{C}})$ and $h(f^{\underline{B}}(\underline{a})) = f^{\underline{C}}(h \cdot \underline{a})$. A homomorphism h is closed iff $h \cdot \underline{a} \in dom(f^{\underline{C}})$ implies $\underline{a} \in dom(f^{\underline{B}})$ and h is an isomorphism iff it is closed and bijective (observe that a bijective homomorphism need not be an isomorphism). We denote by "$\cong$" the isomorphism relation between partial algebras.

Let $\underline{A}$ be any partial algebra. Define the relation $\lesssim$ in $Ext(\underline{A})$ as the transitive closure of the join of the extension relation $\underset{e}{\subset}$ and the isomorphism relation $\cong$ reduced to $Ext(\underline{A})$. The following lemma will be helpful in the characterization of $\lesssim$.

L e m m a 1. Let $h:A \longrightarrow B$ be an isomorphism of a partial algebra $\underline{A} = \langle A,(f^{\underline{A}} : f \in F)\rangle$ onto $\underline{B} = \langle B,(f^{\underline{B}} : f \in F)\rangle$. Then for each $\underline{A}' \in Ext(\underline{A})$ there exists a unique $\underline{B}' \in Ext(\underline{B})$ such that h is an isomorphism of $\underline{A}'$ onto $\underline{B}'$. Similarly, for any $\underline{A}'$ such that $\underline{A} \in Ext(\underline{A}')$ there exists a unique $\underline{B}'$ such that $\underline{B} \in Ext(\underline{B}')$ and h is an isomorphism of $\underline{A}'$ onto $\underline{B}'$.

P r o o f . Let $\underline{A}'$ be a partial algebra such that either $\underline{A}' \in Ext(\underline{A})$ or $\underline{A} \in Ext(\underline{A}')$. For any $f \in F$ and $\underline{a} \in A^{n(f)}$ take

$$f^{\underline{B}'}(h \cdot \underline{a}) = \begin{cases} h(f^{\underline{A}'}(\underline{a})) & \text{if } \underline{a} \in dom(f^{\underline{A}'}) \\ \text{undefined} & \text{otherwise} \end{cases}$$

Being h an isomorphism from $\underline{A}$ onto $\underline{B}$, we obtain well-defined partial operations in $\underline{B}'$ consistent with the operations of $\underline{B}$, i.e. $\underline{B}' \in Ext(\underline{B})$ when $\underline{A}' \in Ext(\underline{A})$ and $\underline{B} \in Ext(\underline{B}')$ when $\underline{A} \in Ext(\underline{A}')$. Clearly this is the only possible definition of operations in B which makes h an isomorphism of $\underline{A}'$ onto $\underline{B}'$.

Observe that $(f^{\underline{B}'} : f \in F)$ is the final structure on B for the mapping h (in the sense of Bourbaki [2] and J.Schmidt [3]).

T h e o r e m 1. Let $\underline{B},\underline{C} \in \mathrm{Ext}(\underline{A})$. Then the following conditions are equivalent:

(i) $\underline{B} \lesssim \underline{C}$

(ii) there is a $\underline{D} \in \mathrm{Ext}(\underline{A})$ such that $\underline{B} \subset_e \underline{D} \cong \underline{C}$

(iii) there is a $\underline{D} \in \mathrm{Ext}(\underline{A})$ such that $\underline{B} \cong \underline{D} \subset_e \underline{C}$

(iv) there is a bijective homomorphism of $\underline{B}$ onto $\underline{C}$.

P r o o f . (i) $\longrightarrow$ (ii). If $\underline{B} \lesssim \underline{C}$, then by the definition of $\lesssim$ there exist partial algebras $\underline{B}_0,\underline{B}_1,\ldots,\underline{B}_n$ such that

$$\underline{B} \cong \underline{B}_0 \subset_e \underline{B}_1 \cong \ldots \subset_e \underline{B}_n \cong \underline{C}.$$

By iterative applications of lemma 1 (in the first step apply it to $\underline{B}$, $\underline{B}_0$ and $\underline{B}_1$) we get a partial algebra $\underline{D} \in \mathrm{Ext}(\underline{A})$ which is an inner extension of $\underline{B}$ isomorphic to $\underline{C}$.

(ii) $\longrightarrow$ (iii). Apply again lemma 1 to $\underline{D}$, $\underline{C}$ and $\underline{B}$ (with $\underline{D} \in \mathrm{Ext}(\underline{B})$ and $\underline{D} \cong \underline{C}$).

