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RELATIVE COGENERATIONS WITH APPLICATIONS TO TREE AUTOMATA AND A THEORY OF ABSTRACT ALGEBRAS WITH FEEDBACK MODIFICATIONS

The usual theory of automata is interpreted in universal algebra as a theory of tree automata. A notion of feedback is used in the theory of automata, in algebraic structure theory of sequential machines and in the control theory [5,6]. In this paper we give an interpretation of the notion of feedback as an automaton used for the modifications of operations. In this way we obtain a theory of abstract algebras with feedback modifications called briefly feedback algebras. Using a theory of relative cogenerations given in § 3 we prove in § 4 that every feedback algebra admits cogenerations. This theorem may be considered as a generalization of my Theorem 1 in [7]. By the notions of congruences of feedback algebras we obtain for each abstract algebra A a set Q(A) of sets $K \subseteq Con(A)$ of congruences of A. For every $K \in Q(A)$ the algebra A admits the relative cogenerations, i.e. for each equivalence $\sim$ of A there is a greatest congruence in K contained in $\sim$. Using the notion of closed feedback algebras we introduce new examples of enrichmental theories of abstract algebras in the sense of [9]. Some applications to tree automata are given in § 3 and 4 ([3],[1]).

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1. Feedback automata and modifications

Let Z be any set and let n be a natural number. We denote by $Fb_n(Z)$ the set of all functions of the following form

$$\alpha : Z^n \times Z^{Z^n} \to Z^{Z^n}$$

such that $\alpha(r,f)(r) = f(r)$ for all $r \in Z^n$ and $f \in Z^{Z^n}$.

The elements of $Fb_n(Z)$ are called the feedback automata of the rank n over Z. The set $Fb_n(Z)$ is a monoid with respect to the multiplication $\alpha \cdot \beta$ given by the formula

$$(\alpha \cdot \beta)(r,f) = \alpha(r, \beta(r,f)).$$

The unit of this monoid is the projection $e_n(r,f) = f$. For each $H \subseteq Fb_n(Z)$ we denote by ext_H the monoid homomorphism from the free monoid H^* generated by H to the monoid $Fb_n(Z)$ which is the unique extension of the inclusion map $i_H : H \hookrightarrow Fb_n(Z)$.

If $\alpha \in Fb_n(Z)$, then a subset $K \subseteq Z^{Z^n}$ is said to be α-closed provided K determines a subautomaton of α, i.e. K has the property:

(s) if $f \in K$ and $r \in Z^n$, then $\alpha(r,f) \in K$.

If a subset K is α-closed for all $\alpha \in H \subseteq Fb_n(Z)$, then K is called H-closed.

We define a feedback modification over a set Z to be a function which for each natural number n determines a feedback automaton of the rank n over Z. The set of all feedback modifications over a set Z is denoted by Fb(Z). Hence Fb(Z) is the set of all functions φ such that $\varphi(n) \in Fb_n(Z)$ for each natural number n. The set Fb(Z) is a monoid with respect to the multiplication $\varphi \cdot \psi$ given by formula

$$(\varphi \cdot \psi)(n) = \varphi(n) \cdot \psi(n) \quad \text{for all } n$$

where $\varphi(n) \cdot \psi(n)$ is the multiplication in the monoid $Fb_n(Z)$.

The unit of $Fb(Z)$ is the function e such that $e(n) = e_n$ is the unit of $Fb_n(Z)$ for all n. If $\varphi \in Fb(Z)$, then for all $r \in Z^n$ and all f in Z^{Z^n} the value $\varphi(n)(r,f)$ will be also briefly denoted by $\varphi(r,f)$. If $S \subseteq Fb(Z)$, then $S(n) \subseteq Fb_n(Z)$ is the set $\{\varphi(n) : \varphi \in S\}$. A subset K of Z^{Z^n} is said to be S-closed provided K is $S(n)$-closed. For every $S \subseteq Fb(Z)$, we denote by Ext_S the monoid homomorphism $Ext_S : S^* \to Fb(Z)$ which is the unique extension of the inclusion map $i_S : S \hookrightarrow Fb(Z)$, where S^* is the free monoid generated by S. The feedback modifioations will be used to modify abstract algebras. An abstract algebra is any pair $A = \langle A_0, A_1 \rangle$ such that A_0 is a set called the universe or support of A, A_1 is a function which for each natural number n determines a set $A_{1,n}$ of n-ary operations in the set A_0; A_1 is called the operation structure of A and the elements of $A_{1,n}$ are said to be the n-ary fundamental operations of A. A structure type of an abstract algebra A is a pair $\underline{\Sigma} = \langle \Sigma, \gamma \rangle$ such that Σ is a family of disjoint sets Σ_n of operation symbols of rank n and γ is a family of surjective mappings $\gamma_n : \Sigma_n \to A_{1,n}$, where n is any natural number. If an abstract algebra A is considered as an algebra of structure type $\underline{\Sigma}$, then for all n, all $\sigma \in \Sigma_n$, the n-ary fundamental operation $\gamma_n(\sigma)$ is denoted by σ_A and A is said to be Σ-algebra or an abstract algebra of (symbol) type Σ with arity function $\xi(\sigma) = n$ for $\sigma \in \Sigma_n$. For the general theory of abstract algebras and for the theory of Σ-algebras see in the papers [2,4,8,9]. If A and B are two abstract algebras, then the relation $A \leq_r B$ means that $A_0 = B_0$ and $A_{1,n} \subseteq B_{1,n}$ for all n, moreover B is called an enrichment of A. For each set Z the set $Enr_Z(AL)$ of all abstract algebras A with $A_0 = Z$ is a complete lattice with respect to the relation $\leq_r$. Let $A \in Enr_Z(AL)$ and let $\varphi \in Fb(Z)$. By A_φ we denote the abstract algebra B such that $B_0 = Z$ and $B_{1,n} = \varphi(Z^n, A_{1,n})$ for each n. The algebra A_φ is called the φ-image of A. If $H \subseteq Fb(Z)$, then the algebra $A_H = \bigcup_{\varphi \in H} A_\varphi$ is said to be the H-image of A. If $A,B \in \mathrm{Enr}_Z(AL)$ and $\varphi \in Fb(Z)$, then (A,φ,B) is called a feedback morphism from A to B provided $A_\varphi \leq_r B$ and we write $\varphi : A \to B$. The compositions of feedback morphisms given by the multiplication in the monoid Fb(Z) is also a feedback morphism and in this way we have obtained a category Fb(AL) called the feedback category of abstract algebras. For two abstract algebras A and B with $A_0 = B_0 = Z$ and a $\varphi \in Fb(Z)$ we define $\varphi_B^{-1}(A)$ to be an abstract algebra C such that $C_0 = Z$ and for all number n

$$C_{1,n} = \left\{ f \in B_{1,n} : \varphi(r,f) \in A_{1,n} \text{ for all } r \right\}.$$

Hence $\varphi_B^{-1}(A) \leq_r A$ or $\varphi : \varphi_B^{-1}(A) \to A$. If $B = \varphi_B^{-1}(A)$ then B is called an inverse φ-image of A. If $C = \varphi_B^{-1}(A)$, then $\varphi_C^{-1}(A) = C$ and thus $\varphi_B^{-1}(A)$ is an inverse φ-image of A.

If $H \subseteq Fb(Z)$, then $H_B^{-1}(A)$ denotes the algebra $\bigcup_{\varphi \in H} \varphi_B^{-1}(A)$.

