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JOINT DISTRIBUTIONS OF OBSERVABLES AND MEASURES WITH INFINITE VALUES

Joint distribution of observables in measures attaining infinite values is ivestigated in the framework of quantum logics. For a logic of a separable Hilbert space, dim $H \geq 3$, it is proved that any σ-finite measure has a carrier, and this result is applied to the problem of the existence of a joint distribution.

1. Introduction

Let us suppose that the set, L, of all experimentally verifiable propositions of physical system forms a quantum logic. According to Varadarajan [1], assume that the quantum logic L is an orthomodular orthocomplemented σ-lattice with the minimal and maximal elements 0 and 1, respectively, and with an orthocomplementation $\perp : a \longmapsto a^\perp$, $a, a^\perp \in L$, which satisfies (i) $(a^\perp)^\perp = a$, for any $a \in L$; (ii) if $a < b$, then $b^\perp < a^\perp$; (iii) $a \vee v^\perp = 1$, for any $a \in L$; (iv) if $a < b$, then $b = a \vee (a^\perp \wedge b)$.

Two elements a and b of L are said to be (i) orthogonal and we write $a \perp b$, if $a < b^\perp$; (ii) compatible and write $a \leftrightarrow b$ if there are three mutually orthogonal elements a_1, b_1, c such that $a = a_1 \vee c$, $b = b_1 \vee c$.

Physical quantities are identified with the observables of the quantum logic. An observable on L is a map x from the set, $B(R_1)$, of all Borel measurable subsets of the real line

R_1, into L such that (i) $x(R_1) = 1$; (ii) $x(E) \perp x(F)$ if $E \cap F = \emptyset$; (iii) $x(\bigcup_{i=1}^{\infty} E_i) = \bigvee_{i=1}^{\infty} x(E_i)$ whenever $E_i \cap E_j = \emptyset$, $i \neq j$. An observable x is bounded if there is a compact subset $C \subset R_1$ such that $x(C) = 1$. Two observables x and y are compatible if $x(E) \leftrightarrow y(F)$ for any $E, F \in B(R_1)$.

Physical states are identified with the states of the quantum logic, that is, a state is map $m: L \rightarrow [0,1]$ with (i) $m(1) = 1$; (ii) $m(\bigvee_{i=1}^{\infty} a_i) = \sum_{i=1}^{\infty} m(a_i)$ whenever $a_i \perp a_j$ for $i \neq j$. The more general notion as the state is a measure. So, we say that a map $m: L \rightarrow R_1 \cup \{\infty\}$ is said to be a measure on L if (i) $m(\bigvee_{i=1}^{\infty} a_i) = \sum_{i=1}^{\infty} m(a_i)$ whenever $a_i \perp a_j$ for $i \neq j$; (ii) $m(0) = 0$.

An element a is a carrier of a measure m if $m(b) = 0$ iff $b \perp a$. It is clear that if a carrier of a measure exists, then it is unique. The measure m is (i) finite if $m(a) < \infty$, for any $a \in L$, or, equivalently, if $m(1) < \infty$; (ii) σ-finite if there is a sequence of mutually orthogonal elements $\{a_i\}_{i=1}^{\infty}$ with $\bigvee_{i=1}^{\infty} a_i = 1$ and $m(a_i) < \infty$ for any i. An observable x is σ-finite with respect to a measure m if there is a sequence $\{E_i\}_{i=1}^{\infty} \subset B(R_1)$ such that $E_i \cap E_j \neq \emptyset$ if $i \neq j$, $m(x(E_i)) < \infty$, for any $i \geq 1$, and $\bigcup_{i=1}^{\infty} E_i = R_1$.

We say that a function m is continuous from below (above) on an element $a \in L$ if, for any $a_1 < a_2 < \ldots$ with $\bigvee_{i=1}^{\infty} a_i = a$ ($a_1 > a_2 > \ldots$ with $\bigwedge_{i=1}^{\infty} a_i = a$ and at least for one n_0 $m(a_{n_0}) < \infty$) we have $m(a) = \lim_i m(a_i)$. Similarly as in [2] we may prove that a finitely additive function on L with $m(0) = 0$ is a measure iff m is continuous from below on any element of L, or, equivalently, m is continuous from above on the minimal element 0.

2. Joint distributions

For an observable x, an event $x(E)$ denotes that the measured value, ξ, of the corresponding physical quantity lies in a Borel subset $E \in B(R_1)$. If a quantum mechanical system is described by a measure m, the expression

$$\mu^{m}_{x_1 \dots x_n}(E_1 \times \dots \times E_n) = m\left(\bigwedge_{j=1}^{n} x_j(E_j)\right), \quad E_j \in B(R_1),\ j=1,\dots,n, \tag{2.1}$$

denotes the measure of the simultaneous measurement of the observable $x_1,\dots,x_n$ which give measured quantities lying in the Borel subsets $E_i \in B(R_1)$, $i=1,\dots,n$.

