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REMARKS ON WEAK AUTOMORPHISMS OF 1-UNARY ALGEBRAS

0. Let $\mathfrak{A} = (A, F)$ be an algebra with the clone $A(\mathfrak{A})$ of algebraic operations (=superpositions of all projections $e_k(x_1,\ldots,x_n) = x_k$ and the operations from F). Any permutation $\mu \in S_A$ induces a permutation $f \longrightarrow f^{\mu} = \mu^{-1} f \mu$ on the set of all operations on the set A. If $A(\mathfrak{A})$ is fixed under this permutation, μ is called a weak automorphism of $\mathfrak{A}$. This notion was introduced by A.Goetz in [2]. Except of special algebras as Boolean and Post algebras ([10]), linear or affine space, algebras with a basis ([1], [7]) and nilpotent groups ([3], [11]) very little is known. Recently L. Polak in [8] has described the group of weak automorphisms of 1-unary algebras $\mathfrak{A} = (A,f)$ as a semi-direct product of the automorphism group of $\mathfrak{A}$ and the group $C = \{k \in Z_n : (A,f)$ is isomorphic to (A,f^k) and $(k,n)=1\}$. Here n is the smallest natural number such that $f^{n+1} = f$. This description seems to be rather unpleasant, because the group C is not given explicitly.

The purpose of this paper is to give more information about the group of weak automorphisms of 1-unary algebras.

I. Since the clone of algebraic operations of the 1-unary algebra $\mathfrak{A} = (A,f)$ consists of $\{1,f,f^2,\ldots\}$ and the powers of projections, any weak automorphism μ of $\mathfrak{A}$ has to induce

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an automorphism of the semi-group $(\{1,f,f^2,\dots\},\cdot,1)$. Thus if $f^{\mu} = f^k$ for some $k>1$, then $f^{n+1} = f$ for some positive n. If there is no such n, then any weak automorphism of $\mathfrak{A}$ is an automorphism of $\mathfrak{A}$. The group of automorphisms of 1-unary algebras were intensively studied in [5] and [6]. Throughout this paper n is the smallest positive integer such that $f^{n+1} = f$ and $WAut(\mathfrak{A})$, $Aut(\mathfrak{A})$ stand for the group of weak automorphisms of $\mathfrak{A}$ and the group of automorphisms of $\mathfrak{A}$, respectively.

P r o p o s i t i o n . Let $\mathfrak{A} = (A,f)$ be a 1-unary algebra with $f^{n+1} = f$. Then:

1. The restriction of f to the image $X = f(A)$ is a permutation of order n.

2. Any $\mu \in WAut(\mathfrak{A})$ preserves the subalgebra (X,f) of $\mathfrak{A}$.

3. The relation $a \sim b$ if and only if $f(a) = f(b)$ is a congruence of $\mathfrak{A}$.

P r o o f . We have $f^{n+1}(a) = f^n(f(a)) = 1(f(a))$ for all $a \in A$, which means that f maps X onto itself and $f^n = 1$. Since $f^k(x) = x$ for all $x \in X$ implies $f^{k+1}(a) = f(a)$ for all $a \in A$, the operation f is a permutation of order n on the set X. Statements 2 and 3 are immediate.

Let $O(x_0) = \{x_0,x_1,\dots,x_{k-1}\}$, where $f(x_i) = x_{i+1}$ ($i+1$ is taken mod k, of course) be the orbit of the element $x_0 \in X$. It is well known (cf.e.g. [4]) that k must divide n, there is no infinite orbit in X and all the orbits form a partition of the set X. If X_{k_i}, $1 \le i \le N$, is the collection of all k_i-element orbits which occur in X, then $n = g.c.d.\{k_1,k_2,\dots,k_N\}$.

