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LATTICES OF FREE EXTENSIONS IN CATEGORIES OF PARTIALLY ALGEBRAIC STRUCTURES

It is not a big risk to say that the class of surjective epimorphisms is big enough if we deal with classical problems of the theory of total algebras. None of the basic concepts and fundamental theorems needs others then surjective epis. Only in some special varieties non-surjective epis are considered - for example, localizations in the theory of rings. In our opinion among all reasons of it the most important are the following. First - each epi in a category of all algebras of a given type is surjective. Second - each algebra in a given variety V is a surjective image of a suitable free V-algebra. And third - in any variety there exists a factorization system (Surjections, Monos).

But working with partial algebras we can not restrict our attention to surjective epis only. One can easily check that none of these three postulates remains true for surjections in the theory of partial algebras. Hence we propose to distinguish another class of epis in categories of partially algebraic structures which may play the same role as surjections play for total algebras. More precisely; in each variety V of partial algebras we distinguish a composition class $E_V = \mathrm{Clepi}_V \cdot \mathrm{Ext}_V$, where Clepi_V is a class of all closed [2]

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(strong in [3]) surjections in V and Ext_V is a class of "free extensions in V" defined in this paper. Free extensions in a category Palg Ω of all partial Ω-algebras are precisely extendable epis. But in many varieties these classes may be different.

What makes the universal algebra succesful is that surjections are described by congruences. The calculus of congruences is a one of the most important tools of this theory. It will be very useful to get a similar tool in the theory of partial algebras. Closed epis have their description - they are described by closed (strong) congruences. Hence in the present paper we deal only with free extensions; to give a framework of a description of lattices of free extensions is a main goal of the present paper. Then combining together descriptions of both classes we can obtain a full description of all epi's from the distinguished class.

Our approach is of the categorical nature. Backgrounds are presented in sections 1 and 2. We use spectral algebraic theories [6] to define free extensions, examine their basic properties and describe lattices of free extensions (section 3). Moreover, we show that our starting point and basic example - extendable epis in Palg Ω is rather a very peculiar one (section 4). Next section is more concrete. First, considering free extensions in Palg Ω we introduce a concept of a saturated initial segment which plays a role of a congruence in a desired description of free extensions. Then, considering free extensions in a given variety $V \subset$ Palg Ω we describe a relationship between free extensions in both categories.

1. Preliminaries - varieties of partial algebras as categories

For all unexplained concepts and notations concerning partial algebras we refer the reader to [2]. The only difference is that closed homomorphisms are called here perfect. We shall use Palg Ω to denote a category of all partial Ω-algebras of a given type $U_\Omega:\mathrm{Palg}\,\Omega \longrightarrow \mathrm{Set}$ is the obvious forgetful functor. ΩX denotes a set of all Ω-terms over a set X, $\underline{\Omega X}$ is an absolutely free total Ω-algebra over X (with a carrier ΩX).

The following observation, due to J.Schmidt [11], forms a base of our categorical description of partial algebras.

T h e o r e m 1.1. For each set X there exsits an ordered family $\{\underline{X}_i\}_{i\in S_\Omega X}$ of partial Ω-algebras such that

i. for each $i \in S_\Omega X$ there is a function $\varrho_i^X : X \longrightarrow U_\Omega \underline{X}_i = X_i$,

ii. if $i \leq j$ in $S_\Omega X$, then there exists a unique homomorphism $\varphi_{ij}:\underline{X}_i \longrightarrow \underline{X}_j$ such that $\varphi_{ij}\cdot\varrho_i^X = \varrho_j^X$
and the following holds: for each $\underline{A}$ in $\mathrm{Palg}\,\Omega$ and a function $h:X \longrightarrow U_\Omega\underline{A}$ there exists a unique $i \in S_\Omega X$ together with a perfect map $\tilde{h}:\underline{X}_i \longrightarrow \underline{A}$ such that $\tilde{h}\cdot\varrho_i^X = h$. Moreover, if $h_1:\underline{X}_j \longrightarrow \underline{A}$ and $h_1\cdot\varrho_j^X = h$, then $j \leq i$ and $h_1 = \tilde{h}\cdot\varphi_{ji}$.

Recall a construction of the family $\{\underline{X}_i\}_{i\in S_\Omega X}$ (called a Ω-spectrum over X). By an initial Ω-segment over X we mean each subset $X_i \subset \Omega X$ containing X and such that together with any term $t \in X_i$ contains each subterm of t. Then the desired family consists of all initial Ω-segments over X endowed with structures of relative subalgebras of $\underline{\Omega X}$.

From now we will assume that Ω is a finitary type.

A class $V \subset \mathrm{Palg}\,\Omega$ is called a variety if V is closed under products, perfect epi images and (closed) subalgebras, i.e. V is a $H_c S_s P$-closed class in the sense of [1]. Each variety is a class of models of a set of formulas of the form

$$(+) \qquad \Big(\dot{\bigvee_{i\in I}}\ \exists t_i\Big) \Longrightarrow p = q \qquad [1]$$

(Recall that $\underline{A}$ in $\mathrm{Palg}\,\Omega$ satisfies $\exists t$ at a given valuation k if the term t is defined in $\underline{A}$ at the valuation k). In other words, each variety is a class of models of "generalized ECE-equations"

$$\Big(\bigvee_{i\in I}\ t_i \overset{e}{=} t_i\Big) \Longrightarrow p \overset{e}{=} q$$

in the sense of P.Burmeister [2].

A variety V is called a $\exists$-variety if for each perfect map $h:\underline{A} \longrightarrow \underline{B}$, $\underline{A} \in V$ provided $\underline{B} \in V$. Each $\exists$-variety is a class of models of a set of formulas of the form

$$(++) \qquad \Big(\bigvee_{i\in I} \exists t_i\Big) \Longrightarrow \exists t.$$

We will identify varieties with the corresponding full subcategories. $F_V:\mathrm{Palg}\Omega \longrightarrow V$ will always denote a left adjoint left inverse to the embedding functor $Z_V:V \longrightarrow \mathrm{Palg}\Omega$. Note that for each $\underline{A}$ in $\mathrm{Palg}\Omega$ the universal arrow $\delta_A:\underline{A} \longrightarrow Z_V F_V \underline{A}$ is epi.

Observe that the Theorem 1.1 remains true if we replace $\mathrm{Palg}\Omega$ by V in each place it appears in the theorem. Let $\{\underline{\tilde{X}}_i\}_{i\in S_V X}$ denote a V-spectrum over a given set X. First define $S_V X$ as a subset of $S_\Omega X$ (with the induced ordering) as follows: for each i in $S_\Omega X$, $i\in S_V X$ iff $\delta_{\underline{X}_i}:\underline{X}_i \longrightarrow Z_V F_V \underline{X}_i$ is perfect. Then put $\underline{\tilde{X}}_i = F_V \underline{X}_i$, $\tilde{\varrho}_i^X = \delta_{\underline{X}_i}\cdot\varrho_i^X$.

By $\exists V$ we denote a smallest $\exists$-variety containing a given variety V.