According to Gudder [3], we say the observables $x_1,\dots,x_n$ have a joint distribution in a measure m if there is a measure $\mu^{m}_{x_1\dots x_n}$ on the set $B(R_n)$ of all Borel subsets of R_n such that (2.1) holds.

Gudder [3] introduced the notion of the joint distribution only for a state (it is named type I joint distribution, too). This type has been studied in [5-12]. Urbanik [4] defined another type of a-joint distribution in a state (type II joint distribution) for the summable self-adjoint operators in a Hilbert space, and Gudder [3] generalized this notion for bounded observables on a sum logic.

If m is a state (or a finite measure), then the joint distribution, if it exists, is determined unambiguously on $B(R_n)$. For a measure m with $m(1) = \infty$, the uniqueness must be studied in more detail.

The notion of joint distribution in a measure may be generalized to any set $\{x_t : t \in T\}$ of observables in a natural way: we say that observables $\{x_t : t \in T\}$ have a joint distribution in a measure m if any finite subset of $\{x_t : t \in T\}$ has one. The generalization of this notion to σ-homomorphisms defined on a measurable space (X,φ) is straightforward (here φ is a σ-algebra of subsets of X and a map $x: \varphi \to L$ is a σ-homomorphism if (i) $x(X) = 1$; (ii) $x(E) \perp x(F)$ if $E \cap F = \emptyset$; (iii) $\left(x \bigcup_{i=1} E_i\right) = \bigvee_{i=1}^{\infty} x(E_i)$, $\{E_i\} \subset \varphi$).

S.P. Gudder in [7] posed the following problem: Can a joint distribution be defined for noncompatible observables?
The answer to that question has been obtained in the papers [5,6,13,14].

In the present note we solve this problem for measures with $m(1) = \infty$. The solution will contain the answer for measures on a Hilbert space logic, too.

In the sequel we suppose that the observables $x_1,\dots,x_n$ are given and for the joint distribution $\mu^m_{x_1\dots x_n}$ of $x_1,\dots,x_n$ in a measure m we shall write simply μ.

L e m m a 2.1. Let observables $x_1,\dots,x_n$ be compatible. Then, for any measure m on L, there is a joint distribution. If at least one observable is σ-finite with respect to m, then the joint distribution is unique.

P r o o f . For compatible observables $x_1,\dots,x_n$, there is a unique σ-homomorphism $x\colon B(R_n) \to L$ such that $x(R_1 \times\dots\times E_i \times\dots\times R_1) = x_i(E_i)$, $i = 1,\dots,n$; see [1, Th. 6.17]. Let us put $\mu(B) := m(x(B))$, $B \in B(R_n)$. Then μ is a well defined joint distribution.

The uniqueness of the joint distribution follows from the uniqueness of the extension of σ-finite measures defined on the set of all rectangles of $B(R_n)$, [2]. Q.E.D.

Define

$$(2.2)\quad a(E_1,\dots,E_n) = \bigvee_{i_1\dots i_n=0}^{1} \bigwedge_{j=1}^{n} x_j\left({}^{i_j}E_j\right),\ E_1,\dots,E_n \in B(R_1),$$

where ${}^{0}E := R_1 - E$, ${}^{1}E := E$.

We put (if it exists)

$$(2.3)\qquad a_0 = \bigwedge \left\{a(E_1,\dots,E_n) : E_1,\dots,E_n \in B(R_1)\right\}.$$

In the paper [13] it is shown that the element a_0 exists, and, moreover, there is a sequence $\left\{a(E_1^k,\dots,E_n^k)\right\}_{k=1}^{\infty}$ such that

$$a_0 = \bigwedge_{k=1}^{\infty} a(E_1^k,\dots,E_n^k). \tag{2.4}$$

The element a_0 is called a commutator of $x_1,\dots,x_n$, and the main properties of the commutator are investigated in [12,13].

