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# SOME VARIETIES OF ALGEBRAS DEFINED BY EXTERNALLY COMPATIBLE IDENTITIES

## 1. Introduction

Let $\underline{V}$ be a variety of F-algebras. Let $w$ and $v$ be F-words. An identity

$$w = v$$

is called externally compatible if $w$ and $v$ are the same variable or

$$w = w_1 \ldots w_n f$$

and

$$v = v_1 \ldots v_n f$$

for some F-words $w_1, \ldots, w_n$, $v_1, \ldots, v_n$ and n-ary operator $f$ in F. (See [3]).

For a variety $\underline{V}$ of F-algebras, where each $f$ in F is at least unary, let $\bar{\underline{V}}$ denote the variety of the same type defined by all externaly compatible identities satisfied in $\underline{V}$. Chromik [3] has given a representation theorem for algebras in $\bar{\underline{V}}$ in the case that all fundamental operations of algebras in $\underline{V}$ are indempotent, and has shown how to construct a basis for the identities satisfied in $\bar{\underline{V}}$ when a basis for $\underline{V}$ is given. To formulate Chromik's theorem in a form more suitable to our needs, let us recall two definitions.

Let $\underline{W}$ be the class of all F-algebras and $\underline{U}$ and $\underline{V}$ be subvarieties of $\underline{W}$. Let $(A,F)$ be an algebra belonging to $\underline{W}$.

A congruence relation $\varphi$ on $(A,F)$ is said to be a $\underline{U}\cdot\underline{V}$-congruence relation if $(A,F)/\varphi$ belongs to the variety $\underline{V}$ and every $\varphi$-class that is a subalgebra of $(A,F)$ belongs to the variety $\underline{U}$. The class of all algebras in $\underline{W}$ having a $\underline{U}\cdot\underline{V}$-congruence relation is called the product class $\underline{U}\cdot\underline{V}$ of $\underline{U}$ and $\underline{V}$. (See [7]).

An algebra $(A,F)$ is called a constant algebra if for each n-ary $f$ in $F$ the identity

$$x_1\ldots x_n f = y_1\ldots y_n f$$

is satisfied in $(A,F)$. In this case we denote

$$0_f(A) := x_1\ldots x_n f.$$

Let $\underline{C}(F)$ be the variety of all constant F-algebras.

1.1. Theorem [3]. Let $\underline{V}$ be a variety of F-algebras. Suppose that for each n-ary $f$ in F, the identity

$$x\ldots xf = x$$

holds in $\underline{V}$. An algebra $(A,F)$ belongs to the variety $\overline{\underline{V}}$ if and only if there is a $\underline{C}(F)\cdot\underline{V}$-congruence relation $\varphi$ on $(A,F)$ such that the $\varphi$-classes are subalgebras and for each n-ary $f$ in F and $a_1,\ldots,a_n$ in A

$$a_1\ldots a_n f = 0_f(a_1\ldots a_n f\varphi). \ \square$$

In this note we give a representation theorem of a similar type for algebras in a variety $\overline{\underline{V}}$ in case $\underline{V}$ satisfies more general conditions than those of Chromik. We show as well that Chromik's method of finding a basis for the identities satisfied in $\overline{\underline{V}}$ from a basis for $\underline{V}$ can be applied in this case, too. Some examples illustrating our results are then given.

For basic algebraic notions we refer the reader to [4], [9]. The notation is similar to that used in [4] and [9]. An F-word in variables $x_1,\ldots,x_n$ is denoted by $x_1\ldots x_n w$ and the corresponding derived operation $\overline{w}$ when applied to elements $a_1,\ldots,a_n$ by $a_1\ldots a_n\overline{w}$. If $w = x_1\ldots x_n w$ and $w = w_1\ldots w_k f$ for some F-words $w_1,\ldots,w_k$ and k-ary $f$ in F, then we write

$x_1 \dots x_n w_1 \dots w_k f$, and we denote the result of the derived operation $\overline{w}$ when applied to elements $a_1, \dots, a_n$ by $a_1 \dots a_n w_1 \dots w_k f$. Note that each externally compatible identity satisfied in $\underline{V}$ has the form

$$x_1 \dots x_i y_1 \dots y_j w_1 \dots w_n f = x_1 \dots x_i z_1 \dots z_k v_1 \dots v_n f$$

for some n-ary $f$ in F and F-words $w_1, \dots, w_n$, $v_1, \dots, v_n$, where one of the sets $\{x_1, \dots, x_i\}$, $\{y_1, \dots, y_j\}$ and $\{z_1, \dots, z_k\}$, or both the latter sets, may be empty.