L e m m a 1.2. i. For any $\underline{A}$ in $\mathrm{Palg}\Omega$, $\underline{A} \in \exists V$ iff $\underline{A}$ satisfies all formulas of the form (++) valid in V,

ii. $S_V X = S_{\exists V} X$, for each set X,

iii. if $V = \exists V$, then for each $i \in S_V X$, $\underline{\tilde{X}}_i = \underline{X}_i$ and for each $\underline{A}$ in $\mathrm{Palg}\Omega$, δ_A is an extendable epi.

iv. The embedding $Z_o:V \longrightarrow \exists V$ has a left adjoint left inverse F_o and each universal arrow $\delta^o_A:\underline{A} \longrightarrow Z_o F_o \underline{A}$ ($\underline{A} \in \exists V$) is perfect and surjective.

Only the second part of iv is nontrivial. We will prove it in the end of section 2.

2. Categorical backgrounds

By a spectral algebraic theory [6] in Set (s.a.t., for short) we mean each 4-tuple $S = (S, J, \varrho, (\ )^*)$ such that

$S:\mathrm{ObSet} \longrightarrow \mathrm{ObPOS}$ (POS =, the category of posets),

$J = (J_X : SX \longrightarrow Set)_{X \in ObSet}$, $\varrho = (\varrho^X : \Delta X \longrightarrow J_X)_{X \in ObSet}$
(Δ is a "constant" functor) and $(\ )^*$ assigns to each pair $(f : X \longrightarrow J_Y j, j)$ a pair $(f,j)^* = (f_j^* : J_X i \longrightarrow J_Y j, i)$ such that the following hold:

i. $f_j^* \cdot \varrho_i^X = f$,

ii. $(\varrho_i^X, i)^* = (id_{J_X i}, i)$ for each set X and $i \in SX$,

iii. if $(f,j)^* = (f_i^*, i)$, $(g,i)^* = (g_i^*, k)$, then $(f_j^* g, j)^* = (f_j^* g_i^*, k)$.

iv. if $j \leq k$ in SY, then $J_Y(j \leq k) \cdot f_j^* = (J_Y(j \leq k) \cdot f)_k^* \cdot J_X(i \leq r)$ for the suitable $i \leq r$ in SX.

Notation. For the simplicity we will write X_i instead of $J_X i$ and φ_{ir} instead of $J_X(i \leq r)$. If it will be no danger of confusion for $(f : X \longrightarrow Y_j, j)$ we will write f^* instead of f_j^*. The equation $dom f^* = X_i$ will always mean $(f,j)^* = (f^*, i)$.

S-algebras are triples $\underline{A} = (A, i \in SA, a : A_i \longrightarrow A)$ (we will also write (A, A_i, a)) such that $a \cdot \varrho_i^A = id_A$ and for each pair $f, g : X \rightrightarrows A_i$, if $af = ag$, then $dom f^* = dom g^*$ and $af^* = ag^*$.

We extend the object function S to a contravariant functor $S : Set^{op} \longrightarrow POS$ as follows; for each $h : X \longrightarrow Y$ and $j \in SY$, $Sh(j) = i$, where $dom(\varrho_j^Y \cdot h)^* = X_i$.

By an S-morphism from (A, A_i, a) to (B, B_j, b) we mean each function $h : A \longrightarrow B$ such that $i \leq Sh(j) = r$ and $b \cdot (\varrho_j^B \cdot h)^* \varphi_{ir} = h \cdot a$. If, moreover, $Sh(j) = i$, then h is called perfect.

Note that for each set X and $i \in SX$, $\underline{X}_i = (X_i, dom(id_{X_i})^*, (id_{X_i})^*)$ is an S-algebra and for $j \leq i$ in SX, φ_{ji} is an S-morphism from $\underline{X}_j$ to $\underline{X}_i$. We shall use S-Alg to denote the category of S-algebras, $U_S : S\text{-Alg} \longrightarrow Set$ is the forgetful functor. But for a simplicity instead of $U_S h$ we will often write simply h.

A s.a.t. S is complete (pointed) if each SX is a complete lattice (has a greatest element i_X) and for each $h : X \longrightarrow Y_k$

with $k = \inf\{j : j \in J\}$, $(k = i_Y)$, $\mathrm{dom}h^* = X_s$, where $s = \inf\{s_j : j \in J\}$ where $X_{s_j} = \mathrm{dom}(\varphi_{kj} \cdot h)^*$, $(s = i_X)$.

S is complete iff S-Alg has all concrete limits [7].

E x a m p l e s . 1. Let $S_\Omega X$ denote a lattice of all initial Ω-segments over a given set X, $J_X(X_i) = X_i = U_\Omega \underline{X}_i$. For each $h: Y \to X_i$ we put $h^* = U_\Omega \tilde{h}$ (compare Theorem 1.1). Then $S_\Omega = (S_\Omega, J, \varrho, (\ )^*)$ is a s.a.t., S_Ω-Alg and PalgΩ are concretely isomorphic ([5], [8]). Concepts of perfect maps in both categories coincide, total S-algebras correspond to total Ω-algebras. Partial Ω-algebra $\underline{A}$ with a carrier A is represented as a triple (A, A_i, a) where A_i is the initial Ω-segment consisting of all terms over A defined in $\underline{A}$ and $a: A_i \to A$ is a valuation map.

We will identify both categories in the sequel.

2. For a given variety $V \subset$ PalgΩ we construct a s.a.t. $S_V = (S_V, \tilde{J}, \tilde{\varrho}, (\ )^*)$, where $S_V X$ is a subset of $S_\Omega X$ defined in section 1, $\tilde{J}_X X_i = U_\Omega F_V \underline{X}_i$ and $\tilde{\varrho}, (\ )^*$ are determined in the obvious way. Again, one can easily show that V and S_V-Alg are concretely isomorphic.

Note that S_Ω and S_V are complete s.a.ts.

We finish this section with some results illustrating the similarity between partial algebras and S-algebras. To save the space the routine proofs are omitted.

L e m m a 2.1. (generalization of Theorem 1.1). Let $h: X \to U_S \underline{A}$ for some $\underline{A}$ in S-Alg. Then there exists a unique $i \in SX$ together with a perfect S-morphism $\tilde{h}: \underline{X}_i \to \underline{A}$ s.t. $\tilde{h} \cdot \varrho_i^X = h$. Moreover, if $h_1: \underline{X}_j \to \underline{A}$ in S-Alg, $h_1 \cdot \varrho_j^X$, then $j \leq i$ and $h_1 = \tilde{h} \cdot \varphi_{ji}$. If, moreover, $f: \underline{A} \to \underline{B}$ is perfect, then $\widetilde{fh} = f \cdot \tilde{h}$.

P r o o f . For $\underline{A} = (A, A_k, a)$, put $\tilde{h} = a \cdot (\varrho_k^A \cdot h)^*$.

L e m m a 2.2. Each perfect S-morphism is uniquely factorizable as a perfect surjective S-morphism followed by a perfect S-mono.

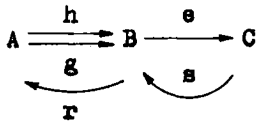
Assume now that S is pointed. We call an S-algebra $\underline{A} = (A, A_i, a)$ total iff $i = i_A$.