Lemma 2.2. Let $x_1,\dots,x_n$ have a joint distribution in m and let a_0 is the commutator of $x_1,\dots,x_n$. Then

$$(2.5)\begin{cases} \text{(i)} \quad m(a(E_1,\dots,E_n)) = m(1),\ E_1,\dots,E_n \in B(R_1), \\ \text{(ii)} \quad m(\bigwedge_{i=1}^{n} x_i(E_i) \wedge \bigwedge_{k=1}^{K} a(E_1^k,\dots,E_n^k)) = m(\bigwedge_{i=1}^{n} x_i(E_i)), \end{cases}$$

for any $E_1,\dots,E_n$, $E_1^k,\dots,E_n^k \in B(R_1)$, $k = 1,\dots,K$, where K may be an integer or ∞;

$$m(\bigwedge_{i=1}^{n} x_i(E_i) \wedge a_0) = r(\bigwedge_{i=1}^{n} x_i(E_i)), \tag{2.6}$$

$$E_1,\dots,E_n \in B(R_1).$$

$$m(a_0) = m(1). \tag{2.7}$$

Proof. The part (i) is evident, and (ii) is same as (2.6) in [6]. (2.6) follows from (2.5) and (2.4). For (2.7) it is sufficient to put $E_1 = E_2 = \dots = E_n = R_1$. Q.E.D.

Lemma 2.3. Let $x_1,\dots,x_n$ have a joint distribution in a measure m. If there is $E \in B(R_1)$ and x_i such that $m(x_i(E)) < \infty$, then

$$m(x_i(E) \wedge a_0^{\perp}) = 0. \tag{2.8}$$

Proof. From the results of the paper [13] there follows that $a_0^{\perp} \leftrightarrow x_j(F)$ for any $F \in B(R_1)$ and any $j = 1,\dots,n$. Hence $a_0^{\perp} \leftrightarrow x_i(E)$ and from (2.6) we have

$$m(x_i(E)) = m(x_i(E) \wedge a_0) + m(x_i(E) \wedge a_0^{\perp}) = m(x_i(E)) + m(x_i(E) \wedge a_0^{\perp}),$$

consequently, (2.8) holds.

L e m m a 2.4. Let $x_1,\dots,x_n$ have a joint distribution in a measure m. If at least one observable is σ-finite with respect to m, then

$$m(a_0^{\perp}) = 0. \tag{2.9}$$

P r o o f . Let $\{E_n\}_{n=1}^{\infty} \subset B(R_1)$ be a sequence with $E_i \cap E_j = \emptyset$, if $i \neq j$, $\bigcup_{n=1}^{\infty} E_n = R_1$, and, for some x_i, $m(x_i(E_n)) < \infty$, $n \geq 1$. Since $a_0^{\perp} \leftrightarrow x_{i(E_n)}$, for any n, then, due to [1, Lemma 6.10], $a_0^{\perp} \wedge \bigvee_{n=1}^{\infty} x_i(E_n) = \bigvee_{n=1}^{\infty} (a_0^{\perp} \wedge x_i(E_n))$. Check

$$m(a_0^{\perp}) = m(a_0^{\perp} \wedge 1) = m\left(a_0^{\perp} \wedge \bigvee_{n=1}^{\infty} x_i(E_n)\right) = \sum_{n=1}^{\infty} m(a_0^{\perp} \wedge x_i(E_n)) = 0,$$

when we use (2.8).

T h e o r e m 2.5. Let $x_1,\dots,x_n$ be observables and let m be a measure. If (2.9) holds, then there is a joint distribution of $x_1,\dots,x_n$ in a measure m. If at least one observable is σ-finite with respect to m, then the joint distribution is unique.

If $x_1,\dots,x_n$ have a joint distribution in m and at least one observable is σ-finite with respect to m, then (2.9) holds.

P r o o f . The first part of Theorem follows from the following. Let a_0 be the commutator of $x_1,\dots,x_n$. Then, according to [13], $x_{i0}(E) := x_i(E) \wedge a_0$, $E \in B(R_1)$, $i=1,\dots,n$, defines an observable x_{i0} of a quantum logic $L_{(0,a_0)} := \{b:\ b \in L,\ b < a_0\}$ (here the greatest element is a_0, an orthocomplementation ",' is defined via $b' := b^{\perp} \wedge a_0$ $(b < a_0)$). Moreover, $x_{1o}, \dots, x_{no}$ are mutually compatible observables. Hence, due to Lemma 2.1, $x_{1o}, \dots, x_{no}$ have a joint distribution in a measure $m_o := m|L(0, a_0)$. From (2.9) we have

$$m\left(\bigwedge_{i=1}^{n} x_i(E_i)\right) = m\left(\bigwedge_{i=1}^{n} x_i(E_i) \wedge a_o\right) + m\left(\bigwedge_{i=1}^{n} x_i(E_i) \wedge a_o^{\perp}\right) =$$

$$= m_o\left(\bigwedge_{i=1}^{n} x_{io}(E_i)\right),$$

which entails that $x_1, \dots, x_n$ have a joint distribution in m.