E x a m p l e 1. Let Σ' and Σ be two (symbol) types for algebras. A feedback transformation from Σ' to Σ over a set Z is a function α such that for each natural number n the value $\alpha(n)$ is a function

$$\alpha(n) : Z^n \times \Sigma'_n \to \Sigma_n.$$

The set of all feedback transformations from Σ' to Σ over a set Z is denoted by $St_Z(\Sigma', \Sigma)$. If $\alpha \in St_Z(\Sigma', \Sigma)$, then each algebra A of the type Σ with $A_0 = Z$ determines an algebra A' of the type Σ' and a feedback modification $_A\alpha \in Fb(Z)$ such that $A'_0 = Z$ and

$$(*) \qquad \sigma'_{A'}(r) = \alpha(r, \sigma')_A(r)$$

$$(**) \qquad {}_A\alpha(r, \sigma'_{A'}) = \alpha(r, \sigma')_A, \quad {}_A\alpha(r,f) = f$$

for all remaining operations f. The algebra A' is an inverse $_A\alpha$-image of A and it is called the α-image of A. If A' is an inverse φ-image of A, where $\varphi \in Fb(A_0)$ and A' is a $\sum'$-algebra, then we can define an $\alpha \in St_{A_0}(\sum', \sum)$ such that

$$\alpha(r, \sigma') = \sigma \quad \text{provided} \quad \varphi(r, \sigma'_{A'}) = \sigma_A$$

and ${}_A\alpha|A' = \varphi|A'$; moreover A' is the α-image of A.

For the second example we define the general product of abstract algebras. If $A^{(t)}$, $t \in T$, is a family of abstract algebras, then the general product of $A^{(t)}$, $t \in T$, is the abstract algebra A such that A_0 is the direct product of sets $A_0^{(t)}$, $t \in T$, or $A_0 = \underset{t\in T}{P} A_0^{(t)}$, and for each n, $A_{1,n} = v\,(\underset{t\in T}{P} A_{1,n}^{(t)})$, where v is a function from $\underset{t\in T}{P} A_{1,n}^{(t)}$ to $A_0^{A_0^n}$ given by the following formula

$$v(\beta)(r)(t) = \beta(t)(r(t)) \quad \text{for} \quad t \in T.$$

E x a m p l e 2. Let $\sum', \sum^{(1)}, \sum^{(2)}, \ldots, \sum^{(m)}$ be (symbol) types for algebras. Let us consider a type $\sum = \sum^{(1)} \times \sum^{(2)} \times \ldots \times \sum^{(m)}$ such that for all n $\sum_n = \sum_n^{(1)} \times \sum_n^{(2)} \times \ldots \times \sum_n^{(m)}$ and $\alpha \in St_Z(\sum', \sum)$, where $Z = Z_1 \times Z_2 \times \ldots \times Z_m$. Moreover, let us consider a system

$$\mathfrak{A} = \{A_1, A_2, \ldots, A_m\}$$

of abstract algebras A_i of type $\sum^{(i)}$ with $A_{i0} = Z_i$. The general product A of algebras $A_1, A_2, \ldots, A_m$ may be considered as an algebra of the type $\sum$ putting for $\sigma = \langle \sigma_1, \sigma_2, \ldots, \sigma_m \rangle \in \sum_n$

$$\sigma_A = v(\langle \sigma_{1A_1}, \sigma_{2A_2}, \ldots, \sigma_{mA_m} \rangle).$$

Now we define an abstract algebra A' of the type $\sum'$ and an feedback modification $\mathfrak{A}^\alpha \in Fb(Z)$ by the following formulas:

(1) $$A'_0 = Z$$

(2) for all n, all $\sigma' \in \sum'_n$, $\sigma'_{A'}(r) = \alpha(r, \sigma')_A(r)$

$$\mathcal{O}\!\iota^{\alpha}(r, \sigma'_{A'}) = \alpha(r, \sigma')_A$$

and for all remaining operations

(3) $$\mathcal{O}\!\iota^{\alpha}(r,f) = f.$$

The algebra A' is called the α-product of $\mathcal{O}\!\iota$. The algebra A' is an inverse $\mathcal{O}\!\iota^{\alpha}$-image of A.

For global feedback modifications we define a general diagonal over sets X and Y to be the set

$$d(X,Y) = \left\{ \langle t,u \rangle \in X^{*} \times Y^{*} : |t| = |u| \right\},$$

where X^{*} and Y^{*} are the free monoids generated by X and Y and $|t|$ is the length of the word t. By $d^{(m)}(X,Y)$ we denote the subset of $d(X,Y)$ of all pairs $\langle t,u \rangle$ with $|t| = |u| = m$ and it is called the m-diagonal over X and Y. The global feedback automaton of rank n over a set Z is a function of the form

$$G_n(Z) : d(Z^n, Fb(Z)) \times Z^{Z^n} \longrightarrow Z^{Z^n}$$

defined by induction as follows:

1° $G_n(Z)(\langle \varepsilon, \varepsilon \rangle, f) = f$, ε empty word,

2° $G_n(Z)(\langle r, \varphi \rangle, f) = \varphi(r,f)$ for $\langle r, \varphi \rangle \in d^{(1)}(Z^n, Fb(Z))$,

3° $G_n(Z)(\langle rt, \varphi\psi \rangle, f) = \varphi(r, G_n(Z)(\langle t, \psi \rangle, f))$.

The global feedback modification over a set Z is a function $G(Z)$ defined on the set of all natural numbers such that $G(Z)(n)$ is the function $G_n(Z)$ for all n. The values $G_n(Z)(t,f)$ will be briefly denoted by $G(Z)(t,f)$ or $G(t,f)$.

2. A theory of abstract algebras with feedback modifications

An abstract algebra with feedback modifications or briefly a feedback algebra is a pair $\langle A,H\rangle$ such that A is any abstract algebra and $H\subseteq Fb(A_0)$. A structure type of a feedback algebra $\langle A,H\rangle$ is any pair $(\underline{\Sigma}, \underline{M})$ such that $\underline{\Sigma}$ is a structure type of A and $\underline{M}$ is a structure type of H, i.e. $\underline{M}:M \longrightarrow H$ is a surjective mapping, where M is a set of feedback symbols. If we consider a feedback algebra $\langle A,H\rangle$ as one of the structure type $(\underline{\Sigma}, \underline{M})$, then for all n, all $\sigma\in\Sigma_n$, the corresponding fundamental operation of A is denoted by σ_A and for each symbol $m\in M$ the feedback modification $\underline{M}(m)\in H$ is denoted by m_H and moreover, the feedback algebra $\langle A,H\rangle$ is said to be (Σ, M)-algebra or a feedback algebra of the (symbol) type (Σ, M). We have three notions of (Σ, M)-homomorphisms.