L e m m a 2.3.i. The full subcategory TotS-Alg of total S-algebra is monadic over Set.

ii. The embedding functor Z_t:TotS-Alg $\longrightarrow$ S-Alg has a left adjoint left inverse F_t:S-Alg $\longrightarrow$ TotS-Alg.

P r o o f . We sketch ii. only. For each $\underline{A} = (A, A_i, a)$ we construct a universal arrow $\gamma_{\underline{A}}:\underline{A} \longrightarrow Z_t F_t \underline{A}$ as follows: let e be a coequalizer of a pair $\left((\varrho^A_{i_A} \cdot a)^*,\ (\varphi_{1i_A})^*\right)$ in TotS-Alg. Then $\gamma_A = e \cdot \varrho^A_{i_A}$. Note that then $e = F_t a$.

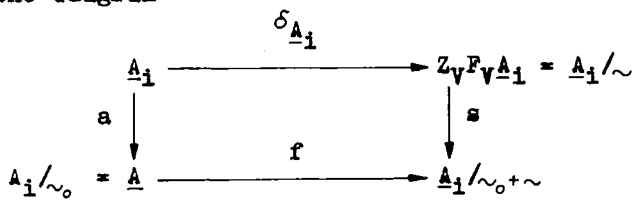
L e m m a 2.4. Assume that $h, g:(A, A_i, a) \rightrightarrows (B, B_j, b)$ are perfect S-morphisms and there exist functions $e:B \longrightarrow C$, $s:C \longrightarrow B$, $r:B \longrightarrow A$



such that $es = id$, $eh = eg$, $hr = id$, $gr = se$. Then there exists a unique S-algebra (C, C_k, c) making e a perfect coequalizer of (h,g).

P r o o f . Put $k = Ss(j)$. Then $(\varrho^C_k \cdot e)^*$ is a coequalizer of $(\varrho^B_j h)^*$, $(\varrho^B_j g)^*$ in Set. But $e \cdot a$ equalizes this pair. Hence $e \cdot a = c \cdot (\varrho^C_k \cdot e)^*$. One can check that (C, C_k, c) is the desired S-algebra.

Now we prove Lemma 1.2.iv. Let $\underline{A} = (A, A_i, a) \in \exists V$ and let $S_{\exists V}$ be the corresponding s.a.t. Then $i \in S_{\exists V} A = S_V A$ and $a:\underline{A}_i \longrightarrow \underline{A}$, $\delta_{\underline{A}_i}:\underline{A}_i \longrightarrow Z_V F_V \underline{A}_i$ are perfect and surjective. Perfect surjective epis correspond to strong (closed) congruences. Consider the diagram



where $\sim + \sim_0$ is a smallest equivalence relation containing $\sim \cup \sim_0$. It is a strong congruence because Ω is a finitary type. Hence a projection f is perfect. Now it is not hard to show that $f = \delta^{o}_{\underline{A}}$.

3. Free extensions-general theory

Recall that an epimorphism of partial Ω-algebras $h:\underline{A} \longrightarrow \underline{B}$ is called extendable if for each $g:\underline{A} \longrightarrow \underline{C}$ with $\underline{C}$ being total, $g = g_1 \cdot h$ for some homomorphism g_1. Equivalently, h is extendable if $F_t h$ is iso (F_t:Palg$\Omega \longrightarrow$ TotPalgΩ - compare Lemma 2.3.ii). This concept is of a pure categorical nature and it has its categorical generalization ([4], def. 37.8). But we want to obtain a factorization system (?, Perfect S-morphisms) hence we need a more precise generalization. To see the problem consider the following. Let $\Omega = \Omega_1 = \{p,q\}$, $V = \mathrm{Mod}(\exists\, p(x) \Longrightarrow x = y)$. Then the only total V-algebra is a one-point algebra. Hence each epi in V is an extendable epi so there is no desired factorization system in V.

Let $S = (S, J, \varrho, (\ )^*)$ be an arbitrary but fixed s.a.t. in Set.

D e f i n i t i o n 3.1. An S-morphism $h:\underline{A} = (A, A_i, a) \longrightarrow \underline{B}$ is called a free extension (of $\underline{A}$) if the commutative square

$$\begin{array}{ccc} \underline{A}_k & \xrightarrow{\ \tilde{h}\ } & \underline{B} \\ {\scriptstyle \varphi_{ik}} \uparrow & & \uparrow {\scriptstyle h} \\ \underline{A}_i & \xrightarrow{\ a\ } & \underline{A} \end{array}$$

($\tilde{h}$ = the perfect extension of $h:A \longrightarrow U_S\underline{B}$ - see Lemma 2.1)

is a pushout in S-Alg.

Note that each free extension is epi, each iso is a free extension and each free extension of a total S-algebra is iso.

L e m m a 3.2. Let (A, A_i, a) be an S-algebra. Then

i. For $h:(A, A_i, a) \longrightarrow (B, B_j, b)$ with $\mathrm{Sh}(j) = k$, h is a free extension iff $\mathrm{dom}(\varrho^A_k \cdot a)^* = \mathrm{dom}(\varphi_{ik})^*$ and $\tilde{h}$ is a perfect coequalizer of this pair in S-Alg.

ii. If $i \leqslant s$ in SA, $\mathrm{dom}(\varrho_s^A \cdot a)^* = \mathrm{dom}(\varphi_{is})^*$ and this pair has a perfect coequalizer $e: \underline{A}_s \longrightarrow \underline{D}$ in S-Alg, then the unique S-morphism $g: \underline{A} \longrightarrow \underline{D}$ s.t. $g \cdot a = e \cdot \varphi_{is}$ is a free extension,

iii. If S is pointed, then for each S-algebra $\underline{A}$, $\gamma_{\underline{A}}: \underline{A} \longrightarrow Z_t F_t \underline{A}$ is a free extension.

P r o o f s of i. and ii. are straightforward. iii. follows from ii. and the construction of $\gamma_{\underline{A}}$.

L e m m a 3.3. Free extensions divide perfect morphisms, i.e. whenever we are given a commutative square in S-Alg

$$\begin{array}{rcccl} (A, A_i, a) = & \underline{A} & \xrightarrow{\quad e \quad} & \underline{B} & \\ f & \downarrow & g & \downarrow & h \\ & \underline{D} & \xrightarrow{\qquad} & \underline{C} & \end{array}$$

where e is a free extension and g is perfect, then there exists $\xi: \underline{B} \longrightarrow \underline{D}$ such that $\xi \cdot e = f$ and $g \cdot \xi = h$.