Repeating the same arguments as those in the proof of Lemma 2.1 we establish the uniqueness of a joint distribution.

The second part of the assertion of Theorem follows from Lemma 2.4.

C o r o l l a r y 2.6. Let a_m be a carrier of a measure m. If $x_1, \dots, x_n$ have a joint distribution in m and at least one observable is σ-finite with respect to m, then

$$a_m < a_o, \tag{2.10}$$

and

$$a_m < a(E_1, \dots, E_n), \quad \text{for any} \quad E_1, \dots, E_n \in B(R_1). \tag{2.11}$$

If (2.10) holds, or equivalently, (2.11) is true, then $x_1, \dots, x_n$ have a joint distribution in m. If at least one observables is σ-finite with respect to m, then the joint distribution is unique.

P r o o f . (2.10) and (2.11) follows from the definition of a carrier, and from Theorem 2.5 and (2.4).

N o t e 1. The condition

$$m(a(E_1, \dots, E_n)^{\perp}) = 0, \quad \text{for any} \quad E_1, \dots, E_n \in B(R_1), \tag{2.12}$$

is the necessary and sufficient condition for $x_1,\dots,x_n$ to have a joint distribution in a state or a finite measure m [5,6,13]. For a measure with $m(1) = \infty$ this condistions is known only in particular cases, see Lemma 2.6 and the following lemma.

P r o p o s i t i o n 2.7. Let a logic L be σ-continuous, that is, for any $a_1 < a_2 < \dots$ and, any a, we have $\left(\bigvee_{i=1}^{\infty} a_i\right) \wedge a = \bigvee_{i=1}^{\infty} (a_i \wedge a)$. Let there hold for a measure m and observables $x_1,\dots,x_n$

$$(2.13)\qquad m\left(\bigwedge_{j=1}^{n} x_j(E_1^j \cup E_2^j)\right) = \sum_{k_1\dots k_n=1}^{2} m\left(\bigwedge_{j=1}^{n} x_j(E_{k_j}^j)\right),$$

$$E_1^j \cap E_2^j = \emptyset,\ E_1^j,\ E_2^j \in B(R_1),\ j=1,\dots,n.$$

If at least one observable is σ-finite with respect to m, then there is a unique joint distribution of $x_1,\dots,x_n$ in m.

P r o o f . It is easy to verify that (2.13) implies that $\mu : E_1 \times \dots \times E_n \longmapsto m\left(\bigwedge_{j=1}^{n} x_j(E_j)\right)$, is a finitely additive function on the set $\mathcal{P}_n$ of all rectangles. The σ-continuity of a logic and the continuity of m from below entail that μ is a σ-additive and σ-finite function on $\mathcal{P}_n$. Therefore it may be extended to a measure on $B(R_n)$. Q.E.D.

The results of all the above assertions may be extended to the set of observables $\{x_t : t \in T\}$ such that there is at most countable subset $\mathcal{A} \subset \bigcup\{R(x_t) : t \in T\}$, where $\mathcal{A}$ generates the minimal sublogic of L containing the set $\bigcup\{R(x_t) : t \in T\}$ (here $R(x) := \{x(E) : E \in B(R_1)\}$). In particular, this is true for a sequence of observables. For given observables $\{x_t : t\in T\}$ we define the commutator, $a_0(T)$, of $\{x_t : t \in T\}$ (if it exists) via

$$(2.14)\qquad a_0(T) = \bigwedge\{a_0(F) : F \text{ is a finite subset of } T\},$$

where $a_0(F)$ is the commutator of observables $x_{t_1},\dots,x_{t_n}$ and $F = \{t_1,\dots,t_n\}$.

From [13] it follows that $a_0(T)$ exists, and moreover, there is a sequence of finite subsets $F_n \subset T$ such that

$$a_0(T) = \bigwedge_{n=1}^{\infty} a_0(F_n). \tag{2.15}$$

T h e o r e m 2.8. Let $\{x_t: t\in T\}$ be a system of observables for which there is at most countable subset $\mathcal{A} \subset \bigcup\{R(x_t): t\in T\}$, where $\mathcal{A}$ generates the minimal sublogic of L containing all $R(x_t)$, $t\in T$. If $\{x_t: t\in T\}$ have a joint distribution in m and at least one observable is σ-finite with respect to m, then

$$m(a_0(T)^{\perp}) = 0. \tag{2.16}$$

If (2.16) holds, then there is a joint distribution of $\{x_t: t\in T\}$. If at least one observable is σ-finite with respect to m, then there is a unique σ-finite measure μ on $\prod_{t\in T} B(R_1)$ such that