2.1. Definition. Let $\langle A,H\rangle$ and $\langle B,H'\rangle$ be two feedback algebras of a (symbol) type (Σ, M).

I. A (Σ,M)-homomorphism from $\langle A,H\rangle$ to $\langle B,H'\rangle$ is any function $h : A_0 \longrightarrow B_0$ such that

(1) h is a Σ-homomorphism from A to B, i.e. for all n, all $\sigma\in\Sigma_n$ and all $r\in A_0^n$ $h(\sigma_A(r)) = \sigma_B(h^n(r))$,

(2) h is a M-homomorphism from H to H', i.e. for all n, $\sigma\in\Sigma_n$, $m\in M$ and $r,q\in A_0^n$, $h(m_H(r,\sigma_A)(q)) =$
$= m_{H'}(h^n(r),\sigma_B)(h^n(q))$

or for all n, for all $\langle r,m\rangle\in d^{(1)}(A_0^n,M)$ the condition (a) holds:

(a) $h(G(A_0)(\langle r,m_H\rangle,\sigma_A)(q)) = G(B_0)(\langle h^n(r),m_{H'}\rangle,\sigma_B)(h^n(q)).$

II. A very strong (Σ,M)-homomorphism from $\langle A,H\rangle$ to $\langle B,H'\rangle$ is any (Σ,M)-homomorphism h from $\langle A,H\rangle$ to $\langle B,H'\rangle$ such that for all n and $\sigma\in\Sigma_n$, all $\langle r,t\rangle\in d(A_0^n,M)$, $q\in A_0^n$:

(b) $h(G(A_0)(\langle r,M_H(t)\rangle,\sigma_A)(q)) = G(B_0)(\langle h^{|t|}(r),M_{H'}(t)\rangle,\sigma_B)(h^n(q)),$

where $M_H : M^* \longrightarrow H^*$ is the monoid homomorphism which is the unique extension of the mapping $\underline{M}(m) = m_H$ for all $m\in M$.

III. A strong (Σ,M)-homomorphism from $\langle A,H\rangle$ to $\langle B,H'\rangle$ is any (Σ,M)-homomorphism h from $\langle A,H\rangle$ to $\langle B,H'\rangle$ such that the condition (b) holds only for all n, $\sigma\in\Sigma_n$ and all $\langle r,t\rangle\in d(A_0^n,M)$ of the following form: $r = r'r'\ldots r' = r'^{|t|}$ for some $r'\in A_0^n$ or for all n, $\sigma\in\Sigma_n$, all $r,q\in A_0^n$ and all $t\in M^*$ holds:

(c) $h(Ext_H(M_H(t))(r,\sigma_A)(q)) = Ext_{H'}(M_{H'}(t))(h^n(r),\sigma_B)(h^n(q))$.

Let us observe that the composition of very strong, strong and usual (Σ,M)-homomorphisms is also a very strong, strong and usual (Σ,M)-homomorphism. In this way we have obtained three categories of feedback algebras of (symbol) type (Σ,M). Those categories will be denoted by $VSFal_{(\Sigma,M)}$, $SFal_{(\Sigma,M)}$ and $Fal_{(\Sigma,M)}$.

For feedback algebras we have four notions of a congruence.

2.2. D e f i n i t i o n . Let $\langle A,H\rangle$ be any feedback algebra and let $\sim$ be any congruence of the abstract algebra A.

I. The relation $\sim$ is called a (usual) congruence of $\langle A,H\rangle$ or a H-congruence of A if it has the following property:

(1) for all $\varphi\in H$, all n, all $f\in A_{1,n}$ and all $r,r',q,q'\in A_0^n$ if $r\sim r'$ and $q\sim q'$, then $\varphi(r,f)(q)\sim\varphi(r',f)(q')$, where $r\sim r'$ means that $r(i)\sim r'(i)$ for all i.

II. The relation $\sim$ is called a very strong congruence of $\langle A,H\rangle$ or a very strong H-congruence of A if it has the following property:

(2) for all n, all $\langle t,\psi\rangle,\langle t',\psi\rangle\in d(A_0^n,H)$, all $f\in A_{1,n}$, $q,q'\in A_0^n$ if $t\sim t'$ and $q\sim q'$, then $G(A_0)(\langle t,\psi\rangle,f)(q)\sim G(A_0)(\langle t',\psi\rangle,f)(q')$ where $t\sim t'$ means that $t(i)\sim t'(i)$ for $i=1,\ldots,|\psi|\cdot n$.

III. The relation $\sim$ is a strong congruence of $\langle A,H\rangle$ or a strong H-congruence of A if it has the following property:

(3) for all n, all $f\in A_{1,n}$, all $r,r',q,q'\in A_0^n$ and all $\varphi\in H^*$ if $r\sim r'$ and $q\sim q'$, then $Ext_H(\varphi)(r,f)(q)\sim Ext_H(\varphi)(r',f)(q')$.

IV. The relation $\sim$ is called a reversible congruence of $\langle A,H\rangle$ or a reversible H-congruence of A if it is a congruence of $\langle A,H\rangle$ and it has the following property:

(4) for each $\varphi\in H$, all n, all $f\in A_0^{A_0^n}$ if $\sim$ is a congruence of all operations $\varphi(r,f)$, where $r\in A_0^n$, then for all $q,q'\in A_0^n$ $\varphi(q,f)(q')\sim\varphi(q',f)(q)$ provided $q\sim q'$.

By a simple verification we obtain

2.3. The relations modh induced by very strong, strong and usual (Σ,M)-homomorphisms $h:\langle A,H\rangle\to\langle B,H'\rangle$ are very strong, strong and usual congruences of $\langle A,H\rangle$.

2.4. For each congruence $\sim$ of $\langle A,H\rangle$ of a (symbol) type (Σ,M) we have the quotient feedback algebra $\langle A/\sim, H_\sim\rangle$ and a surjective (Σ,M)-homomorphism $j_\sim:\langle A,H\rangle\to\langle A/\sim,H_\sim\rangle$, where $A/\sim$ is the quotient Σ-algebra, $j_\sim(a)=[a]_\sim=\{b\in A_0 : b\sim a\}$, $H_\sim=\{\varphi_\sim : \varphi\in H\}$ and $\varphi_\sim([r]_\sim,\sigma_{A/\sim})([q]_\sim)=[\varphi(r,\sigma_A)(q)]_\sim$ and moreover $\varphi_\sim([\bar{r}]_\sim,f)=f$ for the remaining operations f.
If $\sim$ is very strong or strong, then $j_\sim$ is very strong or strong.

Let us observe that

2.5. If $\sim$ is a congruence of a feedback algebra $\langle A,H\rangle$, then $\sim$ has the following properties:

(1) if $f\in A_{1,n}$, $\varphi\in H$ and $r\sim r'$, then $\varphi(r,f)(r')\sim\varphi(r',f)(r)$,

(2) if $f\in A_{1,n}$, $\varphi\in H$, $q\sim q'$, then $\varphi(r,f)(q)\sim\varphi(r,f)(q')$ i.e. $\sim$ is a congruence of the operations $\varphi(r,f)$ for $r\in A_0^n$,

(3) if $f'\in A_0^{A_0^n}$ and $\sim$ is a congruence of all operations $\varphi(r,f')$ where $r\in A_0^n$ and φ is one given fixed element of H, and moreover (1) holds for $f=f'$, then $\sim$ is a congruence of f'.

P r o o f . The property (1) follows from 2.2.(1) for $r\sim r'$ and $r'\sim r$. The property 2 follows from 2.2.(1) for $r=r'$. For (3) assume that $q\sim q'$. Then $f'(q)=\varphi(q,f')(q)\sim$ $\sim\varphi(q,f')(q')\overset{(1)}{\sim}\varphi(q',f')(q)\sim\varphi(q',f')(q')=f'(q')$.

2.6. Let $\langle A,H\rangle$ be any feedback algebra. Then the following propositions hold:

I. If $\sim$ is a congruence of $\langle A,H\rangle$, then $\sim$ is a congruence of A_H.

II. If $\sim$ is a reversible congruence of $\langle A,H\rangle$, then for all $\varphi\in H$ $\sim$ is a congruence of a n-ary operation f if and only if $\sim$ is a congruence of all operations $\varphi(r,f)$, where $r\in A_0^n$.