P r o o f . Consider the commutative diagram in S-Alg:

$$\begin{array}{ccccccl} \underline{A}_k & \xrightarrow{\quad \tilde{e} \quad} & \underline{B} & \xrightarrow{\quad h \quad} & \underline{C} & = & (C, C_p, c) \\ \varphi_{ik} \uparrow & a & \uparrow & f & \uparrow g & & \\ \underline{A}_i & \xrightarrow{\qquad} & \underline{A} & \xrightarrow{\qquad} & \underline{D} & = & (D, D_s, d) \end{array}$$

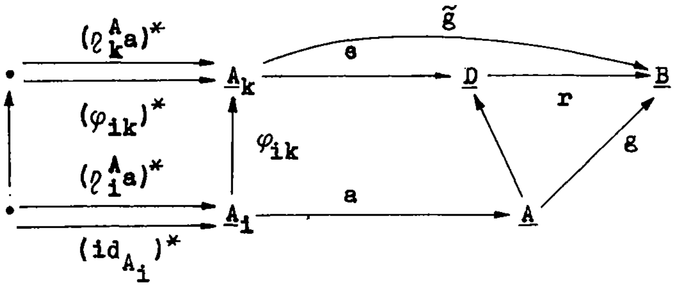
Then $Sf(s) = SfSg(p) = She(p) \geqslant k$, i.e. there exists $\hat{f}: \underline{A}_k \longrightarrow \underline{D}$ such that $\hat{f} \cdot \varrho_k^A = f$ (compare Lemma 2.1). Moreover, $\hat{f} \cdot (\varrho_k^A \cdot a)^* = \hat{f} \cdot (\varphi_{ik})^*$ and, by Lemma 3.2.ii., $\hat{f} = \xi \cdot \tilde{e}$. Then $\xi \cdot e = f$, $g \cdot \xi = h$.

C o r o l l a r y 3.4. If S is pointed and $h: \underline{A} \longrightarrow \underline{B}$ is a free extension, then $F_t h$ is iso and $\gamma_{\underline{A}} = h \cdot \gamma_{\underline{B}}$. In particular, each free extension in PalgΩ is an extendable epi.

By $\mathrm{Ext}_{S\text{-}Alg}$ (or simply Ext) we shall denote a class of free extensions in S-Alg. $\mathrm{Perf}_{S\text{-}Alg}$(Perf) denotes a class of perfect maps.

T h e o r e m 3.5. Assume that coequalizers of pairs of perfect S-morphisms exist in S-Alg. Then a pair ($\mathrm{Ext}_{S\text{-}Alg}$, $\mathrm{Perf}_{S\text{-}Alg}$) forms a factorization system in S-Alg.

P r o o f . By Lemma 3.3 we must show only that each S-morphism has the required factorization. Let $g:(A,A_i,a)\longrightarrow(B,B_j,b)$. Consider the diagram



$dom(\varrho_k^A\cdot a)^* = dom(\varphi_{ik})^*$ because $\tilde{g}$ is perfect (Lemma 2.1). Let e be a coequalizer of this pair. $\tilde{g} = r\cdot e$, hence e is perfect. By Lemma 2.2 perfect coequalizers are surjective, hence r is perfect too. a is a coequalizer of $(\varrho_i^A\cdot a)^*$, $(id_{A_i})^*$, hence $e\cdot\varphi_{ik} = \psi\cdot a$ for some ψ and by Lemma 3.2.ii. ψ is a free extension. Obviously, $g = r\cdot\psi$. The proof is complete.

R e m a r k . It has been proved in [7] that for each complete s.a.t. S, S-Alg is cocomplete. Hence, in particular, Theorem 3.5 is valid for an arbitrary variety of partial algebras.

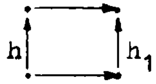
Throughout the rest of this section we assume S to be complete.

L e m m a 3.6. An S-morphism h is a free extension iff h divides each perfect S-morphism.

P r o o f . By Lemma 3.3 it is enough to show the sufficiency. Assume that h divides each perfect map and let $h = f\cdot e$ for some perfect f and a free extension e. Then there exists g s.t. $g\cdot h = e$, $f\cdot e = id$. Thus g is a perfect split mono and epi (because e is epi). Then g is iso and consequently, h is a free extension.

C o r o l l a r y 3.7. (i) For each set X, $i\leq j$ in SX, $\varphi_{ij}\in Ext$, (ii) $e:\underline{X}_i\longrightarrow\underline{B}$ is a free extension if $e = k\cdot\varphi_{ij}$ for some $j\in SX$, $k\in Iso$, (iii) Free extensions form a subcategory of S-Alg.

(iv) if the square



is a pushout in S-Alg, $h \in \mathrm{Ext}$, then $h_1 \in \mathrm{Ext}$,
(v) If $f \cdot e \in \mathrm{Ext}$ and e is epi, then $f \in \mathrm{Ext}$.

Combining together Lemma 2.2 and Theorem 3.5 we obtain that each S-morphism h has a (unique) factorization $h = m \cdot e_1 \cdot e_0$, where m is perfect and injective, e_1 - perfect and surjective, e_0 - free extension. The composition class $E = \mathrm{Perf}_{sur} \cdot \mathrm{Ext}$ is a class of S-epimorphisms which, in our opinion, plays the same role as surjective epis play in the theory of total algebras.

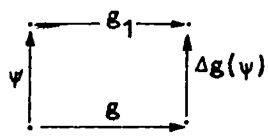
For each S-algebra $\underline{A}$ by $\mathrm{Ext}\underline{A}$ we denote the set of representatives of free extensions of $\underline{A}$. It is a poset with a greatest element $\mathcal{T}_{\underline{A}}$ and a least element $\mathrm{id}_{\underline{A}}$. By Corollary 3.7.iv $\mathrm{Ext}\underline{A}$ is a complete lattice with suprema determined by pushouts. For each i in SA, $\mathrm{Ext}\underline{A}_i$ is a sublattice of SA.

L e m m a 3.8. Let $g: \underline{B} \to \underline{A}$ be perfect. Then there exist monotone functions

$$\mathrm{Ext}\underline{A} \underset{\Gamma_g}{\overset{\Delta_g}{\rightleftarrows}} \mathrm{Ext}\underline{B}$$

such that $\Delta_g \cdot \Gamma_g \leqslant \mathrm{id}$, $\mathrm{id} \leqslant \Gamma_g \cdot \Delta_g$. If, moreover, g is surjective, then $\Delta_g \cdot \Gamma_g = \mathrm{id}$.

P r o o f . For $\varphi \in \mathrm{Ext}\underline{A}$, let $g_0 \cdot \Gamma_g(\varphi)$ be an (Ext,Perf)--factorization of $\varphi \cdot g$. For $\psi \in \mathrm{Ext}\underline{B}$, $\Delta_g(\psi)$ is a unique S-morphism making the square



a pushout. The desired inequations are obvious. If g is surjective, then t square

$$\begin{array}{ccc} \cdot & \xrightarrow{(\eta_k^B\cdot g)^*} & \cdot \\ {\scriptstyle\varphi_{ps}}\uparrow & & \uparrow{\scriptstyle\varphi_{jk}} \\ \cdot & \xrightarrow{(\eta_j^B\cdot g)^*} & \cdot \end{array}$$

is a pushout in S-Alg for each $j \leq k$ in SB. Now using a routine calculation one can show that $(\Gamma_g(\varphi), g)$ together with (φ, g_0) form a pushout in S-Alg. Hence $\Delta_g \cdot \Gamma_g(\varphi) = \varphi$ for each $\varphi \in \mathrm{Ext}\underline{B}$.