$$\mu\Big(\bigcap_{j=1}^{n} \pi_{t_j}^{-1}(E_j)\Big) = m\Big(\bigwedge_{j=1}^{n} x_{t_j}(E_j)\Big),\quad E_1,\dots,E_n\in B(R_1), \tag{2.17}$$

where π_t is the t-th projection from R_1^T onto R_1

P r o o f . It is clear that if $F_1\subset F_2\subset T$, then $a_0(F_2) < < a_0(F_1)$. Let x_{t_0} be σ-finite with respect to m. Then (2.15) implies

$$a_0(T) = \bigwedge_{n=1}^{\infty} a_0(F_n) > \bigwedge_{n=1}^{\infty} a_0(F_n\cup\{t_0\}) > \bigwedge_{n=1}^{\infty} a_0\Big(\bigcup_{i=1}^{n}(F_i\cup\{t_0\})\Big) >$$

$$> a_0(T).$$

Theorem 2.5 entails $m(a_0(B_n)^{\perp}) = 0$, $n\geq 1$, where $B_n = \bigcup_{i=1}^{n} F_i \cup \cup\{t_0\}$. The continuity of m from below gives (2.16).

Conversely, let (2.16) hold. Then, for any finite subset $F \subset T$, we have $m(a_0(F)^{\perp}) = 0$. Now we claim to show that there is a unique μ on $\prod_{t\in T} B(R_1)$ for which (2.17) holds. Let x_{t_0} be σ-finite with respect to m, and let for some $E \in B(R_1)$ have $0 < m(x_{t_0}(E)) < \infty$. Define a system of functions, $\{\mu_F^E$: F is a finite subset of $T\}$, on $\prod_{t\in T} B(R_1)$ via

$$\mu_F^E\Big(\bigcap_{j=1}^{n} \pi_{t_j}^{-1}(E_j)\Big) = m(x_{t_0}(E) \wedge \bigwedge_{j=1}^{n} x_{t_j}(E_j)) , \tag{2.18}$$

where $E_1,\dots,E_n \in B(R_1)$, $F = \{t_1,\dots,t_n\}$. The system $\{\mu_F^E$: F is a finite subset of $T\}$ fulfils the conditions of Kolmogorov's consistence theorem [23], hence, there is a unique measure μ^E on $\prod_{t\in T} B(R_1)$ with (2.18). Define

$$\mu(B) = \sum_{i=1}^{\infty} \mu^{E_i}(B),$$

where $B \in \prod_{t\in T} B(R_1)$ and $\{E_i\}_{i=1}^{\infty}$ is a measurable partition of R_1 with $0 < m(x_{t_0}(E_i)) < \infty$, $i \geq 1$. The function μ is well defined and it is σ-additive and σ-finite. It is easy to check that (2.17) is fulfilled. The uniqueness of μ follows from the extension theorem for σ-finite measure on the set of all cylindrical sets.

3. Hilbert space logic

One of the most important examples of quantum logics is a set, L(H), of all closed subspaces of a Hilbert space H over the real or complex fields C. This is a case of the great importance in quantum mechanics. In this section we apply the general results on existence of a joint distribution in a measure with infinite values showing that any σ-finite measure on L(H) has a carrier for a separable Hilbert space, dim $H \geq 3$.

The famous Gleason theorem [15] asserts that any state m on a separable Hilbert space H, dim $H \geq 3$, is induced by a positive von Neumann operator T via the formula

$$(3.1) \qquad m(P) = tr(TP), \; P \in L(H).$$

Here we identify the subspace P with its orthoprojector T^P onto P. We recall that a bounded operator T on H is said to be an operator with a finite trace if $tr(T) := \sum_{a \in I} (Tx_a, x_a)$ is absolutely convergent series, independent of the used orthonormal basis $\{x_a : a \in I\}$.

The Gleason theorem has been generalized in [16,17] for all bounded signed measures on L(H) for a separable Hilbert space whose dimension is at least 3. Eilers and Horst [18] proved Gleason's theorem for finite measures on L(H) for a non-separable Hilbert space, and Drisch [19] extended (3.1) for bounded signed measures on a logic L(H) of a non-separable Hilbert space whose dimension is a non-real measurable cardinal.