P r o o f . I follows from 2.5.(2). II follows from 2.5.(3). For each feedback algebra $\langle A,H\rangle$ the sets of all strong, very strong and of usual congruences of $\langle A,H\rangle$ will be denoted by $SCon(\langle A,H\rangle)$, $VSCon(\langle A,H\rangle)$ and $Con(\langle A,H\rangle)$ respectively. If those congruences are considered as H-congruences of A, then those sets will be denoted by $SCon_H(A)$, $VSCon_H(A)$ and $Con_H(A)$. Hence we have $SCon(\langle A,H\rangle) = SCon_H(A)$, $VSCon(\langle A,H\rangle) = VSCon_H(A)$ and $Con(\langle A,H\rangle) = Con_H(A)$. Let us observe that the notion of $\{e\}$-congruence of A, where e is the unit of the monoid $Fb(A_0)$, is identical with the usual notion of a congruence of an abstract algebra A. Hence we obtain $Con(A) = Con_{\{e\}}(A)$.

A subalgebra of a feedback algebra $\langle A,H\rangle$ is any feedback algebra $\langle A',H\rangle$, where A' is a subalgebra of A.

2.7. If $\langle A,H\rangle$ is a feedback algebra, $A_H\leqslant_r A'$ and Y is a subalgebra of A', then $\langle Y_0,H\rangle$ is a subalgebra of $\langle A,H\rangle$. If a set X generates A, then for each $H\subseteq Fb(A_0)$ and each abstract algebra A' with $A_H\leqslant_r A'$ the set X generates A'.

P r o o f . Let $\varphi\in H$. Moreover, let $f\in A_{1,n}$. For each $r\in Y_0^n$, $f(r) = \varphi(r,f)(r)\in Y_0$ since Y is a subalgebra of A'. Hence Y_0 is closed with respect to all fundamental operations of A and thus Y_0 may be considered as a subalgebra of A. If $Y\supseteq X$ is a subalgebra of A', then Y_0 may be considered as a subalgebra of A but X generates A, and thus $Y_0 = A_0$ i.e. X generates $A'\geqslant_r A_H$.

If $\langle A^{(t)}, H^{(t)}\rangle$ for $t\in T$ are feedback algebras, then the general product of $\langle A^{(t)},H^{(t)}\rangle$, $t\in T$, is the feedback algebra $\langle A,H\rangle$, where A is the general product of $A^{(t)}$, $t\in T$, and $H = w(\underset{t\in T}{P} H^{(t)})$, $w : \underset{t\in T}{P} H^{(t)}\longrightarrow Fb(A_0)$ is defined by the formula:

(gp) $w(\beta)(r,f)(q)(t) = \beta(t)(r(t),f(t))(q(t)), \quad t \in T.$

2.8. The category $Fal_{(\Sigma,M)}$ has products.

P r o o f . Let $\langle A^{(t)},H^{(t)}\rangle$, $t \in T$, be any family of feedback algebras of a (symbol) type (Σ,M). Let us denote by A the Σ-direct product of Σ-algebras $A^{(t)}$, $t \in T$. For each $m \in M$ we define $m_H \in Fb(A_0)$ putting for all n, $f \in A_0^{A_0^n}$, $r,q \in A_0^n$:

if $f = \sigma_A$ for some $\sigma \in \Sigma_n$ $m_H(r,f)(q) = g$, where for $t \in T$

$$g(t) = m_{H(t)}(r(t), \sigma_{A(t)})\ (q(t))$$

else $m_H(r,f) = f$. Let $H = \{m_H : m \in M\}$. Then the feedback algebra $\langle A,H\rangle$ with the projections $p_t : A_0 \longrightarrow A_0^{(t)}$, $t \in T$ is the product (called direct) of $\langle A^{(t)},H^{(t)}\rangle$, $t \in T$, in the category $Fal_{(\Sigma,M)}$.

3. Relative cogenerations and relative regular equivalences in abstract algebras with applications

Let A be any abstract algebra. If $K \subseteq Con(A)$ is a set of congruences of A and $\sim$ is an equivalence relation of A_0 or $\sim \in Eq(A_0)$ then we say that the relation $\sim$ K-cogenerates in A a congruence $\sim^*$ provided $\sim^*$ is the greatest congruence in K which saturates $\sim$, i.e. which is contained in $\sim$.
Let Σ be a structure type of A. Let us consider a category $Epi_\Sigma(A,K,\sim)$. The objects of $Epi_\Sigma(A,K,\sim)$ are all surjective Σ-homomorphism $h:A \longrightarrow A'$ of A such that $mod h \in K$ and $mod h \subseteq \sim$, where mod h is the congruence of A induced by h. The morphisms in $Epi_\Sigma(K,A,\sim)$ from $h:A \longrightarrow A'$ to $h':A \longrightarrow A''$ are all Σ-homomorphisms $q:A' \longrightarrow A''$ with $h' = q \circ h$. The objects of $Epi_\Sigma(A,K,\sim)$ are called mod$(K,\sim)$ Σ-epimorphisms of A. A object $h:A \longrightarrow A'$ is said to be finite provided A_0' is a finite set. The terminal object of $Epi_\Sigma(A,K,\sim)$ is called a minimal mod$(K,\sim)$ Σ-epimorphism of A.

Hence we obtain

3.1. An equivalence $\sim$ of A_0 K-cogenerates in an abstract algebra A a congruence if and only if A has a minimal mod$(K,\sim)$ Σ-epimorphism, where Σ is the (symbol) type of A.

P r o o f . If $\sim^*$ is the congruence of A K-cogenerated by $\sim$, then $j_{\sim^*}: A \to A/\sim^*$ is a minimal mod$(K,\sim)$ Σ-epimorphism of A. If $h: A \to A'$ is a minimal mod$(K,\sim)$ Σ-epimorphism, then h is the terminal object of the category $Epi_{\Sigma}(A,K,\sim)$ and thus the congruence modh is K-cogenerated by $\sim$.

3.2. D e f i n i t i o n . Let A be any abstract algebra and let $K \subseteq Con(A)$ be any set of congruences of A. We say that K admits in A relative cogenerations or that A admits relative K-cogenerations if every equivalence relation $\sim$ of A_0 K-cogenerates in A a congruence i.e. for each $\sim \in Eq(A_0)$ there is a greatest congruence in K contained in $\sim$.

3.3. T h e o r e m . If an abstract algebra B admits relative K-cogenerations, then for every abstract algebra A with $A \leq_r B$ the algebra A admits the relative K-cogenerations.

P r o o f . Since $A \leq_r B$, therefore $Con(B) \subseteq Con(A)$ and $K \subseteq Con(B) \subseteq Con(A)$. Hence the congruence $\sim^* \in K$ which is K-cogenerated in B by $\sim$, is also K-cogenerated in A by $\sim$.

By Theorem 1 from [7] every abstract algebra B admits cogenerations, i.e. admits relative Con(B)-cogenerations. Hence by 3.3 we easily obtain

3.4. T h e o r e m . For two abstract algebras A and B if $A \leq_r B$, then the algebra A admits relative Con(B)-cogenerations.

3.5. D e f i n i t i o n . Let A be any abstract algebra and let $K \subseteq Con(A)$. Moreover, let Σ be the (symbol) type of A. An equivalence relation $\sim$ of A_0 is said to be mod K Σ-regular in A provided there is a finite mod$(K,\sim)$ Σ-epimorphism of A.

3.6. T h e o r e m . Let us assume that an equivalence $\sim$ of A_0 K-cogenerates a congruence $\sim^*$ in an abstract algebra A of (symbol) type Σ. Then the following conditions are equivalent:

(1) the equivalence $\sim$ is mod $K\sum$-regular in A;
(2) there is in K a congruence of finite index which saturates $\sim$;
(3) the congruence $\sim^*$ of A K-cogenerated by $\sim$ has a finite index.