II. T h e o r e m 1. If μ is a weak automorphism of the algebra $\mathfrak{A}_{k_i} = (X_{k_i},f)$ such that $f^{\mu} = f^k$ on X, then $(k,k_i) = 1$. Conversely, if k is a positive integer such that $(k,k_i) = 1$, then the permutation μ_{k_i} of X_{k_i} defined on each k_i-element orbit $\{x_0,x_1,\dots,x_{k_i-1}\}$ by the formula

(1) $$\mu_{k_i}(x_{pk}) = x_p \quad \text{for} \quad p=0,1,\dots,k_i-1$$

is an element of $WAut(\mathfrak{A}_{k_i})$ with $f^{\mu_{k_i}} = f^k$.

P r o o f . Since the mapping $f \to f^{\mu} = f^k$ has to be a permutation on the set $\{1,f,\dots,f^{k_i-1}\}$, k must be relatively prime to k_i, because if $(f^l)^{\mu} = f$, then $f^{kl} = f$, i.e., $kl \equiv 1 \pmod{k_i}$. On the other hand, the assumption $(k,k_i) = 1$ implies that the mapping defined by (1) is in fact a permutation on the set X_{k_i}. Moreover, we have

$$\mu_{k_i}^{-1} f \mu_{k_i}(x_{pk}) = \mu_{k_i}^{-1} f(x_p) = \mu_{k_i}^{-1}(x_{p+1}) = x_{(p+1)k} = f^k(x_{pk}),$$

where $p+1$, $(p+1)k$ and pk are taken mod k_i. Theorem 1 is thus proved.

T h e o r e m 2. There is an element μ in $WAut(X,f)$ with $f^{\mu} = f^k$ if and only if $(k,k_i) = 1$ for all $i = 1,2,\dots,N$. Moreover, any such weak automorphism μ of (X,f) is of the form $\mu = \mu_0\alpha$, where $\alpha \in Aut(X,f)$ and μ_0 is defined by

(2) $$\mu_0(x) = \mu_{k_i}(x) \quad \text{for} \quad x \in X_{k_i},\ 1 \le i \le N.$$

P r o o f . Observe that any weak automorphism μ of (X,f) maps every orbit $O(x_0)$ onto other orbit $O(x_0')$, because μ permutes f^i, $i = 1,2,\dots$. Thus μ has to preserve the sum of all k_i-element orbit, i.e., X_{k_i} and, consequently, the restriction of μ to the set X_{k_i} is an element of $WAut(\mathfrak{A}_{k_i})$. It follows from Theorem 1 that $(k,k_i) = 1$ for all $i=1,\dots,N$.

Now let μ be a weak automorphism of (X,f) such that $f^{\mu} = f^k$. First of all we check that μ_0 defined in (2) is an element of $WAut(X,f)$. Indeed, if k is relatively prime to each k_i, $1 \le i \le N$, then μ_0 is a permutation of X_{k_i} and accordingly to Theorem 1, it is a weak automorphism μ_{k_i} of the algebra $\mathfrak{A}_{k_i}$, $i = 1,2,\dots,N$, such that $f^{\mu_0} = f^k$.

Let us consider the restriction of the function $\mu_0^{-1}\mu$ to the set X_{k_i}, $k_i > 1$. We have

$$\mu^{-1}\mu_0 f \mu_0^{-1}\mu = \mu^{-1}\mu_{k_i} f \mu_{k_i}^{-1}\mu = \mu^{-1} f^l \mu = f^{kl},$$

where l is a positive integer from the set $\{0,1,\ldots,k_i-1\}$ such that $\mu_{k_i}^{-1} f^l \mu_{k_i} = f$. This gives $kl \equiv 1 \pmod{k_i}$ and, consequently, the function $\mu_0^{-1}\mu$ maps f on f which is enough to be an automorphism of the algebra (X,f). Theorem 2 is thus proved.