For measures on L(H) with $m(H) = \infty$ we need the following notions. A bilinear form is a function $t: D(t) \times D(t) \to C$, where D(t) is a linear submanifold of H named the domain of t such that t is linear in the first argument and antilinear in the second one. If $t(x,y) = t(y,x)$ for all $x, y \in D(t)$, then t is said to be symmetric; if for a symmetric bilinear form t we have $t(x,x) \geq 0$, then t is said to be positive. Let t be a symmetric bilinear form and $B \geq 0$ be a self-adjoint operator. Then $t \circ B$ denotes a symmetric bilinear form defined via $t \circ B(x,y) := t(B^{1/2}x, B^{1/2}y)$, when the corresponding assumptions on the domains of t and $B^{1/2}$ are satisfied. Symmetric bilinear form is said to be a bilinear form with a finite trace if (i) $D(t) = H$; (ii) $t(x,y) = (Tx,y)$ for all $x, y \in H$, where T is an operator with finite trace. We put tr t := tr(T), and we write $t \in Tr(H)$, where Tr(H) is the set of all bounded operators with finite trace.

Lugovaja and Sherstnev [20] proved that, for any σ-finite measure m on L(H) of an infinite-dimensional separable Hilbert space there is a unique symmetric bilinear positive form t with a dense domain such that

$$m(P) = \begin{cases} \operatorname{tr} t \circ P & \text{if } t \circ P \in Tr(H), \\ \infty & \text{otherwise.} \end{cases} \tag{3.2}$$

In the paper [21] this result has been extended to σ-finite f-bounded signed measures on L(H) of a Hilbert space whose dimension is a non-real measurable cardinal.

The joint distribution of observables on L(H) in a state has been studied in [3,5]. It was proved that $x_1,\dots,x_n$ have a join distribution in a state m induced by $T \in Tr(H)$ via (3.1) iff

$$x_{i_1}(E_{i_1}) \dots x_{i_n}(E_{i_n})T = x_1(E_1) \dots x_n(E_n)T, \tag{3.3}$$

for any permutation $(i_1,\dots,i_n)$ of $(1,\dots,n)$ and all $E_1,\dots,E_n \in B(R_1)$.

In the following we shall study the existence of a joint distribution for a measure m on L(H) with $m(H) = \infty$, and the condition analogous to (3.3) will be proved. First of all we begin with a finite-dimensional Hilbert space.

L e m m a 3.1. (Lugovaja-Sherstnev [20]). Let $\dim H = 3$ and let m be a measure on L(H) with $m(H) = \infty$. If there are a one-dimensional Q and a two-dimensional P with $m(Q) < \infty$, $m(P) < \infty$, then $Q < P$.

Denote

$$P_m = \bigvee \{P: m(P) < \infty\}. \tag{3.4}$$

The following lemma has been proved in [21].

L e m m a 3.2. Let $3 \le \dim H < \infty$ and let m be a measure with $m(H) = \infty$. If there is a two-dimensional Q_0 with $m(Q_0) < \infty$, then $m(Q) < \infty$ iff $Q \le P_m$.

L e m m a 3.3. Let $4 \le \dim H < \infty$ and let m be a measure with $m(H) = \infty$. Let there be a three-dimensional Q_0 with $m(Q_0) < \infty$. If $m(M) = m(N) = 0$, then $m(M \vee N) = 0$ (the Jauch--Piron property).

P r o o f . Due to Lemma 3.2, $m(Q)<\infty$ iff $Q\leq P_m$. Hence, $m(N\vee N)<\infty$. Applying the Gleason theorem to $m_o := m|L_{(0,P_m)} = m|L(L_m)$ we see that $m(M\vee N) = 0$.

L e m m a 3.4. Let the conditions of Lemma 3.3 are fulfilled. Then any measure m on $L(H)$ has a carrier.

P r o o f . Let us denote $\mathcal{M} = \{P: m(P) = 0\}$. It is clear that (i) $\mathcal{M} \neq \emptyset$; (ii) if $Q\leq P$, $P\in\mathcal{M}$, then $Q\in\mathcal{M}$; (iii) if $P\perp Q$ and $P,Q\in\mathcal{M}$, then $P\vee Q\in\mathcal{M}$; (iv) if P_x and $P_y\in\mathcal{M}$, then $P_x\vee P_y\in\mathcal{M}$, where P_x denotes the one-dimensional subspace generated by a non-zero vector $x\in H$. Let us put $P_m^o = \bigvee\{P: m(P)=0\}$. Then from Lemma 3.3 and (i)-(iv) we have that $m(P_m^o) = 0$. Define $A_m = P_m^{o\perp}$. Then A_m is a carrier of a measure m. Q.E.D.

We recall that a subset $\mathcal{M}\subset L(H)$ with (i)-(iv), from the last proof, is said to be an ideal.