The conditions (1), (2) and (3) are equivalent for all $K = Con(B)$, where B is any abstract algebra with $A \leq_r B$.

P r o o f . (1) $\Longrightarrow$ (2). Let $h: A \to A'$ be a finite $mod(K,\sim)$ $\sum$-epimorphism of A. Then $card(A_0)$ is the index of $mod\,h \in K$ and it is finite. Moreover, mod h saturates $\sim$.
(2) $\Longrightarrow$ (3). Let a congruence $\sim' \in K$ which saturates $\sim$ have a finite index. Then $\sim' \subseteq \sim^*$, therefore we have a surjective $\sum$-homomorphism $h: A/\sim' \to A/\sim^*$. Hence $card(A/\sim^*) = in(\sim^*) \leq$ $\leq car(A/\sim') = in(\sim')$ i.e. the index of $\sim^*$ is a finite number.
(3) $\Longrightarrow$ (1). Then $j_{\sim^*}: A \to A/\sim^*$ is a finite $mod(K,\sim)$ $\sum$-epimorphism of A i.e. by 3.5 the equivalence $\sim$ is mod $K\sum$-regular in A. The last part of 3.6 is obtained by 3.4.

3.7. D e f i n i t i o n . A subset $Z \subseteq A_0$ is called mod $K\sum$-regular in an abstract algebra A of a (symbol) type $\sum$ provided the equivalence $\sim_Z$ of A_0 induced by Z, i.e. determined by the partition $\{A_0 - Z, Z\}$, is mod K $\sum$-regular in A.

A characterization of mod K $\sum$-regular subsets Z of an abstract algebra A of a (symbol) type $\sum$ is given by 3.6 for $\sim = \sim_Z$. The mod $Con(A)\sum$-regular equivalences and subsets in a $\sum$-algebra A are called $\sum$-regular equivalences and subsets in A. By the last part of 3.6 we immediately obtain

3.8. For a $\sum$-algebra A and any $\sim \in Eq(A_0)$ the following conditions are equivalent:
(1) $\sim$ is $\sum$-regular in A;
(2) there is a congruence of finite index of A which saturates $\sim$;
(3) the congruence of A cogenerated by $\sim$ has a finite index.

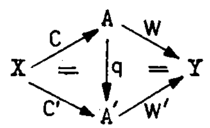
3.9. D e f i n i t i o n . Let $\sum$ be a (symbol) type for algebras. Let X and Y be any sets. A $\sum$-machine over X and Y is any function

$$f : F_{\Sigma}(X)_0 \longrightarrow Y,$$

where $F_{\Sigma}(X)$ is the absolutely free algebra of type Σ freely generated by X. A Σ-tree automaton over X and Y is each triple $\mathfrak{A} = \langle c,A,w\rangle$, where A is an abstract algebra of the type Σ, $c : X \longrightarrow A_0$ is a function called the input function of $\mathfrak{A}$, $w : A_0 \longrightarrow Y$ is a function called the output function of $\mathfrak{A}$. Let $\mathfrak{A} = \langle c,A,w\rangle$ be any Σ-tree automaton over X and Y. The function $b_{\mathfrak{A}} = w \circ c_{\mathfrak{A}}$, where $c_{\mathfrak{A}} : F_{\Sigma}(X) \longrightarrow A$ is the unique Σ-homomorphism which is an extension of c, is called the Σ-machine realizable by $\mathfrak{A}$ and $\mathfrak{A}$ is said to be a Σ-realization of $b_{\mathfrak{A}}$. A Σ-tree automaton $\mathfrak{A} = \langle c,A,w\rangle$ is called reachable if $c_{\mathfrak{A}}$ is a surjective function.

3.10. Definition. Let K be any set of congruences of the algebra $F_{\Sigma}(X)$. A mod K Σ-realization of a Σ-machine f over X and Y is each Σ-tree automaton $\mathfrak{A} = \langle c,A,w\rangle$ over X and Y such that $b_{\mathfrak{A}} = f$ and the congruence induced by $c_{\mathfrak{A}}$ belongs to K. If moreover, $\mathfrak{A}$ is reachable, then $\mathfrak{A}$ is said to be a mod K reachable Σ-realization of f.

For each $K \subseteq Con(F_{\Sigma}(X))$ and each Σ-machine f over X and Y we have a category $Real_K(f)$ of mod K reachable Σ-realizations of the Σ-machine f with morphisms from $\mathfrak{A} = \langle c,A,w\rangle$ to $\mathfrak{A}' = \langle c',A',w'\rangle$ being all Σ-homomorphisms $q : A \longrightarrow A'$ such that



commutes. The terminal object of $Real_K(f)$ is called a minimal mod K Σ-realization of f. A Σ-machine f over X and Y is said to be mod K Σ-regular provided the equivalence modf induced by f is mod K Σ-regular in the algebra $F_{\Sigma}(X)$.

3.11. Theorem:

I. A Σ-machine f over X and Y has a minimal mod K Σ-realization if and only if the equivalence modf induced by f K-cogenerates a congruence in the algebra $F_{\Sigma}(X)$.

II. If a Σ-machine f over X and Y has a minimal mod K Σ-realization, then f has a finite mod K Σ-realization if and only if f is mod K Σ-regular.

III. Let K = Con(B), where B is any abstract algebra with $F_{\Sigma}(X) \leq_r B$. Then we have:

(i) every Σ-machine f over X and Y has a minimal mod K Σ-realization,

(ii) a Σ-machine f over X and Y has a finite mod K Σ-realization if and only if f is mod K Σ-regular.

P r o o f . I. The categories $\mathrm{Real}_K(f)$ and $\mathrm{Epi}_{\Sigma}(F_{\Sigma}(X), K, \mathrm{mod} f)$ are isomorphic and thus by 3.1 we obtain I.

II. Follows from part I by 3.6.

III. Follows from parts I and II by 3.4.

3.12. D e f i n i t i o n . Let K be any set of congruences of the algebra $F_{\Sigma}(X)$. We say that K admits minimal relative Σ-realizations over X if for every set Y every Σ-machine f over X and Y has a minimal mod K Σ-realization.

By 3.11 we immediately obtain

3.13. T h e o r e m . A set K admits minimal relative Σ-realizations over X if and only if the algebra $F_{\Sigma}(X)$ admits relative K-cogenerations.

Now we give a remark on tree automata with different types. Let $\mathcal{O}\!\iota = \langle c, A, w \rangle$ and $\mathcal{O}\!\iota' = \langle c, A', w \rangle$ be a Σ-tree automaton and a Σ'-automaton over X and Y respectively with the same input function $c: X \longrightarrow A_0 = A'_0$ and the same output function $w: A_0 \longrightarrow Y$.

Then using 2.7 we can prove the following proposition

3.14. If $\varphi: A \longrightarrow A'$ is a feedback morphism, i.e. $A_{\varphi} \leq_r A'$, and $\mathcal{O}\!\iota$ is reachable, then $\mathcal{O}\!\iota'$ is reachable and by tree induction we obtain a function $\bar{\varphi}: F_{\Sigma}(X)_0 \longrightarrow F_{\Sigma'}(X)_0$ such that:

1° $\bar{\varphi}(x) = x$ for $x \in X$,

2° for all n, all $\sigma \in \Sigma_n$ we have

$$\bar{\varphi}(\sigma(t_1, t_2, \ldots, t_n)) = \sigma'(\bar{\varphi}(t_1), \bar{\varphi}(t_2), \ldots, \bar{\varphi}(t_n))$$

provided $\varphi(\langle c_{\alpha}(t_1), c_{\alpha}(t_2), \ldots, c_{\alpha}(t_n) \rangle, \sigma_A) = \sigma'_{A'}$,

3^o $b_{\alpha} = b_{\alpha} \cdot \bar{\varphi}$ and $b_{\alpha'}^{-1}(y) = \bar{\varphi}(b_{\alpha}^{-1}(y))$, $y \in Y$.

