DEMONSTRATIO MATHEMATICA
Vol. XX No 1-2 1987

Kazimiera Dyrda

NONE OF THE VARIETY $\mathbf{E}_n$, $n>2$, IS LOCALLY FINITE

In this paper we shall prove that there exists a 2-generated infinite algebra in $\mathbf{E}_2$.

Let us recall some fundamental definitions.

By a BCK-algebra we mean a general algebra $\underline{A} = \langle A,*,0\rangle$ of type $\langle 2,0\rangle$ satisfying the following axioms:

(1) $$((x*y)*(x*z))*(z*y) = 0$$

(2) $$(x*(x*y))*y = 0$$

(3) $$x*x = 0$$

(4) $$0*x = 0$$

(5) if $x*y = 0$ and $y*x = 0$, then $x = y$.

If $x\leq y$ means $x*y = 0$, then the relation $\leq$ is a partial ordering. Recall that a BCK-algebra $\underline{A}$ satisfies condition (S), if for any $a,b\in A$ in the set $\{x\in A : x*a\leq b\}$ a greatest element, $a\circ b$, exists uniquely. In this case we can extend the structure of BCK-algebra and add a new operation $\circ$ satisfying the identity:

(S) $$a*(b\circ c) = (a*b)*c.$$

This paper is based on the lecture presented at the Conference on Universal Algebra held at the Technical University of Warsaw (Wilga), May 22-25, 1986.

By a BCK^S-algebra we mean a general algebra $\underline{A} = \langle A,*,\circ,0\rangle$ of type $\langle 2,2,0\rangle$ satisfying (1)-(5) and (S). Let $\underline{A}$ be a BCK^S-algebra.

By the reflection of $\underline{A}$ (see [8]), which in the sequel will be denoted by $R(\underline{A})$, we shall mean the algebra $\langle A \cup R(A), *_R, \circ_R, 0\rangle$ where the set $R(A) = \{r_a : a \in A\}$ is the image of A under a bijection r such that $r_a \notin A$ for every $a \in A$. The zero-element in $R(\underline{A})$ is the same as in $\underline{A}$ and the operations $*_R$, $\circ_R$ are defined by the following conditions for every $a, b \in A$:

(i) $$a *_R b = a * b$$

(ii) $$r_a *_R r_b = b * a$$

(iii) $$r_a *_R b = r_{a \circ b}$$

(iv) $$b *_R r_a = 0$$

(v) $$a \circ_R b = a \circ b$$

(vi) $$r_a \circ_R r_b = r_0$$

(vii) $$a \circ_R r_b = r_b \circ_R a = r_{b * a}.$$

By E_n, $n \geq 1$, we denote the class of all BCK-algebras satisfying

(E_n) $$x * y^n = x * y^{n+1}, \text{ where}$$
$$x * y^1 = x * y$$
$$x * y^{k+1} = (x * y^k) * y, \quad k \geq 1.$$

The class E_n, $n \geq 1$, is a variety (see [1]). On the other hand, as it is well known, BCK-algebras do not form a variety ([7]). Moreover, E_1 is a well-known variety of algebras determined by the purely implicational fragment of the set INT of all theses of the intuitionistic propositional calculus.

The symbol $\mathbb{E}_n^S$ denotes a subclass of the class BCK^S formed by all BCK^S-algebras which satisfy (E_n).

From the result of Diego [2] and Popiel (see Jankov [6]) it is known that: any free algebra of the variety $\mathbb{E}_1$ free-generated by a finite set is finite.

We shall prove that the theorem of Diego and Popiel does not hold in $\mathbb{E}_n$, $n \geqslant 2$.

The following lemma holds.

L e m m a 1. If $n \geqslant 2$ and $\underline{A} \in \mathbb{E}_n^S$, then $R(\underline{A}) \in \mathbb{E}_n^S$.

Now we define terms α_n as follows:

$$\alpha_1 = p, \text{ where } p \text{ is variable}$$

$$\alpha_{n+1} = (\alpha_n \circ \alpha_n) * \alpha_n, \quad n \geqslant 1.$$

Then we obtain:

L e m m a 2. For every $i,j = 1,2,\ldots$ the following conditions hold:

(i) $BCK^S \models \forall p\,[\alpha_i \geqslant \alpha_{i+1}]$,

(ii) $BCK^S \models \forall p\,[\alpha_i = \alpha_{i+1} \Rightarrow \alpha_i = \alpha_{i+j}]$.

We omit easy proofs of Lemma 1 and Lemma 2.

For every valuation $v : \{p\} \longrightarrow \underline{A}$, $\underline{A} \in BCK^S$, we define a valuation $\overline{v} : \{p\} \longrightarrow R(\underline{A})$ putting: $\overline{v}(p) = r_{v(p)}$.

Then it is easy to verify the following lemma.