C o r o l l a r y 3.9. For each S-algebra $\underline{A} = (A, A_1, a)$, $\Gamma_a : \mathrm{Ext}\underline{A} \longrightarrow \mathrm{Ext}\underline{A}_1 \subset SA$ is an embedding preserving infs and for each $e, f \in \mathrm{Ext}\underline{A}$, $\sup_{\mathrm{Ext}\underline{A}} \{e, f\} = \Delta_a(\sup_{SA}\{\Gamma_a(e), \Gamma_a(f)\})$.

But in general Ext$\underline{A}$ is only a meet-sublattice of SA.

E x a m p l e . Let $\Omega = \Omega_1 = \{p, s_1, s_2, t\}$, $V =$ $= \mathrm{Mod}(\exists s_1(x) \wedge \exists s_2(x) \Longrightarrow \exists tp(x))$, $\underline{A} = (\{x, y\}, p^A(s) = y)$ and nothing more is defined in $\underline{A}$. Let $\underline{B}^i = (\{x, y, s_i x\}, p^i(x) = y$, $s_i^i(x) = s_i x)$ and nothing more is defined $(i = 1,2)$. One can verify that $\Gamma_a : \underline{A} \longrightarrow S_V A$ does not preserve the supremum of $\underline{B}^1$ and $\underline{B}^2$ because $\underline{B}^1 \vee \underline{B}^2 = (\{x, y, s_1 x, s_2 x, ty\}, \tilde{p}(x) = y, \tilde{t}(y) = ty$, $\tilde{s}_i(x) = s_i x$ for $i = 1,2$. (Here we identify free extensions of $\underline{A}$-identity embeddings into $\underline{B}_0$'s - with their codomains).

4. Free extensions in categories of partial monadic algebras

Recall from [5] that a monad $T = (T, \mu^T, \eta^T)$ in Set is called a pb-monad if for each mono $m : X \longrightarrow Y$ the 4-tuples of functions $(m, \eta_X^T, Tm, \eta_Y^T)$ and $(\mu_X^T, TTm, \mu_Y^T, Tm)$ are pullbacks and for each $g : Z \longrightarrow Y$, T preserves pullback of m and g.

We call a subset $B \subset A$ open in $\underline{A} = (A, a) \in \mathrm{Set}^T$ if $a^{-1}(B) \subset TB$ i.e., a pullback of a and $m : B \hookrightarrow A$ has the form

$$(+) \qquad \begin{array}{ccc} TA & \xrightarrow{a} & A \\ {\scriptstyle Tm}\uparrow & & \\ TB & & \uparrow{\scriptstyle m} \\ \uparrow & & \\ C & \xrightarrow{b} & B \end{array}$$

If T is a pb-monad, then for each $\underline{A}$ in Set^T open subsets of $\underline{A}$ form a complete lattice with set-theoretical join and meet as lattice operations.

Starting from a given pb-monad T we construct a s.a.t. $S_T = (S_T, J, \varrho, (\ )^*)$ as follows: for each set X, $S_T X$ consists of all open subsets of (TX, μ_X^T) containing X. For each $X_i \in S_T X$ and $h: Y \longrightarrow X_i = J_X X_i$, h^* is a unique function making the square (where x_i, y_j are suitable embeddings)

$$\begin{array}{ccc} TY & \xrightarrow{\mu^T \cdot T(x_i h)} & TX \\ {\scriptstyle y_j}\uparrow & & \uparrow{\scriptstyle x_i} \\ Y_j & \xrightarrow{h^*} & X_i \end{array}$$

a pullback. Note that S_T is pointed. $\mathrm{Tot}S_T\text{-Alg} = \mathrm{Set}^T$. S_T-algebras are called partial monadic T-algebras. Note also that the s.a.t. S_Ω is constructed just in that way starting from the monad T_Ω corresponding to the category of all total Ω-algebras. Hence $\mathrm{Palg}\,\Omega =$ the category of partial monadic T_Ω-algebras.

We will show that in categories of partial monadic algebras free extensions are precisely extendable epis (with respect to Set^T). To show it we will need the following facts (to save the space we omit the routine proofs of them):

F a c t 4.1. Let B be an open subset of $(A, a) \in \mathrm{Set}^T$ (T is always assumed to be a pb-monad). Then in the diagram (+) C is open in (TB, μ_B^T) and $\underline{B} = (B, C, b)$ is an S_T-algebra. We call $\underline{B}$ an open subalgebra of $\underline{A}$. Moreover, $m: B \longrightarrow A$ is an initial S_T-morphism from $\underline{B}$ to $\underline{A}$ (i.e. whenever $h: U_S \underline{D} \longrightarrow B$ and $m \cdot h$ is an S_T-morphism from $\underline{D}$ to $\underline{A}$, then h is an S_T-morphism from $\underline{D}$ to $\underline{B}$).

F a c t 4.2. For each S_T-algebra $\underline{A} = (A, A_1, a)$ there exists a total S_T-algebra $\underline{A}^1 = (A \sqcup 1, \bar{a}: T(A \sqcup 1) \longrightarrow A \sqcup 1)$ $(1 = \{\emptyset\})$ such that $\underline{A}$ is an open subalgebra of $\underline{A}^1$.

F a c t 4.3. Let $h: \underline{C} \longrightarrow \underline{A}$ be an S_T-morphism between total S_T-algebras and let $m: \underline{B} \hookrightarrow \underline{A}$ be an open subalgebra. Consider a pullback in Set

$$\begin{array}{ccc} U_S\underline{C} = C & \xrightarrow{\;h\;} & A = U_S\underline{A} \\ n \uparrow & & \uparrow m \\ D & \xrightarrow{\;g\;} & B = U_S\underline{B} \end{array}$$

Then D is open in C and g is a perfect map between open subalgebras $\underline{D}$ and $\underline{B}$.

Lemma 4.4. Each S_T-algebra is an open subalgebra of its free completion.

Proof. Let $F_t(A, A_i, a) = (FA, \hat{a})$ and $f:(FA,\hat{a}) \longrightarrow \underline{A}^1 = (A \sqcup 1, \bar{a})$ be an S_T-morphism s.t. $f \cdot \gamma_{\underline{A}} = p_A : A \hookrightarrow A \sqcup 1$. By Facts 4.2 and 4.3 it is enough to show that in the diagram

$$\begin{array}{ccccc} TA & \xrightarrow{\;F_t a\;} & FA & \xrightarrow{\;f\;} & A \sqcup 1 \\ \uparrow & & \uparrow \gamma_{\underline{A}} & & \uparrow p_A \\ A_i & \xrightarrow{\;a\;} & A & \xrightarrow{\;=\;} & A \end{array}$$

the right square is a pullback in Set. Obviously, $f \cdot F_t a \cdot \varrho^T_A = p_A$, hence $f \cdot F_t a = \bar{a} \cdot Tp_A$. It easily implies that the outer square is a pullback. Then the right square is a pullback because $F_t a$ is a surjective ($F_t a$ is surjective since it is a coequalizer in Set^T).

Now we are ready to prove the main theorem of this section.

Theorem 4.5. Let T be an arbitrary pb-monad. Then for each S_T-morphism $e:\underline{A} \longrightarrow \underline{B}$ the following are equivalent:

i. $e \in Ext_{S_T\text{-Alg}}$

ii. $F_T e$ is an isomorphism,

iii. $\underline{B}$ is an open subalgebra of $F_t\underline{A}$, $\gamma_{\underline{A}} = \underline{A} \xrightarrow{\;e\;} \underline{B} \overset{m}{\hookrightarrow} F_t\underline{A}$.

