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ONE MORE CONCEPT OF MULTIADJOINTNESS

A concept of multiadjoint functor introduced here may be treated as a generalization of a "functor having a left multiadjoint" in the sense of Diers [2] and simultaneously it generalizes a Gray's concept of "split fibration with small fibres" [4]. The main idea of our approach is that for a given functor $U : \underline{A} \to \underline{C}$ and $\underline{C}$-object X instead of one "universal arrow" with domain X (adjointness) or "the set universal arrow" with domain X (discrete multiadjointness [2]) we consider a "small category of universal arrow" with domain X being a coreflective subcategory of a comma category $(X \downarrow U)$.

1. Notation and preliminaries

Throughout $U : \underline{A} \to \underline{C}$ is an arbitrary but fixed functor between categories with small hom-sets. For each $X \in \mathrm{ob}\,\underline{C}$, objects of a comma category $(X \downarrow U)$ are called U-morphisms with domain X. U-morphism (f,A) is called U-epi provided that $r,s : A \to B$ and $Ur \cdot f = Us \cdot f$ imply $r = s$. Let E be a class of U-morphisms. By an E-factorization of a U-morphism (f,A) we mean every pair $((e,B),g)$ with $(e,B) \in E$ and $Ug \cdot e = f$. For each $\underline{C}$-morphism $h : Y \to X$, $\hat{h} : (X \downarrow U) \to (Y \downarrow U)$ is a functor such that $\hat{h}(f,A) = (fh,A)$ and $h(\varphi) = \varphi$ for each

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$\varphi \in \mathrm{Mor}(X\downarrow U)$. Each full coreflective subcategory $\underline{D}_1$ of a given category $\underline{D}$ will be described as a triple $(\underline{D}_1, G, \varepsilon)$ where G is a right adjoint right inverse to the embedding $\underline{D}_1 \longrightarrow \underline{D}$ and ε is a counit of this adjointness.

Set, Cat, CAT denote categories of sets, small categories and all categories over given universum $\mathcal{U}$, respectively. For all unexplained notations and symbols of category theory we refer the reader to [6] and [10].

2. Motivations and general theory

We recall at first some well-known concepts of category theory. We describe them in a non-conventional way but this kind of description gives us a possibility to observe their common structure:

(i) $U : \underline{A} \longrightarrow \underline{C}$ is right adjoint [6] iff for each $\underline{C}$-object X, $(X\downarrow U)$ contains a full coreflective subcategory $(\mathbf{1}, G_X, \varepsilon^X)$ such that $\mathbf{1}$ is a terminal category and, moreover, for each $h : Y \longrightarrow X$ in $\underline{C}$, $G_Y \cdot \hat{h} \cdot \varepsilon^X = \mathrm{id}_{G_Y \hat{h}}$.

(ii) U has a left multiadjoint [2] iff each $(X\downarrow U)$ contains a full coreflective subcategory $(\mathcal{D}_X, G_X, \varepsilon^X)$ such that $\mathcal{D}_X$ is a discrete small category and, moreover, for each $h : Y \longrightarrow X$ in $\underline{C}$, $G_Y \cdot \hat{h} \cdot \varepsilon^X = \mathrm{id}_{G_Y \hat{h}}$.

(iii) U is a fibration with small split cleavage [4] iff each $(X\downarrow U)$ contains a full coreflective subcategory $(U^{-1}X, G_X, \varepsilon^X)$ such that $U^{-1}x$ is the fibre of U over X and for each $h : Y \longrightarrow X$ in $\underline{C}$ $G_Y \cdot \hat{h} \cdot \varepsilon^X = \mathrm{id}_{G_Y \hat{h}}$.

This suggests the following definition.

D e f i n i t i o n 2.1. By a spectrum of a functor U we mean a family $\underline{S} = (SX, G_X, \varepsilon^X)_{X \in \mathrm{ob}\, \underline{C}}$ of full coreflective subcategories of each $(X\downarrow U)$, respectively, such that

$$G_Y \cdot \hat{h} \cdot \varepsilon^X = \mathrm{id}_{G_Y \hat{h}}$$

for each $\underline{C}$-morphism $h : Y \longrightarrow X$.

Each spectrum $\underline{S}$ determines a functor $S : \underline{C}^{op} \longrightarrow CAT$ such that for each $\underline{C}$-morphism $h : Y \longrightarrow X$, $Sh : SX \longrightarrow SY$ and $SH = G_Y \cdot \hat{h} \cdot E_X$, where $E_X : SX \hookrightarrow (X \downarrow U)$ is a full embedding. Note that every functor has a trivial spectrum $((X \downarrow U), id_{(X \downarrow U)}, id)$. We call a spectrum $\underline{S}$ small if every SX is a small category. If, moreover, for each $X \in ob\ \underline{C}$ every isomorphism in SX is an identity then we call the spectrum $\underline{S}$ strongly reduced.

D e f i n i t i o n 2.2. A functor U is multiadjoint if it has a small spectrum.

It may happen that a given functor U has more then one small spectrum. Hence it will be convenient to introduce the following definition.