4. Cogenerations in feedback algebras and relative feedback cogenerations in abstract algebras

Now we prove the main theorem.

4.1. Theorem. Let $\langle A,H\rangle$ be any feedback algebra and let $\sim$ by any equivalence relation of the set A_0. Then the following propositions hold:

(1) there is a greatest congruence $\sim^*$ of $\langle A,H\rangle$ contained in $\sim$ and it is the greates H-congruence of A contained in $\sim$;

(2) there is a greatest very strong congruence $\sim^*_{vs}$ of $\langle A,H\rangle$ contained in $\sim$ and it is the greatest very strong H-congruence of A contained in $\sim$;

(3) there is a greatest strong congruence $\sim^*_s$ of $\langle A,H\rangle$ contained in $\sim$ and it is the greatest strong H-congruence of A contained in $\sim$;

(4) there is a greatest reversible congruence $\sim^*_r$ of $\langle A,H\rangle$ contained in $\sim$ and it is the greatest reversible H-congruence of A contained in $\sim$.

Proof. For a set Z, $f \in Z^{Z^n}$ and $\psi \in Fb(Z)^*$ we define in Z a $(|\psi|+1)\cdot n$-ary operation f_ψ by the formula

$$f_\psi(r) = G(Z)(\langle t,\psi\rangle,f)(q),$$

where $r = t\cdot q$ in the monoid $(Z^n)^*$. Let us observe that, by 1^o from the definition of G(Z) which is given in the last part of §1, $f_\varepsilon = f$, where ε is the empty word. Moreover, for $\psi \in Fb(Z)^*$, $\psi \neq \varepsilon$, an n-ary operation f acting on Z we define in Z a 2n-ary operation ${}_\psi f$ by the defining formula

$${}_\psi f(r) = Ext(\psi)(t,f)(q),$$

where Ext : $Fb(Z)^* \longrightarrow Fb(Z)$ is the unique monoid homomorphism which is the extension of the identity inclusion $1_{Fb(Z)}$, t and q belong to Z^n and $r = t\cdot q$ in the monoid $(Z^n)^*$. We remark that ${}_\psi f(r) = f_\psi(u)$, where for some t and q in Z^n we have $r = t \cdot q$ and $u = t^{|\psi|} \cdot q$ in the monoid $(Z^n)^*$. If F is a set of operations acting on Z and $\psi \in Fb(Z)^*$, then

$$(F)_\psi = \left\{ f_\psi : f \in F \right\} \quad \text{and} \quad {}_\psi(F) = \left\{ {}_\psi f : f \in F \right\}.$$

Using the above definitions we give the proof of (1)-(4).

P r o o f (1). We define $Op_H^{(1)}(A) = \bigcup_{n=0}^{\infty} \bigcup_{\substack{\psi \in H \\ \psi = \varepsilon}} (A_{1,n})_\psi$ and $Op_H^{(1)}(A)_m = \left\{ f : f \in Op_H^{(1)}(A) \text{ and } f \text{ is m-ary} \right\}$. Let $A^{(H)}$ be the abstract algebra B such that $B_0 = A_0$ and for all natural n $B_{1,n} = Op_H^{(1)}(A)_n$. From the construction of $A^{(H)}$ we obtain

$$(1^*) \qquad A \leqslant_r A^{(H)} \quad \text{and} \quad Con(A^{(H)}) = Con_H(A) = Con(\langle A,H \rangle).$$

Since $A \leqslant_r A^{(H)}$, therefore by 3.4 the algebra A admits relative $Con(A^{(H)})$-cogenerations. Let $\sim^*$ be the congruence of A $Con(A^{(H)})$-cogenerated by $\sim$. Hence, by (1^*), $\sim^*$ is the greatest congruence of $\langle A,H \rangle$ contained in $\sim$ and it is also the greatest H-congruence of A contained in $\sim$.

P r o o f (2). We define $Op_H^{(2)}(A) = \bigcup_{n=0}^{\infty} \bigcup_{\psi \in H^*} (A_{1,n})_\psi$ and $Op_H^{(2)}(A)_m = \left\{ f \in Op_H^{(2)}(A) : f \text{ is m-ary} \right\}$. Let $A^{vs(H)}$ be the abstract algebra B such that $B_0 = A_0$ and $B_{1,n} = Op_H^{(2)}(A)_n$ for all natural number n. From the construction of $A^{vs(H)}$ it follows that

$$(2^*) \qquad A \leqslant_r A^{vs(H)} \quad \text{and} \quad Con(A^{vs(H)}) = VSCon_H(A) = VSCon(\langle A,H \rangle).$$

Since $A \leqslant_r A^{vs(H)}$ therefore by 3.4 the algebra A admits the relative $Con(A^{vs(H)})$-cogenerations. Let $\sim^*_{vs}$ be the congruence of A which is $Con(A^{vs(H)})$-cogenerated by $\sim$. Hence by (2^*) the congruence $\sim^*_{vs}$ is the greatest very strong congruence of $\langle A,H \rangle$ contained in $\sim$ and it is also the greatest very strong H-congruence of A contained in $\sim$.

P r o o f (3). We define $Op_H^{(3)}(A) = \bigcup_{n=0}^{\infty} \bigcup_{\psi \in H^*, \psi \neq \varepsilon} (A_{1,n} \cup \psi(A_{1,n}))$ and $Op_H^{(3)}(A)_m = \{ f \in Op_H^{(3)}(A) : f \text{ is } m\text{-ary}\}$. We define $A^{s(H)}$ to be the abstract algebra B such that $B_0 = A_0$ and $B_{1,n} = Op_H^{(3)}(A)_n$ for all natural number n. From the definition of $A^{s(H)}$ we obtain

$$(3^*) \qquad A \leqslant_r A^{s(H)} \quad \text{and} \quad Con(A^{s(H)}) = SCon_H(A) = SCon(\langle A,H \rangle).$$

Since $A \leqslant_r A^{s(H)}$ therefore by 3.4 the algebra A admits relative $Con(A^{s(H)})$-cogenerations. Let $\sim_s^*$ be the congruence A of A which is $Con(A^{s(H)})$-cogenerated by $\sim$. Hence by (3^*) the congruence $\sim_s^*$ is the greatest strong congruence of $\langle A,H \rangle$ contained in $\sim$ and it is also the greatest strong H-congruence of A contained in $\sim$.

P r o o f (4). We denote by $RCon(\langle A,H \rangle)$ and $RCon_H(A)$ the sets of all reversible congruences of $\langle A,H \rangle$ and of all reversible H-congruence of A. Moreover, for each $\varphi \in Fb(A_0)$ we denote by $R_\varphi^{(n)}$ the set of all n-ary operations f acting on A_0 satisfying the following condition:

(r) for every H-congruence $\equiv$ of A if $\equiv$ is a congruence of all operations $\varphi(r,f)$ with $r \in A_0^n$, then $\varphi(q,f)(q') \equiv \varphi(q',f)(q)$ for all q and q' in A_0^n with $q \equiv q'$.