Proof. i. $\Longrightarrow$ ii. See Corollary 3.4.

ii. $\Longrightarrow$ iii. See Lemma 4.4.

iii. $\Longrightarrow$ i. Let $\underline{A} = (A, A_i, a)$. By Fact 4.3 we may assume that pullback of $F_t a : TA \longrightarrow FA = U_S F_t \underline{A}$ and $m : B \hookrightarrow FA$ has the form

$$\begin{array}{ccc} TA & \xrightarrow{F_t a} & FA \\ \uparrow & & \uparrow m \\ S_T A \ni A_k & \xrightarrow{g} & B \end{array}$$

and g is a perfect map from $\underline{A}_k$ to $\underline{B}$. g is surjective because $F_t a$ is. Obviously, $A_i \subset A_k$. We show that the square

$$\begin{array}{ccc} \underline{A}_k & \xrightarrow{g} & \underline{B} \\ \varphi_{ik} \uparrow & & \uparrow e \\ \underline{A}_i & \xrightarrow{a} & \underline{A} \end{array}$$

is a pushout in S_T-Alg. Let $h:\underline{A}_k \longrightarrow \underline{X}$, $f:\underline{A} \longrightarrow \underline{X}$, $h \cdot \varphi_{ik} = f \cdot a$. Then $F_t h = F_t f \cdot F_t a$ (because $\varphi_{ik} \in$ Ext) and $\gamma_{\underline{X}} \cdot h = F_t f \cdot \bar{m} \cdot g$. But g is surjective and $\gamma_{\underline{X}}$ is an initial mono. Hence there exists S_T-morphism $\psi : \underline{B} \longrightarrow \underline{X}$ s.t. $\psi \cdot g = h$. Obviously, $\psi \cdot e = f$. Now the theorem follows directly from Definition 3.1 because, obviously, $g = \tilde{a}$.

C o r o l l a r y 4.6. For each S_T-algebra $\underline{A} = (A, A_1, a)$ the lattice Ext$\underline{A}$ is isomorphic to a lattice of all open subsets of $F_t\underline{A}$ containing A. Γ_a:Ext$\underline{A} \longrightarrow S_T A$ is a lattice embedding. Since partial Ω-algebras are partial monadic T_Ω-algebras we obtain:

P r o p o s i t i o n 4.7. Let Ω be an arbitrary finitary type. Then for each $\underline{A} \in$ PalgΩ, $\text{Ext}_{\text{Palg}}\ \underline{A}$ is a complete, distributive, algebraic lattice.

E x a m p l e s . Let V be a regular variety [10], $V = \text{Mod}(t_1^i = t_2^i)_{i \in I}$. Then the corresponding monad T(V) is a pb-monad and $S_{T(V)}$-Alg is (concretely isomorphic to) a variety of partial Ω-algebras V_p, where $V_p = \text{Mod}\ (\exists t_1^i \Longleftrightarrow \exists t_2^i, \exists t_1^i \Longrightarrow t_1^i = t_2^i)_{i \in I}$. Theorem 4.5 and Proposition 4.7 remain true for these varieties of partial algebras.

The category of locally compact spaces may be regarded as a category of partial monadic β-albebras where β is an ultrafilter monad [9]. Total algebras are compact spaces, perfect S_β-morphisms are perfect maps (and this is a reason of our proposal to replace the name "closed" by "perfect"). By Corollary 4.6, the lattice of free extensions of a given locally compact space H is isomorphic to the lattice of open subsets of its Stone-Čech compactification containing H.

5. Free extensions in varieties of partial algebras

We follow the notation established in section 1. Throughout we assume that V is an arbitrary but fixed variety of partial Ω-algebras and Ω is a finitary type. First note that because Z_V preserves perfect morphisms, $F_V : \mathrm{Palg}\,\Omega \to V$ preserves free extensions. Moreover,

P r o p o s i t i o n 5.1. For each $h : \underline{A} \to \underline{B}$ in V the following are equivalent

i. $h \in \mathrm{Ext}_V$,

ii. An $(\mathrm{Ext}_{\mathrm{Palg}\Omega}, \mathrm{Perf}_{\mathrm{Palg}\Omega})$-factorization of h in $\mathrm{Palg}\,\Omega$ has the form $h = (i \cdot \delta_{\underline{C}}) \cdot e$, for some $\underline{C}$ in $\mathrm{Palg}\,\Omega$ with $\delta_{\underline{C}} : \underline{C} \to Z_V F_V \underline{C}$ perfect, $i \in \mathrm{Iso}$ and $e \in \mathrm{Ext}_{\mathrm{Palg}\Omega}$.

P r o o f . Let $h \in \mathrm{Ext}_V$, $h = g \cdot e$ be an (Ext,Perf)-factorization of h in $\mathrm{Palg}\Omega$. Then $g = d \cdot \delta_{\underline{C}}$ for some $d : F_V \underline{C} \to \underline{B}$. Note that $d, \delta_{\underline{C}}$ are perfect. But $\delta_{\underline{C}} \cdot e$ is epi in V and $d \cdot (\delta_{\underline{C}} \cdot e) = h \in \mathrm{Ext}V$. By Corollary 3.7.v, $d \in \mathrm{Ext}V$. Hence d is iso.

The converse follows from Lemma 3.6 because Z_V preserves perfect maps. Let $\underline{A} \in \mathrm{Palg}\,\Omega$ and $\delta_{\underline{A}} = e_1 \cdot e_0 : \underline{A} \to \underline{C} \to Z_V F_V \underline{A}$ be an (Ext,Perf)-factorization of δ_A in $\mathrm{Palg}\Omega$. Then $\underline{C} = Z_{\exists V} F_{\exists V} \underline{A}$ and $e_0 : \underline{A} \to Z_{\exists V} F_{\exists V} \underline{A}$, $e_1 : F_{\exists V} \underline{A} \to Z_0 F_0 (F_{\exists V} \underline{A})$ are suitable universal arrows (we omit the obvious proof). Hence, by Theorem 5.1 we have

C o r o l l a r y 5.2. For each variety $V \subset \mathrm{Palg}\Omega$,

i. $\mathrm{Ext}_{\exists V} = \mathrm{Ext}_{\mathrm{Palg}\,\Omega} \cap \exists V$,

ii. for each $\underline{A}$ in V, the lattices $\mathrm{Ext}_V \underline{A}$ and $\mathrm{Ext}_{\exists V} \underline{A}$ are isomorphic.

T h e o r e m 5.3. Let $\underline{A} = (A, A_1, a) \in \mathrm{Palg}\Omega$. Then $\mathrm{Ext}_{\mathrm{Palg}\Omega} \underline{A} \cong \left\{ k \in S_\Omega A :\ k \geq i \text{ and } \mathrm{dom}(\varrho_k^A \cdot a)^* = \mathrm{dom}(\varphi_{ik})^* \right\}$.