D e f i n i t i o n 2.3. A pair $(U, \underline{S})$ such that $\underline{S}$ is a small spectrum of U is called multiadjointness. If $(U, \underline{S})$ is a multiadjointness we call U multiadjoint with respect to $\underline{S}$. A multiadjointness $(U, \underline{S})$ is strongly reduced if $\underline{S}$ is a strongly reduced spectrum.

One can prove that every adjointness and discrete multiadjointness is strongly reduced. A small split cleavage of U is strongly reduced if U is amnestic i.e. every $\underline{A}$-isomorphism f with Uf being $\underline{C}$-identity, must be an identity in $\underline{A}$.

The definition of multiadjoint functor introduced above makes easy to note a connection with all previous concepts mentioned in the begining of this section. But for applications it will be more convenient to have its modified version. Using obvious one-to-one correspondence between functors $F : \underline{D} \longrightarrow (X \downarrow U)$ and pairs $(J_X : \underline{D} \longrightarrow \underline{A},\ \varrho^X : \Delta X \longrightarrow UJ_X)$ (where Δ is a "constant" functor into the functor category) one can prove the following proposition which can be treated as a "local definition" of multiadjoint functor.

P r o p o s i t i o n 2.4. $U : \underline{A} \longrightarrow \underline{C}$ is multiadjoint iff for each $\underline{C}$-object X there exists a small category SX together with a functor $J_X : SX \longrightarrow \underline{A}$ and a natural transformation $\varrho^X : \Delta X \longrightarrow UJ_X$ such that the following hold:

(i) Each U-morphism $(f:X \to UA,\ A)$ has a distinguished ϱ-factorization $((\varrho^X_i, J_X I),\ (f,A)^o)$ such that for each ϱ-factorization $((\varrho^X_j, J_X j), g)$ of (f,A) there exists a unique $\varphi:\ j \to i$ in SX with $g = (f,A)^o \cdot J_X(\varphi)$.

(ii) $(\varrho^X_i, J_X i)^o = id_{J_X i}$, for each $\underline{C}$-object X and $i \in$ ob SX.

(iii) For each $h : Y \to X$ in $\underline{C}$ and $(f:X \to UA, A)$, with $(f,A)^o : J_X i \to A$, $(fh,A)^o = (f,A)^o \cdot (\varrho^X_i h, J_X i)^o$.

P r o o f . We omit technical details of the proof. Note only that if U satisfies (i)-(iii) above, then the function

$$(f,A) \rightsquigarrow (\varrho^X_i,\ J_X i), \quad \text{where} \quad J_X i = \mathrm{dom}(f,A)^o,$$

defines a right adjoint right inverse functor to the embedding $SX \to (X \downarrow U)$ and $\{(f,A)^1\}_{(f,A) \in ob(X \downarrow U)}$ is a counit of this adjointness.

Conversely, for a given spectrum $\underline{S} = (SX, G_X, \varepsilon^X)_{X \in ob\ \underline{C}}$ of U one can define $(f,A)^o = \varepsilon^X_{(f,A)}$ for each $(f,A) \in (X \downarrow U)$. Note also that for each $h : Y \to X$ in $\underline{C}$ and $i \in$ ob SX, $Sh(i) = G_Y \cdot \hat{h} \cdot E_X(i) = j$ where $J_Y j = \mathrm{dom}(\varrho^X_j h,\ J_X i)^o$.

Hence we can describe a small spectrum of U as a 4-tuple $(S, J, \varrho, (\ )^o)$ where $S : \underline{C}^{op} \to \mathbf{Cat}$, $J = (J_X : SX \to \underline{A})$ $\varrho = (\varrho^X : \Delta X \to UJ_X)_{X \in ob\ \underline{C}}$ and $(\ )^o$ assings to each U-morphism (f,A), an $\underline{A}$-morphism $(f,A)^o$ and conditions (i)-(iii) of proposition above are satisfied. $(f,A)^o$ is called a universal ϱ-extension of (f,A).

Now it is easy to observe that our concept of multiadjointness is stronger then that one introduced by Tholen (compare Theorem 2.4 (iv) [11]).

Obviously a multiadjointness $(U, (S, J, \varrho, (\ )^o))$ is an adjointness iff each SX is a terminal category. It is a split cleveage iff $\varrho^X_i = id_X$ for each $\underline{C}$-object X and $i \in$ ob SX. In general, for a given multiadjointness $(U, (S, J, \varrho, (\ )^o))$ one can define a category $\underline{C}_S$ as follows [4]: objects of $\underline{C}_S$ are pairs (X,i) with $X \in$ ob $\underline{C}$ and $i \in$ ob SX and

$$\underline{C}_S((X,i),(Y,j)) = \{(h,\varphi);\ h \in \underline{C}(X,Y),\ \varphi \in SX(i, Shj)\}$$

(recall that $Shj = k$, when $J_X k = \mathrm{dom}(\varrho^Y_j, J_Y j)^o$) and the composition is given by rule $(g,\psi)\cdot(h,\varphi) = (gh, Sh(\psi)\cdot\varphi)$.
A morphism (h,φ) is strong cartesian iff φ is an identity.
Then U can be factorized as $U = P\cdot G$, where $G : \underline{A} \longrightarrow \underline{C}_S$ is a right adjoint (to $F : \underline{C}_S \longrightarrow \underline{A}$ such that $F(X,i) = J_X i$, $F(h,\varphi) = (\varrho^Y_j h, J_Y j)^o \cdot J_X(\varphi))$ and $P : \underline{C}_S \longrightarrow \underline{C}$ (an obvious projection) is a small split fibration i.e. there exists a split cleveage $(P,(S',J',\varrho',(\ )^*))$.

