DEMONSTRATIO MATHEMATICA
Vol. XX No 1-2 1987

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THE SUBDIRECTLY IRREDUCIBLE DIAGONALIZABLE ALGEBRAS

The notion of a diagonalizable algebra was introduced by Magari ([2]). In this paper the author describes some algebraic properties of these algebras. For further information about diagonalizable algebras see [1], [3], [4].

It is known (see [2]) that the only simple diagonalizable algebra is the two-element algebra. In this paper we characterize all subdirectly irreducible diagonalizable algebras.

An algebra $\underline{A} = \langle A, \vee, \wedge, -, \tau, 0, 1\rangle$ of type $\langle 2,2,1,1,0,0\rangle$ is called a diagonalizable algebra, if it satisfies the following axioms:

A1. $\langle A, \vee, \wedge, -, 0, 1\rangle$ is a Boolean algebra,

A2. $\tau(x\wedge y) = \tau(x)\wedge\tau(y)$,

A3. $\tau(\tau(x) \rightarrow x) = \tau(x)$, where the symbol $x \rightarrow y$ denotes $-x\vee y$,

A4. $\tau(1) = 1$.

It is known (see [2]) that the following properties hold in every diagonalizable algebra:

w1. $\tau(x)\leq\tau(\tau(x))$,

w2. if $x\leq y$ then $\tau(x)\leq\tau(y)$,

w3. if $\tau(x)\leq x$ then $x = 1$.

A τ-filter of a diagonalizable algebra $A = \langle A,\vee,\wedge,-,\tau,0,1\rangle$ is a filter F of the Boolean algebra $\langle A,\vee,\wedge,-,0,1\rangle$ which satisfies the condition:

This paper is based on the lecture presented at the Conference on Universal Algebra held at the Technical University of Warsaw (Wilga), May 22-25, 1986.

$$x \in F \text{ implies } \tau(x) \in F \text{ (see [2]).}$$

For each element x and for any τ-filters F_1, F_2 of a diagonalizable algebra A we have:

1) $[\tau(x))$ is a τ-filter,
2) if $\tau(x) = 1$ then $[x)$ is a τ-filter of A,
3) $F_1 \cap F_2$ is a τ-filter,
4) a filter generated by $F_1 \cup F_2$ is a τ-filter.

Let the symbol $\rho(F)$ denote the congruence relation of a Boolean algebra which is induced by a filter F.

It is easy to prove that the following holds.

Lemma 1. The following conditions are equivalent for every diagonalizable algebra A^2 and each relation $\rho \subseteq A^2$

1) ρ is a congruence relation on A,
2) there exists a τ-filter F of A such that

$$\rho = \rho(F).$$

Lemma 2. Let $F_\tau(A)$ be the lattice of all τ-filters of a diagonalizable algebra A and C(A) be the congruence lattice of A. Then the mapping $f : F_\tau(A) \longrightarrow C(A)$, $f(F) = \rho(F)$ is a lattice isomorphism.

Lemma 3. Let A be a diagonalizable algebra. A τ-filter F is an atom in the $F_\tau(A)$ if and only if there exists a co-atom x in A such that $F = [x)$ where $\tau(x) = 1$.

Proof. Let F be an atom in the $F_\tau(A)$, $x \in F$, $x \neq 1$ and consider the τ-filter $[\tau(x))$. Let us notice that $[\tau(x)) \subseteq F$ and since by w3, $x \notin [\tau(x))$, it follows that $[\tau(x)) \neq F$. Since F is an atom in the $F_\tau(A)$ it follows that $\tau(x) = 1$. Whence $[x)$ is a τ-filter. Obviously $\{1\} \neq [x) \subseteq F$. So we get $[x) = F$. Suppose on the contrary that there is y in A such that $x < y < 1$. Then by w2, $\tau(y) = 1$. By the argument analogous to the previous one we get $[y) = F$ whence $x = y$, a contradiction.