(iii) $\longrightarrow$ (iv). If h is an isomorphism of $\underline{B}$ onto $\underline{D}$, then $id_{\underline{D}} \cdot h$ is a bijective homomorphism of $\underline{B}$ onto $\underline{C}$ (where $id_{\underline{D}}$ is the identity embedding of $\underline{D}$ into $\underline{C}$).

(iv) $\longrightarrow$ (i). Let $\underline{A} = \langle A,(f^{\underline{A}} : f \in F)\rangle$. If h is a bijective homomorphism of $\underline{B}$ onto $\underline{C}$, then let $\underline{B}'$ be an inner extension of $\underline{B}$ such that for any $f \in F$ and $\underline{a} \notin \mathrm{dom}(f^{\underline{B}})$

$$f^{\underline{B}'}(\underline{a}) = \begin{cases} h^{-1}(f^{\underline{C}}(h \cdot \underline{a})) & \text{if } h \cdot \underline{a} \in \mathrm{dom}(f^{\underline{C}}) \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Then we have $\underline{B} \subset_e \underline{B}' \cong \underline{C}$ which by the definition of $\lesssim$ yields $\underline{B} \lesssim \underline{C}$.

It follows from the definition of the relation $\lesssim$ that it is a quasi-order in $\mathrm{Ext}(\underline{A})$. Let $\approx$ be the induced equivalence relation in $\mathrm{Ext}(\underline{A})$ i.e. for any $\underline{B},\underline{C} \in \mathrm{Ext}(\underline{A})$,

$$\underline{B} \approx \underline{C} \quad \text{iff} \quad \underline{B} \lesssim \underline{C} \text{ and } \underline{C} \lesssim \underline{B}$$

(or equivalently iff $\underline{C}$ is isomorphic to some extension of $\underline{B}$ and vice versa).

Observe that $\underline{B} \approx \underline{C}$ does not imply that $\underline{B}$ and $\underline{C}$ are isomorphic i.e. the existence of bijective homomorphisms both ways does not imply isomorphism. Let $\underline{A} = \langle N, s^{\underline{A}} \rangle$ with $n(s) = 1$ and $dom(s^{\underline{A}}) = \{3k : k \in N\} \cup \{3k+1 : k \in N-\{0\}\}$. Let

$$s^{\underline{A}}(n) = \begin{cases} n+1 & \text{for } n=3k \text{ for some } k \in N \\ n & \text{for } n=3k+1 \text{ for some } k \in N-\{0\}. \end{cases}$$

Take $\underline{B} = \langle N, s^{\underline{B}} \rangle \in Ext(\underline{A})$ with

$$dom(s^{\underline{B}}) = \{3k : k \in N\} \cup \{3k+1 : k \in N\}$$

and $s^{\underline{B}}(3k) = s^{\underline{B}}(3k+1) = 3k+1$ for all $k \in N$.

Then id_N is a bijective homomorphism of $\underline{A}$ onto $\underline{B}$. To get a bijective homomorphism from $\underline{B}$ onto $\underline{A}$ take any bijective mapping h' of $\{3k+2 : n \in N\}$ $(=N-dom(s^{\underline{B}}))$ onto $\{0,1\} \cup \{3k+2 : k \in N\}$ and define h as follows:

$$h(n) = \begin{cases} n+3 & \text{for } n \in dom(s^{\underline{B}}) \\ h'(n) & \text{for } n \notin dom(s^{\underline{B}}). \end{cases}$$

$\underline{A}$ and $\underline{B}$ are not isomorphic, since $1 \in s^{\underline{A}}(N) - dom(s^{\underline{A}})$, while $s^{\underline{B}}(N) - dom(s^{\underline{B}}) = \emptyset$.

In the sequel we shall investigate some conditions which imply the equality of the relations $\approx$ and $\cong$ or – looking at it from the other end – some consequences of the fact that this equality does not hold. Before that, observe that the relation $\lesssim$ in $Ext(\underline{A})/\approx$ such that

$$[\underline{B}] \lesssim [\underline{C}] \quad \text{iff } \underline{B} \lesssim \underline{C}$$

is an order in $Ext(\underline{A})/\approx$, which by theorem 1 can be equivalently defined by

$[B] \leq [C]$ iff there exists $\underline{B}' \in [\underline{B}]$ and $\underline{C}' \in [\underline{C}]$ with $\underline{B}' \subseteq_e \underline{C}'$.