Let α and β be a unit and counit of the adjointness $F \dashv G$.

Note that

(i) for each $Z \in \mathrm{ob}\ \underline{C}_S$, α_Z is a strong cartesian morphism,

(ii) $SX = S'X$, $J_X = F\cdot J'_X$ and $\varrho^X = P\alpha_{J'_X}$ for each $\underline{C}$-object X,

(iii) $(f,A)^o = \beta_A\cdot F((f,GA)^*)$ for each U-morphism (f,A).

P r o p o s i t i o n 2.5. A (right adjoint, split fibration) - factorization of $(U,(S,J,\varrho,(\ )^o))$ is determined uniquely up to isomorphism i.e. if $(G_1,F_1,\alpha_1,\beta_1)$, $(P_1,(S_1,J_1,\varrho_1,(\ )^*))$ is a pair consisting of an adjointness and a split fibration such that $U = P_1\cdot G_1$ and conditions (i)-(iii) above are satisfied, then there exists an isomorphism K such that $K\cdot G = G_1$ and $P_1\cdot K = P$.

We omit a proof since it only generalizes a method used in Proposition 1.1 [3]. We finish this section with a theorem on preservation of limits.

P r o p o s i t i o n 2.6. Let $(U,(S,J,\varrho,(\ )^o))$ be a multiadjointness and assume that $\underline{A}$ has limits of a given type. Then U preserves limits of this type iff every SX has limits of this type. Moreover, if these conditions are satisfied, then for each $h : Y \longrightarrow X$ in $\underline{C}$ $Sh : SX \longrightarrow SY$ preserves limits of a given type.

P r o o f . Let $H : \underline{D} \longrightarrow \underline{A}$ be a diagram of a given type with a limit cone $(w_d : A \longrightarrow Hd)_{d\in \mathrm{ob}\ \underline{D}}$. Having an admissible family $(f_d : X \longrightarrow UHd)_{d\in \mathrm{ob}\ \underline{D}}$ in $\underline{C}$ let $J_X i_d$ be the domain of $(f_d,Hd)^o$. Thus we may define a functor $\hat{H} : \underline{D} \longrightarrow SX$ such that $\hat{H}d = i_d$ for each $d \in \mathrm{ob}\ \underline{D}$.

Let $(\varphi_d : i \longrightarrow i_d)_{d\in ob\, \underline{D}}$ be a limit cone of H in SX. Then there exists s : $J_X i \longrightarrow A$ determined uniquely by the family $((f_d, Hd)^o \cdot J_X(\varphi_d) : J_X i \longrightarrow Hd)_{d \in ob\, \underline{D}}$ i.e. $w_d \cdot Us \cdot \varrho_i^X = f_d$ for each $d \in ob\, \underline{D}$. Thus $Us \cdot \varrho_i^X$ is a $\underline{C}$-morphism we are looking for. The proof of uniqueness is obvious.

To prove the converse assume that H : $\underline{D} \longrightarrow SX$ is a diagram of a given type. Then lim $J_X H$ exists in $\underline{A}$. Now, by Theorem 2 p.117 of [6] for each $\underline{C}$-object X, lim $E_X H$ exists in $(X \downarrow U)$ and for each h : Y $\longrightarrow$ X in $\underline{C}$, $\hat{h}$ preserves it. Moreover, since $G_Y \cdot \hat{h} \cdot \varepsilon_{\lim H} = id : G_Y \hat{h} E_X G_X \lim H \longrightarrow G_Y \hat{h} \lim H$ we have

$$Sh \lim H = G_Y \hat{h} E_X G_X \lim H = G_Y \hat{h} \lim E_X H = \lim G_Y \hat{h} E_X H = \lim ShH.$$

The proof is complete.

3. Regular multiadjointness

Throughout this section we assume that $(U,(S,J,\varrho,(\ )^o))$ is a strongly reduced multiadjointness. For each $\underline{A}$-object A, we write ε_A instead of $(id_{UA}, A)^o$.

D e f i n i t i o n 3.1. An $\underline{A}$-morphism g : A $\longrightarrow$ B is called S-regular iff $g \cdot \varepsilon_A = (Ug, B)^o$.

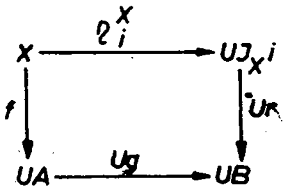
We shall use S-Reg to denote the class of all S-regular morphisms.