We define $Op_H^{(4)}(A) = Op_H^{(1)}(A) \cup \bigcup_{n=0}^{\infty} \bigcup_{\varphi \in H} (R_\varphi^{(n)})_\varphi$ and $Op_H^{(4)}(A)_m = \{ f \in Op_H^{(4)}(A) : f \text{ is } m\text{-ary}\}$. We define $A^{r(H)}$ to be the abstract algebra B such that $B_0 = A_0$ and $B_{1,n} = Op_H^{(4)}(A)_n$ for all natural number n.

From the construction of the algebra $A^{r(H)}$ it follows that

$$(4^*) \qquad A \leqslant_r A^{r(H)} \quad \text{and} \quad Con(A^{r(H)}) = RCon_H(A) = RCon(\langle A,H \rangle).$$

Since $A \leqslant_r A^{r(H)}$ therefore by 3.4 the algebra A admits the relative $Con(A^{r(H)})$-cogenerations. Let $\sim_r^*$ be the congruence of A which is $Con(A^{r(H)})$-cogenerated by $\sim$. Hence by (4*) the relation $\sim_r^*$ is the greatest reversible congruence of $\langle A,H\rangle$ contained in $\sim$ and it is also the greatest reversible H-congruence of A contained in $\sim$. This finishes our proof of Theorem 4.1.

The abstract algebra A is equivalent under cogenerations to the feedback algebra $\langle A,\{e\}\rangle$, where e is the unit of the monoid $Fb(A_0)$. Hence Theorem 4.1 may be considered as a generalization of my Theorem 1 from [7].

4.2. D e f i n i t i o n . Let A be any abstract algebra. If $K \subseteq Con(A)$, then K is said to be a feedback set of congruences of A provided there is a set $H \subseteq Fb(A_0)$ with $K \in \{Con_H(A), VSCon_H(A), SCon_H(A), RCon_H(A)\}$. The set of all feedback sets of congruences of A is denoted by $Q(A)$.

By 4.1 we immediately obtain:

4.3. T h e o r e m . For every abstract algebra A, for each feedback set K of congruences of A, i.e. for $K \in Q(A)$, the algebra A admits relative K-cogenerations.

By 3.13 and 4.3 we have

4.4. T h e o r e m . For every $K \in Q(F_\Sigma(X))$ the set K admits minimal relative Σ-realizations over X.

For any feedback algebra $\langle A,H\rangle$ of a type (Σ,M) we can define in an analogous way as in § 3 the categories $Epi_{(\Sigma,M)}(\langle A,H\rangle,\sim)$, $VSEpi_{(\Sigma,M)}(\langle A,H\rangle,\sim)$ and $SEpi_{(\Sigma,M)}(\langle A,H\rangle,\sim)$ of usual, very strong and strong (Σ,M)-epimorphisms of $\langle A,H\rangle$ which saturate an equivalence $\sim \in Eq(A_0)$.

From 4.1 it follows that

4.5. The categories $Epi_{(\Sigma,M)}(\langle A,H\rangle,\sim)$, $VSEpi_{(\Sigma,M)}(\langle A,H\rangle,\sim)$ and $SEpi_{(\Sigma,M)}(\langle A,H\rangle,\sim)$ have terminal objects. We have the following isomorphisms of categories:

$Epi_{(\Sigma,M)}(\langle A,H\rangle,\sim) \simeq Epi_\Sigma(A, Con_H(A),\sim)$

$VSEpi_{(\Sigma,M)}(\langle A,H\rangle,\sim) \simeq Epi_\Sigma(A,VSCon_H(A),\sim)$ and

$SEpi_{(\Sigma,M)}(\langle A,H\rangle,\sim) \simeq Epi_\Sigma(A,SCon_H(A),\sim)$.

5. Feedback enrichmental theories of abstract algebras

5.1. **Theorem.** For each feedback algebra $\langle A,H\rangle$ there is least closed feedback algebra $\langle A',H\rangle$ with $A\leq_r A'$.

Proof. For each n, we define by induction a sequence $Z_0^{(n)},Z_1^{(n)},\ldots$ through the formulas

$$Z_0^{(n)} = A_{1,n} \quad \text{and} \quad Z_{j+1}^{(n)} = Z_j^{(n)} \cup \bigcup_{\varphi\in H} \varphi(A_0^n, Z_j^{(n)})$$

We define $A' = Cl_H(A)$ to be such an abstract algebra that $A'_0 = A_0$ and for each n $A'_{1,n} = \bigcup_{j=0}^{\infty} Z_j^{(n)}$. By the construction the feedback algebra $\langle A',H\rangle$ is closed and it is the least closed feedback algebra with $A\leq_r A'$.

Let us observe that

5.2. For each $H\subseteq Fb(Z)$, the function $Cl_H : Enr_Z(Al) \longrightarrow Enr_Z(AL)$ is a monad of the poset category $Enr_Z(AL)$ with respect to the relation $\leq_r$. The Cl_H-monad algebras are H-closed abstract algebras.

5.3. **Definition.** A feedback selection is any mapping fb such that for each set Z $fb(Z)\leq Fb(Z)$. Any feedback selection fb defines a mapping $\overline{fb} : Enr(AL) \longrightarrow Enr(AL)$ such that for any abstract algebra A, $\overline{fb}(A) = Cl_{fb(A_0)}(A)$.

By 5.2 we obtain

5.4. **Theorem.** For every feedback selection fb the mapping $\overline{fb}: Enr(AL) \longrightarrow Enr(AL)$ is a monad of the poset category Enr(AL) of all abstract algebras under the relation $\leq_r$ i.e. $\overline{fb}$ is an enrichmental theory of abstract algebras.

The theories of the form $\overline{fb}$ are said to be the usual feedback enrichmental theories of abstract algebras.

5.5. **Definition.** Let $\langle A,H\rangle$ be any closed feedback algebra and let Op(A) be the set of all fundamental (basic) operations of A. Then $\langle A,H\rangle$ is called right complete if it has the property

(1) if $f\in Op(A)$, then $f_\psi \in Op(A)$ for each $\psi\in H^*$.

$\langle A,H\rangle$ is called left complete if it has the property:

(2) if $f\in \mathrm{Op}\,A$, then ${}_{\psi}f\in \mathrm{Op}(A)$ for each $\psi\in H^{*}$ (we assume that ${}_{\varepsilon}f = f$ if ε is empty word).

The algebra $\langle A,H\rangle$ is called complete if it is right and left complete.

5.6. Theorem. For every feedback algebra $\langle A,H\rangle$ there is a least right complete, left complete and complete feedback algebra $\langle A',H\rangle$ with $A\leq_{r} A'$.