P r o o f . Assume that $(\varrho_k^A \cdot a)^*$, $(\varphi_{ik})^* : (A_i)_1 \rightrightarrows A_k$. By Lemma 3.2.ii it is enough to show that this pair has a perfect coequalizer. We shall use here Lemma 2.4. First we define a function $r: A_k \longrightarrow (A_i)_1$. For each $p \in \Omega A_i$ define inductively Fun(p) as follows:

if $p \in A_i$, then Fun(p) = 0,

if $p = q[t_1,\ldots,t_n]$; $t_1,\ldots,t_n \in \Omega A_i$, $q \in \Omega_n$, then

$$\mathrm{Fun}(p) = \sum_{i=1}^{n} \mathrm{Fun}(t_i) + 1$$

(we will use square brackets to denote Ω-terms over A_i and usual brackets for terms over A).

Now using an algebraic induction one can show that for each $t \in A_k \subset \Omega A$ there exists a unique $r(t) \in (A_i)_1$ such that $(\varphi_{ik})^*(r(t)) = t$ and if $p \in (A_i)$, $(\varphi_{ik})^*(p) = t$, then $p = r(t)$ or $\mathrm{Fun}(p) > \mathrm{Fun}\, r(t)$. Moreover, if $t = q(t_1,\ldots,t_n) \in A_k$, then

$$r(t) = \begin{cases} a(q(t_1,\ldots,t_n) & \text{if } q(t_1,\ldots,t_n) \in A_i \subset A_k \\ q[r(t_1),\ldots,r(t_n)] & \text{otherwise.} \end{cases}$$

Hence $r: A_k \longrightarrow (A_i)_1$ is defined, $(\varphi_{ik})^* \cdot r = \mathrm{id}$.

Define an equivalence relation $\sim$ on A_k as follows: for each $t, t_1 \in A_k$ $t \sim t_1$ iff $r(t) = f[p_1,\ldots,p_k]$, $t_1 = f[s_1,\ldots,s_k]$ $p_1,\ldots,p_k,s_1,\ldots,s_k \in A_i$, f is a term symbol and $a(p_i) = a(s_i)$ for $i = 1,2,\ldots,k$.

Let $p \in (A_i)_1$. We claim that $(\varrho_k^A \cdot a)^*(p) \sim (\varphi_{ik})^*(p)$. It is obvious if $p \in A_i \subset (A_i)_1$. Let $p = q[p_1,\ldots,p_n]$, $p_1,\ldots,p_n \in (A_i)_1$, $q \in \Omega_n$ and

$$\hat{\hat{p}}_j = (\varrho_k^A \cdot a)^*(p_j) \sim (\varphi_{ik})^*(p_j) = \hat{p}_j \text{ for } j=1,2,\ldots,n.$$

Then

$$r((\varrho_k^A \cdot a)^*(p)) = r(q(\hat{\hat{p}}_1,\ldots,\hat{\hat{p}}_n), \quad r((\varphi_{ik})^*(p)) = r(q(\hat{p}_1,\ldots,\hat{p}_n)).$$

Note that

$q(\hat{\hat{p}}_1,\ldots,\hat{\hat{p}}_n) \in A_i$ iff $q(\hat{p}_1,\ldots,\hat{p}_n) \in A_i$ and then their values in $\underline{A}$ are equal.

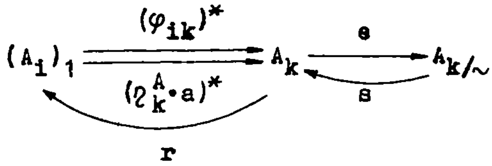
Thus, if it is the case, $(\varrho_k^A \cdot a)^*(p) \sim (\varphi_{ik})^*(p)$. Otherwise,

$$r((\varrho_k^A \cdot a)^*(p)) = q[r(\hat{\hat{p}}_1),\dots,r(\hat{\hat{p}}_n)] \quad \text{and}$$

$$r(\varphi_{ik})^* p = q[r(\hat{p}_1),\dots,r(\hat{p}_n)].$$

Now, directly from the definition of $\sim$ one can show that also in that case $(\varrho_k^A \cdot a)^*(p) \sim (\varphi_{ik})^*(p)$.

Let $e: A_k \longrightarrow A_k/_\sim$ be a projection and $s: A_k/_\sim \longrightarrow A_k$ be a function such that for each $t \in A_k$, $s[t]_\sim = f(a(s_1),\dots,a(s_n))$ iff $r(t) = f[s_1,\dots,s_n]$ $(s_1,s_2,\dots,s_n \in A_i$, f is a term symbol). One can check that the diagram



fulfils the assumptions of Lemma 2.4.

The proof is complete.

This Theorem is a stronger version of Lemma 3.2.11 - the equation $\operatorname{dom}(\varrho_k^A \cdot a)^* = \operatorname{dom}(\varphi_{ik})^*$ implies the existence of a perfect coequalizer of this pair. But it is true only for finitary partial algebras. It fails to be true for infinitary case.

E x a m p l e . Let $P = (P,\mu^P,\varrho^P)$ be a power set monad. It is a pb-monad $\mathcal{A} \subset PA = 2^A$ is open in (PA,μ_A^P) if $\emptyset \in \mathcal{A}$, $\{a\} \in \mathcal{A}$ for each $a \in A$ and for $A_0 \subset A_1$, $A_0 \in \mathcal{A}$ provided that $A_1 \in \mathcal{A}$. Partial P-algebras are posets with a least element and with every upper bounded subset having a supremum. A poset $(A,\leq)$ satisfying these properties is represented as a triple $(A,\mathcal{A}_i,a)$, where $\mathcal{A}_i$ is a family of all upper bounded subsets of $(A,\leq)$, $a(A_0) = \sup A_0$ for each A_0 in $\mathcal{A}_i$. For each $\mathcal{A}_j \supset \mathcal{A}_i$, $\operatorname{dom}(\varrho_j^A \cdot a)^* = \operatorname{dom}(\varphi_{ij})^*$ iff for each family $\{A_k\}_{k\in K} \subset \mathcal{A}_i$, $\bigcup A_k \in \mathcal{A}_j$ iff $\{\sup A_k : k \in K\} \in \mathcal{A}_j$. $h:(A,\leq) \longrightarrow (B,\leq)$ is an S_P-morphism if h preserves existing suprema. h is perfect if it reflects them (i.e. if $A_0 \subset A$ and $\sup h(A_0)$ exists, then $\sup A_0$ exists in $(A,\leq)$).

Consider an S_p-algebra $\underline{A} = (A \cup \{0\}, \leq)$ where 0 is a least element of $\underline{A}$ and A is a disjoint sum of $\{D_n \cup C_n : n \in N\}$, $D_n = \{d_1^n, d_2^n, d_3^n, d_4^n\}$, $C_n = \{c_1^n, c_2^n\}$ for each $n \in N$ and $\sup\{d_1^n, d_3^n\} = c_1^n = \sup\{d_1^{n+1}, d_2^{n+1}\}$, $\sup\{d_2^n, d_4^n\} = c_2^n = \sup\{d_3^{n+1}, d_4^{n+1}\}$.