Note that $(U,(S,J,\varrho,(\ )^o))$ is a discrete multiadjointness iff S-Reg = $\underline{A}$. If it is a small split fibration then the notions of S-Regular morphism and strong cartesian morphism coincide.

P r o p o s i t i o n 3.2. For an arbitrary $\underline{A}$-morphism g : A $\longrightarrow$ B the following are equivalent

(i) g is S-regular,

(ii) Each commutative square



has a unique diagonal $d : J_X i \to A$ making both triangles commutative,

(iii) $Ug \cdot (f,B)^o = g \cdot (f,A)^o$ for each U-morphism (f,A),

(iv) Gg is strong cartesian in $\underline{C}_S$ (compare section 2).

Proof is straightforward.

P r o p o s i t i o n 3.3 (Properties of S-Reg):

(i) S-Reg is a subcategory of $\underline{A}$, Iso $\underline{A} \subset$ S-Reg.

(ii) Each U-morphism (f,A) has at most one ϱ-factorization $((\varrho^X_i, J_X i), g)$ with $g \in$ S-Reg and, if it is the case, $g = (f,A)^o$.

(iii) For each $\underline{C}$-object X and $\varphi \in$ Mor SX, $J_X \varphi$ is S-regular iff φ is an identity.

The proof is of routine nature and will be omitted.

D e f i n i t i o n 3.4. A (strongly reduced) multiadjointness $(U,(S,J,\varrho,(\ )^o)$ is called regular if every universal ϱ-extension is S-regular.

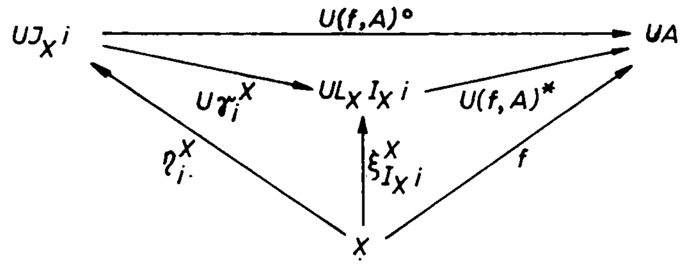
Note that each discrete multiadjointness and small split cleavage is regular. In general we have

P r o p o s i t i o n 3.5. A multiadjointness $(U,(S,J,\varrho,(\ )^o)$ is regular iff the class of universal ϱ-extensions is closed under composition.

P r o o f . The "left-to-right" implication follows directly from Proposition 3.3 (i). To prove the converse assume that for a given U-morphism (f,A), $(f,A)^o : J_X i \to A$. Then $(f,A)^o \cdot \varepsilon_{J_X i}$ is a universal ϱ-extension of $(U(f,A)^o, A)$ i.e., $(f,A)^o$ is S-regular.

It follows directly from Proposition 3.2 (ii) that each regular multiadjoint functor is a morphism - (E,M)-functor [9] for $E = \varrho$ and M = S-Reg. Conversely, if U is a morphism-(E,M)-functor then U is regular multiadjoint with M = S-Reg if $\underline{C}$ is E-colocally small and M is a subcategory of $\underline{A}$.

We call two spectra $\underline{S} = (S,J,\varrho,(\ )^o)$ and $\underline{T} = (T,L,\xi,(\ )^*)$ of a given functor U isomorphic if for each $\underline{C}$-object X there exists an isomorphism $I_X : SX \to TX$ together with a natural isomorphism $\gamma^X : J_X \to L_X T_X$ such that for each U-morphism (f,A), $(f,A)^* \cdot \gamma^X_i = (f,A)^o$ i.e. the following diagram



is commutative.

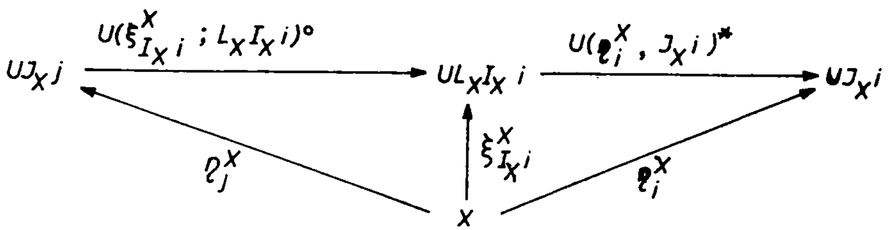
T h e o r e m 3.6. Suppose that multiadjointness $(U,\underline{S})$ and $(U,\underline{T})$ are regular. Then the following are equivalent

(i) The spectra $\underline{S}$ and $\underline{T}$ are isomorphic,

(ii) S-Reg = T-Reg.

P r o o f . (i) implies (ii). Note that for each $\underline{A}$-object A, $(id_{UA},A)^{*}\cdot\gamma_{i}^{UA} = (id_{UA},A)^{o}$ for suitable $i \in$ ob SUA. Thus if $g : A \longrightarrow B$ is T-regular, then $g\cdot(id_{UA},A)^{*} = (Ug,B)^{*}$ and $g\cdot(id_{UA},A)^{o} = g\cdot(id_{UA},A)^{*}\cdot\gamma_{i}^{UA} = (Ug,B)^{*}\gamma_{i}^{UA} = (Ug,B)^{o}$ i.e. g is S-regular. In the same way we prove S-Reg $\subset$ T-Reg.