R e m a r k . For $k = 1$ we get $\mathrm{Aut}(X,f) = \mathrm{Aut}(\mathfrak{A}_{k_1}) \times \mathrm{Aut}(\mathfrak{A}_{k_2}) \times \ldots \times \mathrm{Aut}(\mathfrak{A}_{k_N})$. In turn, if $\{x_t\}_{t \in T_i}$, $1 \leq i \leq N$, is a selector of all orbits in A_{k_i}, then the algebra $(O(x_t),f)$ can be identify with the algebra $(\{0,1,\ldots,k_i-1\},\sigma)$, where σ is a permutation defined by $\sigma(i) = i+1 \pmod{k_i}$. Thus $\mathrm{Aut}(O(x_t),f) = Z_{k_i}$ for all $i = 1,2,\ldots,N$ and the group of automorphisms of $\mathfrak{A}_{k_i}$ is (isomorphic to) the wreath product of Z_{k_i} and the symmetric group S_{T_i} on the set T_i (cf. [5]).

III. T h e o r e m 3. A weak automorphism μ of (X,f) can be extended to a weak automorphism $\bar{\mu}$ of the algebra $\mathfrak{A} = (A,f)$ with $f^{\bar{\mu}} = f^k$ if and only if the cardinality of the sets $[a]_\sim$ and $f^{-1}(\{\mu f^k(a)\})$ is the same for all $a \in X' = A - X$.

P r o o f . By statement 2 of the Proposition, any weak automorphism $\bar{\mu}$ of $\mathfrak{A}$ maps the elements of X' onto itself. Thus, if $\bar{\mu}$ is an extension of $\mu \in \mathrm{WAut}(X,f)$, then, of course, $f^\mu = f^k$. Moreover, since $k > 0$, μ is a permutation on X and the last equality gives $f(\bar{\mu}(a)) = \mu(f^k(a))$ for all $a \in X'$. According of statement 3 of the Proposition the relation $\sim$ is a congruence of $\mathfrak{A}$ and any weak automorphism of $\mathfrak{A}$ has to permute the abstract classes of $\sim$. Thus $\bar{\mu}([a]_\sim) = f^{-1}(\{\mu f^k(a)\})$.

Suppose now that $\mu \in \mathrm{WAut}(X,f)$ with $f^{\mu} = f^{k}$ satisfies the condition $\mathrm{card}([a]_{\sim}) = \mathrm{card}(f^{-1}(\{\mu f^{k}(a)\}))$ for all $a \in X'$. Let ψ_a be a bijection of the set $[a]_{\sim}$ onto the set $f^{-1}(\{\mu f^{k}(a)\})$, where a runs through a selector T of abstract classes of $\sim$ in the set X'. We put

$$\bar{\mu}(x) = \begin{cases} \mu(x) & \text{if } x \in X, \\ \psi_a(x) & \text{if } x \in [a]_{\sim}, \quad a \in T. \end{cases}$$

Clearly, $\bar{\mu}$ is a permutation of A. We have also $\psi_a(x) \in f^{-1}(\{\mu f^{k}(a)\})$, which is equivalent to $f(\psi_a(x)) = \mu f^{k}(a)$. Thus for any $x \in [a]_{\sim}$, $a \in T$, we have

$$\mu^{-1} f \bar{\mu}(x) = \bar{\mu}^{-1} f \psi_a(x) = \bar{\mu}^{-1} \mu f^{k}(a) = f^{k-1}(f(a)) = f^{k-1}(f(x)) = f^{k}(x)$$

for all $x \in A$. Therefore $\bar{\mu}$ is a weak automorphism of $\mathfrak{A}$, as required.

Corollary 1. Since any bijection ψ_a of $[a]_{\sim}$ onto $f^{-1}(\{\mu f^{k}(a)\})$ is a superposition of a fixed bijection and some permutation of the set $[a]_{\sim}$, the set of all extensions of μ mentioned in Theorem 3 can be identified with the Cartesian product $\prod_{[a]\in T} S_{[a]}$.

Corollary 2. If all classes $[a]_{\sim}$, $a \in T$, have the same cardinality and $f(X') = X$, then each weak automorphism of (X,f) can be extended to a weak automorphism of the algebra $\mathfrak{A}$.

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