It follows that x is a co-atom of A. The converse implication is obvious.

L e m m a 4. Let A be a diagonalizable algebra and let an element $x_0 \in A \setminus \{1\}$ has the following property:
(E) for each x in $A \setminus \{1\}$: $x \leq x_0$ or $\tau(x) \leq x_0$.
Then:
1) $\tau(x_0) = 1$,
2) x_0 is a co-atom of $\underline{A}$,
3) for each co-atom $x \in A$

$$\tau(x) = 1 \text{ implies } x = x_0,$$

4) x_0 is the unique element of the set $A \setminus \{1\}$ having the property (E).

P r o o f . 1. Suppose $\tau(x_0) \neq 1$. Then by (E) and w1, $\tau(x_0) \leq x_0$. Whence by w3, $x_0 = 1$, what is impossible.

2. Let $x_0 \leq y < 1$. By w2, we have $\tau(x_0) \leq \tau(y)$. Since $\tau(x_0) = 1$ it follows that $\tau(y) = 1$. By (E) we have $y \leq x_0$ or $\tau(y) \leq x_0$. The first case implies $x_0 = y$, the second gives $x_0 = 1$, what is impossible.

3. Let x be a co-atom of $\underline{A}$ such that $\tau(x) = 1$. By (E) we have $x \leq x_0$. Since both x and x_0 are co-atoms, it follows that $x = x_0$. The condition (4) is an easy consequence of (1), (2) and (3).

T h e o r e m . A diagonalizable algebra $\underline{A}$ is subdirectly irreducible if and only if there is $x_0 \neq 1$ in A such that for each $x \neq 1$ in A $x \leq x_0$ or $\tau(x) \leq x_0$.

P r o o f . Let A be a subdirectly irreducible algebra and F the unique atom of the lattice $F_\tau(A)$. By Lemma 3, $F = [x_0)$ where $\tau(x_0) = 1$ for some co-atom x_0 in A. Let $x \in A \setminus \{1\}$. If $\tau(x) = 1$, then $[x)$ is a τ-filter. Since $\underline{A}$ is subdirectly irreducible algebra $[x_0) \subseteq [x)$. Consequently $x \leq x_0$. Analogously for $\tau(x) \neq 1$ we have $\tau(x) \leq x_0$.

Now let $x_0 \in A \setminus \{1\}$ and for each $x \neq 1$ in A $x \leq x_0$ or $\tau(x) \leq x_0$. Using Lemma 4 we get that x_0 is a co-atom and $\tau(x_0) = 1$. Hence $[x_0)$ is a τ-filter and obviously $[x_0) \neq \{1\}$. We will show that for every τ-filter $F \neq \{1\}$ of $\underline{A}$ $[x_0) \subseteq F$. Let $y \in F \setminus \{1\}$. From the assumption it follows that $y \leq x_0$ or $\tau(y) \leq x_0$. Hence $x_0 \in F$ and $[x_0) \subseteq F$, what completes the proof.

C o r o l l a r y 1. If a diagonalizable algebra $\underline{A}$ is subdirectly irreducible, then there exists exactly one co-atom x_0 of $\underline{A}$ such that $\tau(x_0) = 1$.

C o r o l l a r y 2. A finite diagonalizable algebra $\underline{A}$ is subdirectly irreducible if and only if there exists exactly one co-atom x_0 of A such that $\tau(x_0) = 1$.

E x a m p l e . Let $\underline{A}$ be a Boolean algebra and x_0 its co-atom. We define a operation τ on A by

$$\tau(x) = \begin{cases} 1 & \text{for } x \in \{x_0, 1\} \\ x_0 & \text{for } x \notin \{x_0, 1\} \end{cases} .$$

It is easy to show that this satisfies the conditions A2-A4. Therefore $\underline{A}$ is a diagonalizable algebra. By the above theorem the algebra $\underline{A}$ is subdirectly irreducible.

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Received April 3, 1986.