(ii) implies (i). For each $\underline{C}$-object X and $i \in$ ob SX let $I_X i$ be a unique object in TX such that $L_X I_X i$ is a domain of $(\varrho_{i}^{X},J_X i)^{*}$. Thus from the diagram



and Proposition 3.3 (i), (iii) one can easily deduce that $j = i$ and $(\xi^{X}_{I_X i}, L_X I_X i)^{o} = \gamma_{i}^{X}$ is a natural isomorphism. Now it is a routine to verify that I_X can be extended to an isomorphism of categories and that considered spectra are isomorphic.

Hence the spectrum of a regular multiadjointness $(U,\underline{S})$ is determined uniquely (up to isomorphism) by the subcategory

S-Reg of its regular morphisms. Thus we will often say "U is multiadjoint with respect to a subcategory $\underline{D} \subset \underline{A}$" instead of "there exists a regular multiadjointness $(U,\underline{S})$ and S-Reg $= \underline{D}$".

4. Po-multiadjointness

D e f i n i t i o n 4.1. A spectrum $(S,J,\varrho,(\ )^o)$ of U is called po-spectrum if for each $\underline{C}$-object X, SX is a partially ordered set. A multiadjointness $(U,(S,J,\varrho,(\ )^o))$ is called po-multiadjointness if $(S,J,\varrho,(\ )^o)$ is a po-spectrum.

Note that every po-multiadjointness is strongly reduced.

P r o p o s i t i o n 4.2. $(U,(S,J,\varrho,(\ )^o)$ is a po--multiadjointness iff ϱ_i^X is U-epi for each $\underline{C}$-object X and $i \in$ ob SX.

P r o p o s i t i o n 4.3. Let $\underline{A}$ be a complete category and U preserves all limits. Then each small spectrum $\underline{S}$ of U is a po-spectrum. More precisely, for each multiadjointness $(U,\underline{S})$ and $\underline{C}$-object X, SX is a complete lattice.

Proof follows directly from Proposition 2.6 and Proposition 3 [6] p.110.

Below we prove that under some assumptions weaker then in Proposition 4.3, we can "reduce" a given small spectrum of U to a po-spectrum. The crucial point of our consideration is the following lemma.

L e m m a 4.4. Let $\underline{D}$ be a small category with equalizers and limits of arbitrary chains. Then there exists a full coreflective subcategory $\underline{D}_1$ of $\underline{D}$ such that $\underline{D}_1$ is a poset.

P r o o f . For each $\underline{D}$-object d, the class of all parallel pairs of morphisms with domain d,

$$P(d) = \left\{ d \overset{f_1^i}{\underset{f_2^i}{\rightrightarrows}} d_i \right\}$$

is a set. Thus using equalizers and limits of chains one can construct a limit of a diagram P(d) i.e. morphism

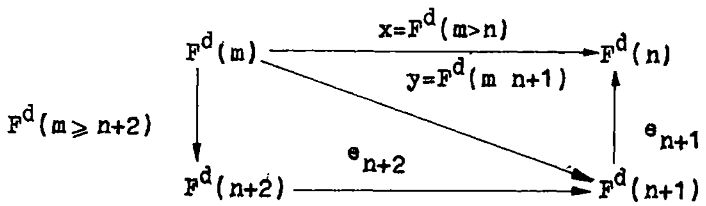
$$e_1 = \mathrm{eq}\ P(d) : d_1 \longrightarrow d$$

such that $f_1^i e_1 = f_2^i e_1$ for each $(f_1^i, f_2^i) \in P(d)$ and for every $h : d_0 \longrightarrow d$ there exists a unique h_1 with $h = e_1 \cdot h_1$ provided that h equalizes each pair of $P(d)$.

Let α be an ordinal number with $\mathrm{card}\,\alpha > \mathrm{card}(\mathrm{ob}\,\underline{D})$. We define a chain $F^d : \underline{\alpha}^{op} \longrightarrow \underline{D}$ ($\underline{\alpha}$ is a chain of ordinals less then α treated as a category) as follows $F^d(0) = d$, $F^d(n+1 \longrightarrow n) = e_{n+1} = \mathrm{eq}\ P(F^d(n)) : F^d(n+1) \longrightarrow F^d(n)$ and for each limit ordinal $\beta < \alpha$

$$\left(F^d(\beta) \xrightarrow{F^d(\beta \to n)} F^d(n)\right)_{n < \beta}$$

is a limit cone of a chain $F^d|_\beta : \underline{\beta}^{op} \longrightarrow \underline{D}$. By the smallness assumption there exist $n < m < \alpha$ such that $F^d(n) = F^d(m)$. If $m = n+1$ then obviously $F^d(k) = F^d(m)$ for each $k \geqslant n+1$. If $m > n+1$, then we have a commutative diagram



Note that $e_{n+1} \cdot y \cdot x : F^d(n) \longrightarrow F^d(n)$. So we have $e_{n+1} \cdot y \cdot x \cdot e_{n+1} = e_{n+1}$ and $y \cdot x \cdot e_{n+1} = \mathrm{id}$. Thus y is split epi and, consequently e_{n+2} is an isomorphism i.e. $F^d(k) = F^d(m)$ for each $k \geqslant n+1$.