Let $(A, \mathcal{A}_i, a)$ be the triple representation of $\underline{A}$ and let be an open subset of (PA, μ_A^P) consisting of all finite subsets of A. Then, obviously, $\mathcal{A}_i \subset \mathcal{A}_j$ and $\mathrm{dom}(\varrho_j^A \cdot a)^* = \mathrm{dom}(\varphi_{ij})^*$. But a coequalizer of this pair cannot be perfect. Indeed, if $h: \underline{\mathcal{A}}_j \longrightarrow \underline{B}$ is an S_p-morphism equalizing this pair, then $h(D_1) = h\cdot(\varphi_{ij})^*(\{d_1^1, d_3^1\}, \{d_2^1, d_4^1\}) = h\cdot(\varrho_j^A \cdot a)^*(\{d_1^1, d_3^1\}, \{d_2^1, d_4^1\}) = h(C_1) = h\cdot(\varrho_j^A \cdot a)^*(\{d_1^2, d_2^2\}, \{d_3^2, d_4^2\}) = \ldots = h(D_2) = h(C_2) = \ldots = h(C_n) = h(D_n) \ldots$. Hence $h(A)$ is upper bounded in $\underline{B}$ i.e., it has a supremum in $\underline{B}$. But A has no supremum in $\underline{\mathcal{A}}_j$ because A is infinite. Hence h is not perfect.

D e f i n i t i o n 5.4. Let Ω be a finitary type and $\underline{A} = (A, A_i, a) \in \mathrm{Palg}\Omega$. We call an initial Ω-segment $A_j \in S_\Omega A$ $\underline{A}$-saturated iff $A_i \subset A_j$ and for each $p \in \Omega A$, if $p = f(t_1, \ldots, t_n)$, where $t_1, \ldots, t_n \in A_i$ and f is a term symbol, then $f(t_1, \ldots, t_n) \in A_j$ iff $f(a(t_1), \ldots, a(t_n)) \in A_j$.

Obviously, A_i and ΩA are $\underline{A}$-saturated. A-saturated initial Ω-segments form a sublattice of $S_\Omega A$. We denote it by $\mathrm{SAT}_{\underline{A}}$. One can check that A_j is $\underline{A}$-saturated iff $\mathrm{dom}(\varrho_j^A \cdot a)^* = \mathrm{dom}(\varphi_{ij})^*$. Hence, by Theorem 5.4 we obtain

T h e o r e m 5.5. Let Ω be a finitary type, $\underline{A} \in \mathrm{Palg}\Omega$. Then $\mathrm{Ext}_{\mathrm{Palg}\Omega}\underline{A}$ is isomorphic to $\mathrm{SAT}_{\underline{A}}$. If $V \subset \mathrm{Palg}\,\Omega$ is a variety, $\underline{A} \in V$, then $\mathrm{Ext}_V \underline{A} = \mathrm{Ext}_{\mathbf{I}V}\underline{A}$ is isomorphic (as a meet-semilattice) to $\mathrm{SAT}_{\underline{A}} \cap S_{\mathbf{I}V}A$ (A is a carrier of $\underline{A}$).

We finish our paper with two lemmas which give us simple examples of results which can be obtained by using method proposed here.

L e m m a 5.5. For each variety $V \subset \mathrm{Palg}\Omega$ the following are equivalent

i. For each $\underline{A}$ in V, $Ext_V\underline{A}$ is (isomorphic to) a sublattice of $S_\Omega A$,

ii. For each set A, $S_V A$ is a sublattice of $S_\Omega A$.

iii. There exists a set ψ of formulas of the form $\exists t^i \Longrightarrow \exists p^i$ such that $\exists V = Mod\,\psi$.

If any of these condition is valid, then for each $\underline{A}$ in V, $Ext_V\underline{A}$ is distributive and algebraic.

P r o o f . i. $\Longrightarrow$ ii. because $S_V A = Ext_V(F_V\underline{A}^o)$, where $\underline{A}^o$ is a discrete partial algebra with a carrier A.
ii. $\Longrightarrow$ i. follows from Theorem 5.4. iii. $\Longrightarrow$ ii. needs only a straightforward verification. ii. $\Longrightarrow$ iii. Assume that $\exists t_1(x_1,\dots,x_n) \wedge \exists t_2(x_1,\dots,x_n) \Longrightarrow \exists p(x_1,\dots,x_n)$ is valid in V (compare Lemma 1.2.i.) and there exist V-algebras $\underline{A}_i$ together with valuations $h_i : X = \{x_1,\dots,x_n\} \longrightarrow U_\Omega \underline{A}_i$ such that $\underline{A}_i$ does not satisfy the formula $\exists t_i \Longrightarrow \exists p$ at these valuations $(i = 1,2)$. Let $\underline{X}_i = dom\,\tilde{h}_i$ for $i = 1,2$. Then $X_1, X_2 \in S_V X$, but $X_1 \cup X_2 \notin S_V X$. A contradiction. The last assertion follows from Proposition 4.7.

L e m m a 5.6. For each variety $V \subset Palg\,\Omega$ the following are equivalent.

i. For each $\underline{A}$ in V, $Ext_V\underline{A}$ is (isomorphic to) a meet-sub-semilattice closed under suprema of directed subsets,

ii. For each set A, $S_V A$ is closed under directed suprema within $S_\Omega A$,

iii. $\exists V$ is axiomatizable within $Palg\,\Omega$ i.e., there exists a set ψ' of formulas of the form $\exists t_1 \wedge \exists t_2 \wedge \dots \wedge \exists t_n \Longrightarrow \exists p$ such that $V = Mod\,\psi'$. If any of those condition is valid, then for each $\underline{A}$ in V $Ext_V\underline{A}$ is an algebraic lattice.

We omit the proof because it needs the same method as the proof of Lemma 5.5.

REFERENCES

[1] H. A n d r e k a , I. N e m e t i : Generalization of the concept of variety and quasivariety to partial algebras through category theory, Diss.Math. 204 (1983).

[2] P. Burmeister: Introduction to Theory and Application of partial algebras, part I: A model Theoretic Approach to Partial Algebras, Akademie-Verlag, Berlin, 1986.

[3] G. Grätzer: Universal algebra, 2nd ed., Springer-Verlag (1979).

[4] H. Herrlich, G. Strecker: Category Theory. Allyn and Bacon Inc., Boston (1973).

[5] G. Jarzembski: Partial monadic algebras, Bull. Acad. Polon. Sci., 30, No 9-10 (1982), 407-413.

[6] G. Jarzembski: Partially monadic functors, J. Pure Appl. Algebra 32 (1984), 315-326.

[7] G. Jarzembski: Spectral algebraic theories and their algebras as a categorical conterpart of partial algebras, Seminarberichte Hagen 19 (1984), 225-243.

[8] A. Kurpiel, B. Wojdyło: Description of partial algebras by segments, Coll. Math. Societatis J. Bolai 43 (1983), 283-292.

[9] E. Manes: A triple theoretic construction of compact algebras, Lect. Notes in Math. 80 (1969), 91-118.

[10] J. Płonka: On equational classes of abstract algebras defined by regular equations, Fund. Math. 64 (1969), 241-247.

[11] J. Schmidt: A homomorphism theorem for partial algebras, Coll. Math., 21 (1970, 5-21.

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