Proof. For a set F of operations acting on Z and for $\varphi\in \mathrm{Fb}(Z)$ we define $\bar{\varphi}(F)$ to be the set $\{\varphi(r,f)$: there is $n, r\in Z^{n}$ and $f\in F\}$. We define by induction three sequence, $F_i^{(r)}$, $F_i^{(l)}$ and $F_i^{(c)}$, $i = 0,1,\dots$, of sets of operations acting on A_0 through the formulas:

(1) $$F_0^{(r)} = F_0^{(l)} = F_0^{(c)} = \mathrm{Op}(A),$$

(2) $$F_{i+1}^{(r)} = F_i^{(r)} \cup \bigcup_{\varphi\in H} \bar{\varphi}(F_i^{(r)}) \cup \bigcup_{\psi\in H^{*}} (F_i^{(r)})_{\psi},$$

(3) $$F_{i+1}^{(l)} = F_i^{(l)} \cup \bigcup_{\varphi\in H} \bar{\varphi}(F_i^{(l)}) \cup \bigcup_{\psi\in H^{*}} {}_{\psi}(F_i^{(l)}),$$

(4) $$F_{i+1}^{(c)} = F_i^{(c)} \cup \bigcup_{\varphi\in H} \bar{\varphi}(F_i^{(c)}) \cup \bigcup_{\psi\in H^{*}} ((F_i^{(c)})_{\psi} \cup {}_{\psi}(F_i^{(c)})).$$

For $t\in\{r,l,c\}$ we put $\mathrm{Op}_H^{(t)}(A) = \bigcup_{i=0}^{\infty} F_i^{(t)}$ and $\mathrm{Op}_H^{(t)}(A)_n = \{f\in \mathrm{Op}_H^{(t)}(A)$: f is n-ary$\}$. Moreover, for $t\in\{r,l,c\}$ we define $A^{(t)} = \mathrm{Cl}_H^{(t)}(A)$ to be the abstract algebra $B^{(t)}$ such that $B_0^{(t)} = A_0$ and $B_{1,n}^{(t)} = \mathrm{Op}_H^{(t)}(A)_n$ for all n. From the construction it follows that $\langle A^{(r)},H\rangle$, $\langle A^{(l)},H\rangle$ and $\langle A^{(c)},H\rangle$ are the least right complete, left complete and complete feedback algebras with $A\leq_{r} A^{(r)}$, $A\leq_{r} A^{(l)}$ and $A\leq_{r} A^{(c)}$.

From 5.6 we obtain

5.7. For $t\in\{r,l,c\}$, $H\subseteq \mathrm{Fb}(Z)$ the mappings $\mathrm{Cl}_H^{(t)}:\mathrm{Enr}_Z(AL)\longrightarrow \mathrm{Enr}_Z(AL)$ are the monads of the poset category $\mathrm{Enr}_Z(AL)$ of abstract algebras with universe Z under $\leq_{r}$.

5.8. **Definition.** For $t \in \{r,l,c\}$ and for each feedback selection fb we define $\overline{fb}^{(t)}$ to be a mapping $\overline{fb}^{(t)}: Enr(AL) \longrightarrow Enr(AL)$ such that

$$\overline{fb}^{(t)}(A) = Cl^{(t)}_{fb(A_0)}(A)$$

for every abstract algebra A.

By 5.7 we immediately obtain

5.9. **Theorem.** For every feedback selection fb and for $t \in \{r,l,c\}$ the mapping $\overline{fb}^{(t)}: Enr(AL) \longrightarrow Enr(AL)$ is a monad of the poset category Enr(AL) of all abstract algebras under the relation $\leqslant_r$ i.e. $\overline{fb}^{(t)}$, for $t \in \{r,l,c\}$, is an enrichmental theory of abstract algebras.

The theories of the form $\overline{fb}^{(r)}$, $\overline{fb}^{(l)}$ and $\overline{fb}^{(c)}$ are called right complete, left complete and complete feedback enrichmental theories of abstract algebras.

6. Feedback adjunctions

The categories $Fal_{(\Sigma,M)}$, $VSFal_{(\Sigma,M)}$ and $SFal_{(\Sigma,M)}$ will be denoted by $Fal^{(1)}_{(\Sigma,M)}$, $Fal^{(2)}_{(\Sigma,M)}$ and $Fal^{(3)}_{(\Sigma,M)}$.

6.1. **Theorem.** For $i = 1,2,3$ the forgetful functor $U^{(i)}: Fal^{(i)}_{(\Sigma,M)} \longrightarrow Set$, $U^{(i)}(\langle A,H\rangle) = A_0$, $U^{(i)}(h) = h$, has a left adjoint functor $F^{(i)}: Set \longrightarrow Fal^{(i)}_{(\Sigma,M)}$.

Proof. Let ξ be the arity function of Σ. We put $M^+ = M^* - \{\varepsilon\}$. Let us consider three types $\Sigma^{(1)} = \Sigma \cup M \times \Sigma$, $\Sigma^{(2)} = \Sigma \cup M^+ \times \Sigma$ and $\Sigma^{(3)} = \Sigma \cup M^+ \times \Sigma$ with the arity functions $\xi^{(1)}$, $\xi^{(2)}$ and $\xi^{(3)}$ such that $\xi^{(1)}(\langle m,\sigma\rangle) = 2\xi(\sigma)$, $\xi^{(2)}(\langle \psi,\sigma\rangle) = (|\psi|+1)\xi(\sigma)$, $\xi^{(3)}(\langle \psi,\sigma\rangle) = 2\xi(\sigma)$ and $\xi^{(i)}(\sigma) = \xi(\sigma)$ for $i = 1,2,3$. We denote by $W^{(1)}$ and $W^{(3)}$ the $\Sigma^{(1)}$-variety and $\Sigma^{(3)}$-variety defined by all equations of the form $\langle m,\sigma\rangle(xx) = \sigma(x)$, where n is an arbitrary natural number, $\sigma \in \Sigma_n$, $m \in M$ and x 's a one-to-one n-ary sequence of variables. Let $W^{(2)}$ be the $\Sigma^{(2)}$-variety defined by all equations of the form

$$\langle m,\sigma\rangle(xx) = \sigma(x)$$

$$\langle m\psi,\sigma\rangle(xyx) = \langle\psi,\sigma\rangle(yx)$$

where n is arbitrary natural number, $\sigma\in\sum_n$, $m\in M$, $\psi\in M^+$ and x and y are one-to-one n-ary and $|\psi|\cdot n$-ary sequences of variables. For every set Y and $i = 1,2,3$ we denote by $B^{(i)}$ the free algebra in $W^{(i)}$ freely generated by Y. Let us denote by $K^{(1)} = \left\{m_{K^{(1)}} : m\in M\right\}\subseteq Fb(B_0^{(1)})$ such that for arbitrary n, all $\sigma\in\sum_n$, $m\in M$ and $r,q\in B_0^{(1)n}$ we have

$$m_{K^{(1)}}(r, \sigma_{B^{(1)}})(q) = \langle m,\sigma\rangle_{B^{(1)}}(rq).$$

We define by induction $K^{(2)} = \left\{m_{K^{(2)}} : m\in M\right\}\subseteq Fb(B_0^{(2)})$ through the defining formulas

$$m_{K^{(2)}}(r,G(B_0^{(2)})(\langle t,\psi\rangle, \sigma_{B^{(2)}})(q) = \langle m\psi,\sigma\rangle_{B^{(2)}}(rtq)$$

where n is arbitrary, $\sigma\in\sum_n$, $m\in M$, $\langle t,\psi\rangle\in d(B_0^{(2)},K^{(2)})$ and $r,q\in B_0^{(2)n}$. Moreover, we define by induction the set $K^{(3)} = \left\{m_{K^{(3)}} : m\in M\right\}\subseteq Fb(B_0^{(3)})$ through the defining formulas

$$m_{K^{(3)}}(r,G(B_0^{(3)})(\langle r^{|\psi|},\psi\rangle,\sigma_{B^{(3)}})(q) = \langle m\psi,\sigma\rangle_{B^{(3)}}(rq)$$

where n is arbitrary, $\sigma\in\sum_n$, $m\in M$, $\psi\in K^{(3)*}$ and $r,q\in B_0^{(3)n}$. Then for $i = 1,2,3$ $F^{(i)}(Y) = \langle B^{(i)}|\sum, K^{(i)}\rangle$, where $B^{(i)}|\sum$ is the $\sum$-retract of $B^{(i)}$. From the construction of $F^{(i)}$ it follows that $F^{(i)}$ is a left adjoint functor to $U^{(i)}$. This finishes our proof of Theorem 6.1.

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