For each $\underline{D}$-object d we shall use $\hat{d}$ to denote an object $F^d(n)$ such that $F^d(k) = F^d(n)$ for each $n \leqslant k < \alpha$. Note that for every $d' \in \mathrm{ob}\,\underline{D}$ there exists at most one morphism from $\hat{d}$ to d'. Now it is not hard to verify that the full subcategory generated by objects $\{\hat{d},\ d \in \mathrm{ob}\,\underline{D}\}$ is that one we are looking for.

T h e o r e m 4.5. Let $\underline{A}$ be a category with equalizers and limits of arbitrary chains. Assume also that U preserves limits of these types. Then the following are equivalent.

(i) U has a small spectrum.

(ii) U has a small po-spectrum.

P r o o f . The only nontrivial implication is (i)$\Rightarrow$(ii). Let $(S,J,\varrho,(\ )^{o})$ be a small spectrum of U. Then by Proposition 2.6 for each $\underline{C}$-object X, SX has equalizers and limits of chains. Hence by Lemma 4.4 there exist coreflective full subcategories $\widehat{SX}\subset SX$. For each $i\in \mathrm{ob}\ SX$, by $\varphi_i : \hat{i}\longrightarrow i$ we denote a coreflector of i in SX. Let us consider a 4-tuple $(\hat{S},\hat{J}=J_{\hat{S}},\hat{\varrho}=\varrho_{\hat{S}},(\ )^{*})$ where $(f,A)^{*}=(f,A)^{o}\cdot J_X(\varphi_i)$ for each $(f : X\longrightarrow UA,\ A)$ with $J_X i=\mathrm{dom}(f,A)^{o}$. Obviously $(\hat{S},\hat{J},\hat{\varrho},(\ )^{*})$ satisfies (i) and (ii) of Proposition 2.4. Let $h : Y\longrightarrow X$ in $\underline{C}$ and $i\in \mathrm{ob}\ SX$. Let $F^{i} : \underline{\alpha}^{op}\longrightarrow SX$ be a chain that constructs $\hat{i}$, i.e. $\hat{i}=\lim F^{i}$. Then, by Proposition 2.6, $\mathrm{Sh}\ \hat{i}=\lim \mathrm{Sh}F^{i}$.

$$\widehat{\mathrm{Sh}\hat{i}}\leq \widehat{\mathrm{Sh}i}\xrightarrow{\ \varphi_{\mathrm{Sh}i}\ }\mathrm{Sh}i\xleftarrow{\ \mathrm{Sh}\varphi_i\ }\mathrm{Sh}\hat{i}=\lim \mathrm{Sh}F^{i}$$

For each $k<\alpha$ there exists exactly one morphism $\psi_k : \widehat{\mathrm{Sh}i}\longrightarrow \mathrm{Sh}F^{i}(k)$. It follows that there exists a unique $\psi : \widehat{\mathrm{Sh}i}\longrightarrow \mathrm{Sh}\hat{i}$ and consequently $\widehat{\mathrm{Sh}i}=\widehat{\mathrm{Sh}\hat{i}}$. Now for each U-morphism (f,A) with $\mathrm{dom}(f,A)^{o}=J_X i$ we have $(fh,A)^{*}=$

$=(f,A)^{o}\cdot(\varrho^{X}_{i}h,J_X i)^{o}\cdot J_Y(\varphi_{\mathrm{Sh}i})=(f,A)^{o}\cdot J_X(\varphi_i)\cdot(\varrho^{X}_{\hat{i}}h,J_X\hat{i})^{*}=$

$=(f,A)^{*}\cdot(\varrho^{X}_{\hat{i}}h,\ J_X\hat{i})^{*}$ i.e., condition (iii) of Proposition 2.4 is also satisfied. The proof is complete.

5. Examples

The first group of examples is based on the following observation.

P r o p o s i t i o n 5.1. Assume, that $\underline{C}$ has a factorization system (E,M) and $\underline{C}$ is colocally small with respect to E. Then for every full subcategory $\underline{D}$ of $\underline{C}$, closed with respect to M (i.e. $B \in ob\, \underline{D}$ provided that there is $m : B \to A$ with $m \in M$ and $A \in ob\, \underline{D}$), the full embedding $\underline{D} \hookrightarrow \underline{C}$ has a small spectrum consisting of suitable full subcategories of E_X = = small skeleton of $\{e \in E,\ dom(e) = X\}$.

If $E \subset Epi\, \underline{C}$ then the spectrum described above is a po--spectrum.