L e m m a 3. (i) $\overline{v}(\alpha_1) > v(\alpha_1)$.

(ii) $\overline{v}(\alpha_{n+1}) = v(\alpha_n)$, for every $n = 1,2,\ldots$

P r o o f . The condition (i) follows immediately from the definition. We easily prove (ii) by induction on n.

L e m m a 4. For $i \neq j$, we have $\alpha_i \neq \alpha_j$ in the 1-generated free algebra of the variety $\mathbb{E}_2^S$, so this algebra must be infinite.

P r o o f . We define algebras $\underline{A}_n$, $n \geqslant 1$, by putting

$$A_1 = \langle \{0,c,b,a\}, *, \circ, 0 \rangle$$

$*$	0	c	b	a
0	0	0	0	0
c	c	0	0	0
b	b	c	0	0
a	a	c	c	0

$\circ$	0	c	b	a
0	0	c	b	a
c	c	a	a	a
b	b	a	a	a
a	a	a	a	a

and $\underline{A}_{n+1} = R(\underline{A}_n)$, $n \geqslant 1$. It is easy to see that for every $n \geqslant 1$ we have $\underline{A}_n \in \mathbb{E}_2^S$.

Let $v_n : \{p\} \longrightarrow \underline{A}_n$ be valuations such that

$$v_1(p) = b \quad \text{and} \quad v_{n+1} = \overline{v}_n, \quad n \geqslant 1.$$

First let us observe that for every $n = 1,2,\ldots$

$$v_n(\alpha_1) > v_n(\alpha_2) > \ldots > v_n(\alpha_n) > v_n(\alpha_{n+1}).$$

For $n = 1$ we have $v_1(\alpha_1) > v_1(\alpha_2)$. Let us suppose for induction that in the algebra $\underline{A}_n$:

$$v_n(\alpha_1) > v_n(\alpha_2) > \ldots > v_n(\alpha_n) > v_n(\alpha_{n+1}).$$

Now let $v_{n+1} : \{p\} \longrightarrow \underline{A}_{n+1}$ be the valuation defined by $v_{n+1} = \overline{v}_n$.

Then by Lemma 3 we have $v_{n+1}(\alpha_1) > v_n(\alpha_1)$ and $v_{n+1}(\alpha_{k+1}) = v_n(\alpha_k)$, $k = 1,2,\ldots,n+1$. This implies that

$$v_{n+1}(\alpha_1) > v_{n+1}(\alpha_2) > \ldots > v_{n+1}(\alpha_{n+1}) > v_{n+1}(\alpha_{n+2}).$$

Let us suppose now that $i \neq j$ and $i < j$. We consider the algebra $\underline{A}_j$ and the valuation $v_j : \{p\} \longrightarrow \underline{A}_j$. Then we obtain

$$v_j(\alpha_1) > v_j(\alpha_2) > \ldots > v_j(\alpha_i) > \ldots > v_j(\alpha_j) > v_j(\alpha_{j+1}),$$

which means $v_j(\alpha_i) \neq v_j(\alpha_j)$.

Lemma 5. If a BCK^S-algebra $\underline{A}$ is $(*,\circ)$-generated by a set G, then $R(\underline{A})$ is $(*)$-generated by the set $G \cup \{z_0\}$.

Now it is easy to see that the following theorem holds.

Theorem. There exists a 2-generated infinite algebra in E_2.

Corollary. None of the variety E_n, $n \geqslant 2$, is locally finite.

Proof. It is easy to see that

$$E_1 \subset E_2 \subset E_3 \subset \dots .$$

Hence by the theorem we obtain the above corollary.

REFERENCES

[1] W.H. Cornish: Varieties generated by finite BCK-algebras, Bull. Austral. Math. Soc., 22 (1980) 411-430.

[2] A. Diego: Sur les algébres de Hilbert, Collection de logique mathématique, Ser. A, fasc. 21, Paris 1966.

[3] K. Dyrdał: On simple algebras in E_n^S, Bull. Sec. Logic, 13, no. 1 (1984), 25-30.

[4] K. Iseki: BCK-algebras with condition (S), Math. Jap. 24 (1979) 107-119.

[5] K. Iseki, S. Tanaka: An introduction to the theory of BCK-algebras, Math. Jap. 23 (1978) 1-26.

[6] V.A. Jankov: Conjunctively indecomposable formulas in propositional calculi, Izv. Akad. Nauk SSSR Ser. Mat. 33 (1969) 18-38. Math. USSR Izv. 3 (1969) 17-36.

[7] A. Wroński: BCK-algebras do not form a variety, Math. Jap. 28 (1983) 211-213.

[8] A. Wroński: Reflections and distensions of BCK-algebras, Math. Jap. 28 (1983) 215-225.

INSTITUTE OF MATHEMATICS, PEDAGOGICAL UNIVERSITY,
25-406 KIELCE, POLAND

Received June 5, 1986.