T h e o r e m 3.5. Let the conditions of Lemma 3.3 be fulfilled. If, for $x_1,\dots,x_n$, we have

$$(3.5)\qquad x_{i_1}(E_{i_1}) \dots x_{i_n}(E_{i_n})A_m = x_1(E_1) \dots x_n(E_n)A_m,$$

for any permutation $(i_1,\dots,i_n)$ of $(1,\dots,n)$ and any $E_1,\dots,E_n\in B(R_1)$, where A_m is a carrier of a measure m, then $x_1,\dots,x_n$ have a joint distribution in m. Moreover, the condition (3.5) is equivalent to

$$(3.6)\qquad A_{x_{i_1}} \dots A_{x_{i_n}} A_m = A_{x_1} \dots A_{x_n} A_m,$$

for any permutation $(i_1,\dots,i_n)$ of $(1,\dots,n)$, where A_x is an Hermitean operator corresponding to an observable x.

P r o o f . It is known [22] that (3.5) implies $(x_1(E_1)\wedge\dots\wedge x_n(E_n))A_m = x_1(E_1)\dots x_n(E_n)A_m$. Hence

$$a(E_1,\dots,E_n)A_m = \sum_{i_1\dots i_n=0}^{1} x_1({}^{i_1}E_1)\dots x_n({}^{i_n}E_n)A_m =$$

$$= I\,A_m = A_m,$$

where I is the identical operator on H. Therefore $a(E_1,\dots,E_n) \geq \geq A_m$, for all $E_1,\dots,E_n$, consequently, $A_0 > A_m$, where A_0 is the commutator of $x_1,\dots,x_n$ and $m(A_0^{\perp}) = 0$. Repeating the first part of the proof of Theorem 2.5 we finish the proof.

We see that measures with $m(H) = \infty$ on a finite-dimensional Hilbert space are in some sense "pathological". More useful information we may obtain in an infinite-dimensional separable Hilbert space.

L e m m a 3.6. Any σ-finite measure on L(H) of an infinite-dimensional separable Hilbert space has a carrier. Moreover, if $m(M_a) = 0$ for any $a \in A$, then $m\left(\bigvee_{a\in A} M_a\right) = 0$.

P r o o f . If $m(H) < \infty$, then the assertion follows immediately from Gleason's theorem.

Let now $m(H) = \infty$. Define $\mathcal{M} = \{P: m(P) = 0\}$. We claim to show that $\mathcal{M}$ is an ideal of L(H). For that it is necessary to show that if $P_x, P_y \in \mathcal{M}$, then $P_x \vee P_y \in \mathcal{M}$. We may limit ourselves with $P_x \perp P_y$, $P_x \neq P_y$. The σ-finiteness of m entails that there is at least one three-dimensional P such that $m(P) < \infty$ and $Px \neq 0$, $Py \neq 0$. Then there is $z \in P$ such that $z \perp x$ and $z \perp y$. Applying the Lugovaja-Sherstnev lemma to a three-dimensional space $Q := P_z \vee P_x \vee P_y$ we have that $m(P_x \vee P_y) < \infty$; if not, then $m(Q) = \infty$ and $P_x \leq P_z \oplus P_y$. Using the Gleason theorem for a finite measure $m_0 = m|L(Q)$ we have $m(P_x \vee P_y) = 0$.

Now we show that if $P_{y_1},\dots,P_{y_n} \in \mathcal{M}$, then $P := P_{y_1} \vee \dots \vee P_{y_n} \in \mathcal{M}$. Lemma 3.2 implies that $m(P) < \infty$ and Lemma 3.3 entails that $m(P) = 0$.

Define the submanifold D generated by the ideal $\mathcal{M}$ via $D = \{x : P_x \in \mathcal{M}\} \cup \{0\}$ and let M be a subspace of H generated by D. Then $M = \bigvee\{P: m(P) = 0, \dim P < \infty\}$. The separability of a Hilbert space implies that there is a sequence of finite-dimensional subspaces of H, $\{P_n\}_{n=1}^{\infty}$, with $m(P_n) = 0$, such that $M = \bigvee_{n=1}^{\infty} P_n$. $\{P_n\}_{n=1}^{\infty}$ may be choosen such that $P_1 < P_2 < \dots$.

The continuity of m from below entails $m(M) = 0$. The element $A_m = M^{\perp}$ is a carrier of a measure m. The last assertion is now evident.

N o t e 2. The author does not know whether Lemma 3.6 holds for a non-separable Hilbert space whose dimension is a non-real measurable cardinal. For that it is necessary and sufficient to show that $m(M)< \infty$. For more details, see the proof of Lemma 3.9.

The following elementary Lemma has been proved in [5].