5.1.1. The category of groups Gr, commutative rings CRng or, in general, each variety or quasivariety V has a factorization system (Surjective Epi, Mono). Then the full embeddings Finite Gr $\hookrightarrow$ Gr, Finite CRng $\hookrightarrow$ CRng, Finite V $\hookrightarrow$ V are po-multiadjoint functors with spectra consisting of all congruence of a finitary index.

5.1.2. Full embeddings Dom $\hookrightarrow$ CRng, Red $\hookrightarrow$ CRng, Prim $\hookrightarrow$ CRng etc..., where Dom, Red, Prim etc... are full subcategories of integral domains, reduced rings, primary rings, etc..., are multiadjoint functors. The spectra of a commutative ring A relatively to these embeddings are prime, semiprime, primary etc... spectra of A considered here as a poset [2].

5.1.3. Let Locc be a full subcategory of CRng with local rings as objects. Then the full embedding Locc $\hookrightarrow$ CRng is po-multiadjoint. The spectrum of a commutative ring A is a prime spectrum of A (with dualized order).

5.1.4. An embedding of a full subcategory of all metrizable spaces into a category of all topological spaces is po-multiadjoint.

5.2. The second group of examples is based on the observation that some compositions of po-multiadjoint functors are po-multiadjoint.

5.2.1. Let TopVect R be a category of all topological real vector spaces and linear continuous maps, Vect(R) be a category of all real vector spaces and linear maps. Consider a commutative diagram of functors

$$\begin{array}{ccc} \mathrm{TopVect}(R) & \xrightarrow{Z_1} & \mathrm{Vect}(R) \\ {\scriptstyle U_1}\downarrow & & \downarrow{\scriptstyle U} \\ \mathrm{Top} & \xrightarrow{U'} & \mathrm{Set} \end{array}$$

One can easily prove, that Z_1 is po-multiadjoint. Then UZ_1 is a po-multiadjoint functor.

5.2.2. Let Norm(R) be a category of all normed real vector spaces and linear maps f with $\|f\| \leqslant 1$. The topology associated with a norm gives the functor U_2 : Norm(R) $\longrightarrow$ TopVect(R). U_2 is a regular po-multiadjoint functor (for a description of its spectrum see [2]). Regular morphisms with respect to this spectrum are norm-preserving linear maps. It has been proved by Manes [7] that U_1 is right adjoint. Then both compositions U_1U_2 and Z_1U_2 are po-multiadjoint.

5.2.3. Let Ban(R) be a full subcategory of Norm(R) with real Banach spaces as objects. The full embedding $E:\mathrm{Ban}(R) \hookrightarrow \mathrm{Norm}(R)$ has a left adjoint and the unit of this adjointness is point-wise a linear, norm-preserving map. Then the composition U_1U_2E is a regular po-multiadjoint functor.

5.2.4. Let Ab be a category of abelian groups and OrdAb be a category of ordered abelian groups and order preserving homomorphisms. Then the forgetful functor U:OrdAb $\longrightarrow$ Ab is po-multiadjoint (for a description of its spectrum see [2]).

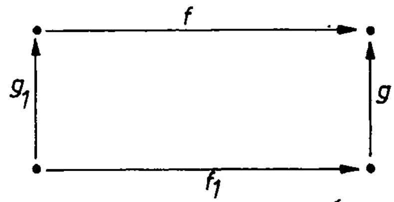
Regular morphisms with respect to this spectrum are proper order preserving homomorphisms, i.e. $\forall x, \ (fx \geqslant 0 \Longrightarrow x \geqslant 0)$. Then the composition OrdAb $\longrightarrow$ Ab $\longrightarrow$ Set is a regular po-multiadjoint functor.

5.2.5. Let Grph be a category of graphs i.e., objects are 4-tuples $(X, A, (d,c): X \rightrightarrows A)$ and morphisms from $(X, A, (d,c))$ to $(Y, B, (d_1, c_1))$ are given by pairs of functions (f,g) such that $d_1 \cdot f = g \cdot d$ and $c_1 \cdot f = g \cdot c$. Let U:Grph $\longrightarrow$ Set, where $U(X, A, (d,c)) = X$, $U(f,g) = f$. The (epi, mono)-factorization system in Set induces a natural factorization system in Grph; $(f,g) = (f, m_g) \cdot (id, e_g)$, where $g = m_g \cdot e_g$ is an (epi, mono)-factorization of g in Set. It follows that U is a po-multiadjoint functor with a spectrum $(S,J,\varrho,(\ )^{o})$ where SX is a poset of equivalence relations on the set $X\times\{0,1\}$.

Since U': Cat ⟶ Grph is right adjoint, then the composition UU': Cat ⟶ Set is a po-multiadjoint functor.

Regular morphisms for this spectrum are functors with injective object-function.