Recall now that a subset $B \subset A$ is convex in an ordered set $\langle A, \leq \rangle$ iff for any $b, c \in B$ and $a \in A$, if $b \leq a \leq c$, then $a \in B$. The intersection of any family of convex subsets of A is again convex, so for any $C \subset A$ there exists a least convex subset containing C. We shall call it a convex closure of C.

T h e o r e m 2. Let A be any partial algebra. Then the equivalence classes of $\approx$ in $\mathrm{Ext}(\underline{A})$ are precisely the convex closures of the equivalence classes of $\cong$ in the ordered set $\langle \mathrm{Ext}(\underline{A}), \underset{e}{\subset} \rangle$.

P r o o f . We shall prove that for any $\underline{B} \in \mathrm{Ext}(\underline{A})$, $[\underline{B}]_{\approx}$ is the convex closure of $[\underline{B}]_{\cong}$. First observe that $[\underline{B}]_{\approx}$ is convex. Indeed, if for some $\underline{C},\underline{D} \in [B]_{\approx}$ and $\underline{E} \in \mathrm{Ext}(\underline{A})$ we have $\underline{C} \underset{e}{\subset} \underline{E} \underset{e}{\subset} \underline{D}$, then by theorem 1 there exists an algebra $\underline{D}' \in \mathrm{Ext}(\underline{A})$ such that $\underline{D} \underset{e}{\subset} \underline{D}' \cong \underline{C}$. Hence $\underline{C} \lessapprox \underline{E}$ and $\underline{E} \lessapprox \underline{C}$, which implies $\underline{E} \approx \underline{C}$ and $\underline{E} \in [\underline{B}]_{\approx}$.

To show that $[\underline{B}]_{\approx}$ is the convex closure of $[\underline{B}]_{\cong}$ it is now enough to prove that for any $\underline{C} \in [\underline{B}]_{\approx}$ there exist $\underline{B}', \underline{B}'' \in [\underline{B}]_{\cong}$ such that $\underline{B}' \underset{e}{\subset} \underline{C} \underset{e}{\subset} \underline{B}''$. But if $\underline{B} \approx \underline{C}$, then again by theorem 1 there exist $\underline{B}', \underline{B}'' \in \mathrm{Ext}(\underline{A})$ such that

$$\underline{B} \cong \underline{B}' \underset{e}{\subset} \underline{C} \underset{e}{\subset} \underline{B}'' \cong \underline{B},$$

which completes the proof.

From Theorem 2 we deduce the following simple corollaries:

C o r o l l a r y . If all the isomorphism classes in $\mathrm{Ext}(\underline{A})$ are convex, then $\approx = \cong$ in $\mathrm{Ext}(\underline{A})$.

C o r o l l a r y . If $\underline{A}$ has no isomorphic extensions, then $\approx = \cong$ in $\mathrm{Ext}(\underline{A})$.

T h e o r e m 3. Let $\underline{B},\underline{C} \in \mathrm{Ext}(\underline{A})$. If $\underline{B} \approx \underline{C}$ and $\underline{B} \ncong \underline{C}$, then there exists a $\underline{D} \in \mathrm{Ext}(\underline{A})$ such that $\underline{D}$ is a proper inner extension of $\underline{B}$ and $\underline{D} \cong \underline{B}$.

P r o o f . Assume $\underline{B} \approx \underline{C}$ and $\underline{B} \ncong \underline{C}$ for $\underline{B},\underline{C} \in \mathrm{Ext}(\underline{A})$. By theorem 1 there exist $\underline{B}', \underline{C}' \in \mathrm{Ext}(\underline{A})$ such that

$$(*) \qquad \underline{B} \underset{e}{\subset} \underline{C}' \cong \underline{C} \quad \text{and} \quad \underline{C} \underset{e}{\subset} \underline{B}' \cong \underline{B},$$

where $\underline{C}'$ is a proper extension of $\underline{B}$ (since $\underline{B} \ncong \underline{C}$). Moreover, there exists an algebra $\underline{D} \in \mathrm{Ext}(\underline{A})$ such that

$$\underline{C}' \underset{e}{\subset} \underline{D} \cong \underline{B}'$$

since, by (*) $\underline{C}' \subsetneq \underline{B}'$. Thus

$$\underline{B} \subset_e \underline{C}' \subset_e \underline{D} \cong \underline{B}$$

and D is a proper extension of $\underline{B}$.