L e m m a 3.7. Let $M_1,\dots,M_n \in L(H)$, where H is an arbitrary Hilbert space. Let $(i_1,\dots,i_n)$ be any permutation of $(1,\dots,n)$. If $0 \neq f \in {}^{i_1}M_1 \wedge \dots \wedge {}^{i_n}M_n$, where ${}^0M := M^{\perp}$, ${}^1M := M$, then

$$(3.7) \qquad M_{j_1} \dots M_{j_n} f = M_1 \dots M_n f,$$

for any permutation $(j_1,\dots,j_n)$ of $(1,\dots,n)$.

T h e o r e m 3.8. Let H be an infinite-dimensional separable Hilbert space. If $x_1,\dots,x_n$ have a joint distribution in m and at least one observable is σ-finite with respect to m, then (3.5) holds. If, additionally, $x_1,\dots,x_n$ are bounded observables, then (3.6) holds.

If m is σ-finite and, for $x_1,\dots,x_n$ there holds (3.5), then $x_1,\dots,x_n$ have a joint distribution in m. If at least one observable is σ-finite with respect to m and (3.5) holds, then the joint distribution is unique.

P r o o f . Since at least one observable is σ-finite with respect to m, we see that m is σ-finite measure, consequently, the carrier of m exists. Due to Lemma 2.6.

$$A_m \leq A_0 \leq a(E_1,\dots,E_n),$$

where A_0 is the commutator of $x_1,\dots,x_n$ defined by (2.4). Therefore if $f \in A_m$, then $f \in a(E_1,\dots,E_n)$ and f is a finite linear combination of vectors from $x_1\left({}^{j_1}E_1\right) \wedge \dots \wedge x_n\left({}^{j_n}E_n\right)$ for some $j_1,\dots,j_n = 0, 1$. Due to Lemma 3.7,

$$x_{i_1}(E_{i_1}) \dots x_{i_n}(E_{i_n})f = x_1(E_1) \dots x_n(E_n)f,$$

for any permutation of $(i_1,\dots,i_n)$ of $(1,\dots,n)$, and, consequently, (3.5) holds.

For bounded observables, (3.6) is a consequence of the spectral theorem for Hermitean operators.

The second part of the proof is analogous to the proof of Theorem 3.5.

N o t e 3. Theorems 3.5 and 3.8 have been proved in [3,5] for states using the consequence of the Gleason theorem that any state is a mixture of pure states. For measures with infinite values this is not true, in general. In our proof we use the new approach: the existence of carriers for σ-finite measures.

In the following the previous Theorem will be extended to a non-separable Hilbert space. We recall that a cardinal I is said to be non-real measurable if there is no positive measure ν, $\nu \neq 0$ on the power set of I with $\nu(\{a\}) = 0$ for each $a \in I$.

P r o p o s i t i o n 3.9. Let H be a Hilbert space whose dimension is a non-real measurable cardinal. Let m be a measure on L(H) with $m(H) = \infty$. Let us put $A^{\perp} = \vee\{P: m(P)=0\}$. If at least one observable is σ-finite with respect to m and $x_1,\dots,x_n$ have a joint distribution in m, then

$$x_{i_1}(E_{i_1}) \dots x_{i_n}(E_{i_n})A = x_1(E_1)\dots x_n(E_n)A, \tag{3.8}$$

for any permutation $(i_1,\dots,i_n)$ of $(1,\dots,n)$ and all $E_1,\dots,E_n \in B(R_1)$.

If $m(A^{\perp}) < \infty$, m is σ-finite, and (3.8) holds, then $x_1,\dots,x_n$ have a joint distribution in m. If at least one observable is σ-finite with respect to m, then the joint distribution is unique.

P r o o f . The first part of the proposition is similar to that in Theorem 3.8.

In the second part we show that $m(A^{\perp})<\infty$ implies $m(A^{\perp})=0$, that is, A will be a carrier of m. The generalized Gleason theorem for a non-separable Hilbert space [21] entails that there is a unique operator $T\in Tr(H)$ such that $m(P) = tr(TP)$ whenever $P<A^{\perp}$. The operator T has a form $T = \sum_i \lambda_i f_i \otimes \bar{f}_i$, where $f_i \perp f_j$, if $i \neq j$, $\|f_i\| = 1$, $f_i \in H$, $\lambda_i > 0$, for any i, $f \otimes \bar{\bar{f}}: x \longmapsto (x,f)f$, $x\in H$. Hence $m(P) = 0$ iff $P\perp f_i$ for any i (here $P\perp f_i$ denotes that $x\perp f_i$, for all $x\in P$). Hence, $A^{\perp}\perp f_i$, for any i, so that, $m(A^{\perp}) = 0$. For the rest of the proof we apply Lemma 2.6.

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