5.3. Partial monadic algebras [5]. Throughout we will write $pb(f,g) = (f_1,g_1)$ if the square is a pullback:



By a pb-monad over topos $\underline{C}$ (Alexandroff's monad in [8]) we mean a monad $\underline{T} = (T,\mu,\varrho)$ such that for each monomorphism $m: X \longrightarrow Y$ in $\underline{C}$

$$pb(\mu_Y, TTm) = (\mu_X, Tm),\quad pb(Tm,\varrho_Y) = (m,\varrho_X)$$

and for each $f: Z \longrightarrow Y$, T preserves pullback of (f,m). A partial monadic $\underline{T}$-algebra is a pair $(A,(a,a_0))$ where $(a,a_0): TA \longrightarrow A$ is a partial morphism such that $(a,a_0)\circ\varrho_A = id_A$ and $(a,a_0)\circ(Ta,Ta_0) = (a,a_0)\circ\mu_A$ where $\circ$ is a composition of partial morphisms defined by pullback. Morphisms from $(A,(a,a_0))$ to $(B,(b,b_0))$ are given by $\underline{C}$-morphisms $h: A \longrightarrow B$ such that $h\ (a,a_0) \leqslant (b,b_0)\circ Th$, where $\leqslant$ is an obvious order on the set of parallel partial morphisms.

Let $PMA(\underline{T})$ be the category of all partial monadic $\underline{T}$-algebras and their morphisms. Then the obvious forgetful functor $U: PMA(\underline{T}) \longrightarrow \underline{C}$ is regular po-multiadjoint with a spectrum defined as follows:

For every object X in $\underline{C}$, SX is a poset of all subobjects of TX $x_i: X_i \longrightarrow TX$ such that:

(i) ϱ_X factors through x_i, i.e. $\varrho_X = x_i\cdot\varrho_i^X$

(ii) $pb(\mu_X, x_i) = (x_i^1, Tx_i \cdot x_i^o)$ for some $x_i^o : Z \longrightarrow TX_i$.

Note that $\underline{X}_i = (X_i, (x_i^1, x_i^o))$ is a partial monadic $\underline{T}$-algebra. Then we define a functor $J_X : SX \longrightarrow PMA(\underline{T})$ by $J_X(x_i) = \underline{X}_i$. Let $f : X \longrightarrow U(A,(a,a_0)) = A$. Its universal ϱ-extension is a morphism $a \cdot f_1 : (X_f, (x_f^1, x_f^o)) \longrightarrow (A,(a,a_0))$, where $(f_1, x_f) = pb(Tf, a_0)$.

One can prove that $h : (A,(a,a_0)) \longrightarrow (B,(b,b_0))$ is S-regular iff $h \circ (a,a_0) = (b,b_0) \circ Th$. In particular, every universal ϱ-extension is S-regular.

5.3.1. For all categories $\underline{C}_i$ listed below, monads $\underline{T}_i$ $(i=1,\dots,5)$ over Set, which satisfy the equation $\underline{C}_i = Set^{\underline{T}_i}$ are pb-monads:

$\underline{C}_1$ = Alg Ω = category of all algebras of a given type Ω,

$\underline{C}_2$ = Smgr = category of semigroups,

$\underline{C}_3$ = Latt = category of complete lattices with maps preserving joins as morphisms i.e. $\underline{T}_3$ = power-set monad,

$\underline{C}_4$ = CompHaus = category of compact Hausdorff spaces i.e. $\underline{T}_4$ = ultrafilter monad.

$\underline{C}_5$ = CLatt = category of continuous lattices and maps preserving directed joins and arbitrary meets as morphisms, i.e. $\underline{T}_5$ = filter monad [1].

Then, the categories of partial monadic $\underline{T}_i$-algebras are isomorphic to the following ones:

PMA $\underline{T}_1$ = category of all partial algebras of a given type Ω.

PMA $\underline{T}_2$ = category of partial semigroups.

PMA $\underline{T}_3$ = category of complete ordered sets [2] i.e., posets in which every upper-bounded subset has a join, with maps preserving all existing joins.

PMA $\underline{T}_4$ = category of locally compact Hausforff spaces.

PMA $\underline{T}_5$ = category of continuous complete posets i.e. posets $(X,\leq)$ such that $(X, \leq^{op}) \in$ ob PMA$(\underline{T}_3)$ and for every family $\{D_i,\ i \in I\}$ of directed subsets $\bigwedge \{\bigvee D_i,\ i \in I\} = \bigvee \{\bigwedge\{d_i,\ i \in I\};\ (d_i,\ i \in I) \in \prod\{D_i,\ i \in I\}\}$ in that sense, that the left hand side exists iff the right hand side exists and then, they are equal. Morphism are maps which preserve directed joins and all existing meets.

5.3.2. Let $\underline{T}$ be a pb-monad. A pair $(A,(a,a_0) : TA \longrightarrow A)$ is called a weak partial monadic $\underline{T}$-algebra, if $(a \cdot a_0) \circ \varrho_A = id_A$ and $(a,a_0) \circ \mu_A \leq (a,a_0)$ (Ta,Ta_0). Let WPMA($\underline{T}$) be the category of all weak partial monadic $\underline{T}$-algebras with morphisms defined in the same way as in PMA($\underline{T}$). Let $U' : \text{WPMA } \underline{T} \longrightarrow \underline{C}$ be the obvious forgetful functor. We define a spectrum of U' in the same way as in 5.3. One can easily prove that this multiadjointness is a po-multiadjointness but not regular.

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