C o r o l l a r y . Let $\underline{B} \in \mathrm{Ext}(\underline{A})$ for some partial algebra $\underline{A}$. If $[\underline{B}]_{\approx}$ contains two non-isomorphic extensions of $\underline{A}$, then for any $\underline{C} \in [\underline{B}]_{\approx}$ there exists a proper inner extension of $\underline{C}$ isomorphic with $\underline{C}$.

For the proof, observe that if $\underline{B}', \underline{B}'' \in [\underline{B}]_{\approx}$ and $\underline{B}' \not\cong \underline{B}''$, then for any $\underline{C} \in [\underline{B}]_{\approx}$, $\underline{C} \not\cong \underline{B}'$ or $\underline{C} \not\cong \underline{B}''$.

C o r o l l a r y . Let $\underline{B} \in \mathrm{Ext}(\underline{A})$. If $[\underline{B}]_{\approx}$ contains two non-isomorphic inner extensions of $\underline{A}$, then it contains a countable (strictly increasing with respect to $\subset_e$) chain $(\underline{B}_n : n \in N)$ of pairwise isomorphic inner extensions of $\underline{A}$.

The proof consists in an inductive generalization of the proof of Theorem 3.

The following is just an equivalent formulation of Theorem 3.

C o r o l l a r y . If no algebra $\underline{C} \in \mathrm{Ext}(\underline{A})$ is isomorphic to its proper inner extension, then $\approx = \cong$ in $\mathrm{Ext}(\underline{A})$.

L e m m a 2. Let $\underline{B}, \underline{C} \in \mathrm{Ext}(\underline{A})$ for some partial algebra $\underline{A}$. If $\underline{B} \approx \underline{C}$, then for all $f \in F$,

$$\mathrm{card}(\mathrm{dom}(f^{\underline{B}})) = \mathrm{card}(\mathrm{dom}(f^{\underline{C}})) \quad \text{and}$$

$$\mathrm{card}(A^{n(f)} - \mathrm{dom}(f^{\underline{B}})) = \mathrm{card}(A^{n(f)} - \mathrm{dom}(f^{\underline{C}}))$$

(where A is the carrier of all algebras involved).

P r o o f . If $\underline{B} \approx \underline{C}$, then by Theorem 1 there exists a $\underline{D} \in \mathrm{Ext}(\underline{A})$ such that $\underline{B} \cong \underline{D} \subset_e \underline{C}$. Thus for all $f \in F$,

$$\mathrm{card}(\mathrm{dom}(f^{\underline{B}})) = \mathrm{card}(\mathrm{dom}(f^{\underline{D}})) \leqslant \mathrm{card}(\mathrm{dom}(f^{\underline{C}})).$$

Analogously we obtain the inverse inequality and a dual argument proves the second claim of the lemma.

T h e o r e m 4. Let $\underline{B} \in Ext(\underline{A})$ for some partial algebra $\underline{A}$. If $[\underline{B}]_{\approx}$ contains two non-isomorphic extensions of $\underline{A}$, then there exists a fundamental operation $f \in F$ such that for any $\underline{C} \in [B]_{\approx}$ the domain of $f^{\underline{C}}$ is both infinite and co-infinite.

The proof follows directly from Lemma 2 and the first corollary following Theorem 3, since proper inner extension increases the cardinality of a finite domain of an operation or decreases the cardinality of a finite complement of a domain.

C o r o l l a r y . Let $\underline{A}$ be a partial algebra. If the domain of every operation in $\underline{A}$ is cofinite, then $\approx = \cong$ in $Ext(\underline{A})$.

This last result suggests that the "gap" between $\approx$ and $\cong$ may contain some information on the "degree of partiality" of a partial algebra (we may consider an algebra $\underline{A}$ with cofinite domains of operations as an algebra with little partiality). On the other hand, the case of two partial algebras $\underline{B}$ and $\underline{C}$, for which there exist bijective homomorphisms both ways but no isomorphism, can also be inscribed in the context presented in this paper, since both algebras may be considered as inner extensions of a common discrete algebra (on a set equipotent to the carriers of $\underline{B}$ and $\underline{C}$). This may be useful for the characterization of such cases.

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