## 2. Main results

In this section only algebras without nullary fundamental operations are considered. This is not a serious restriction, since instead of a constant operation $c$, we can always introduce a unary operation $c'$ satisfying the identity $c'(x) = c'(y)$, without essential changes in the structure of the algebras in question. (See [11]).

To formulate the main theorem, let us introduce some definitions.

Define the following sequence of F-words

$$x_1 \dots x_n u_1 := x_1 \dots x_n f,$$

$$x_1 \dots x_n u_2 := x_1 \dots x_n f \dots x_1 \dots x_n f\ f = \\ = x_1 \dots x_n u_1 \dots x_1 \dots x_n u_1\ f$$

and

$$x_1 \dots x_n u_{i+1} := x_1 \dots x_n u_i \dots x_1 \dots x_n u_i\ f.$$

Note that for $i \geq 3$

(2.1) $$x_1 \dots x_n u_i = x_1 \dots x_n u_2 \dots x_1 \dots x_n u_2\ u_{i-2}.$$

Now let $\underline{V}$ be a variety of F-algebras and $\tilde{\underline{V}}$ the class of all F-algebras (A,F) satisfying the following condition:

(2.2) there is a congruence relation $\varphi$ on (A,F) such that

(i) $(A,F)/\varphi$ belongs to the variety $\underline{V}$,

(ii) for each n-ary f in F and $a, a_1,\dots,a_n$, $b_1,\dots,b_n$ in A, if $a_1\dots a_n f$ and $b_1\dots b_n f$ are in $a^\varphi$, then

$$b_1\dots b_n f = a_1\dots a_n f.$$

Let us denote

(2.3) $$O_f(a\varphi) := a_1\dots a_n f.$$

2.4. Theorem. Let $\underline{V}$ be a variety of F-algebras satisfying the identities

(a) $$x_1\dots x_n f = x_1\dots x_1 u \;\dots\; x_n\dots x_n u\; f$$

and

$(b_i)$ $$x_1\dots x_n f = x_1\dots x_n u_i \;\dots\; x_1\dots x_n u_i\; f$$

for some natural number $i \geq 2$, each n-ary f in F, and some F-word u that is not just a variable.

Let B be a basis for the identities satisfied in $\underline{V}$. Let $\bar{B}$ consist of the identities (a), $(b_i)$, all externally compatible identities of B and all the identities

$$w\dots wu = v\dots vu$$

for each not externally compatible identity w = v of B.

Then the following three classes of F-algebras coincide:

(i) the variety $\underline{W}$ defined by $\bar{B}$,

(ii) the class $\tilde{\underline{V}}$,

(iii) the variety $\bar{\underline{V}}$.

Proof. A. $\underline{W} \subseteq \tilde{\underline{V}}$. Let (A,F) be in the variety $\underline{W}$. It is easy to see that the relation $\varphi$ defined on A by

$$a\; \varphi\; b \text{ if and only if } a\dots a\bar{u} = b\dots b\bar{u}$$

is an equivalence relation. Since by (a) for n-ary f in F and $a_i\; \varphi\; b_i$, $i = 1,\dots,n$, we have

$(a_1 \ldots a_n f) \ldots (a_1 \ldots a_n f)\bar{u} =$

$= ((a_1 \ldots a_1 \bar{u}) \ldots (a_n \ldots a_n \bar{u}) f) \ldots ((a_1 \ldots a_1 \bar{u}) \ldots (a_n \ldots a_n \bar{u}) f)\bar{u} =$

$= ((b_1 \ldots b_1 \bar{u}) \ldots (b_n \ldots b_n \bar{u}) f) \ldots ((b_1 \ldots b_1 \bar{u}) \ldots (b_n \ldots b_n \bar{u}) f)\bar{u} =$

$= (b_1 \ldots b_n f) \ldots (b_1 \ldots b_n f)\bar{u},$

it follows that $\varphi$ is a congruence relation.

Let us note that substituting $x_1 \ldots x_n f$ for each $x_i$, $i = 1, \ldots, n$, in (a) we get the following identity:

(c)
$$x_1 \ldots x_n u_2 = (x_1 \ldots x_n f \ \ldots \ x_1 \ldots x_n f \ u) \ldots (x_1 \ldots x_n f \ \ldots \ x_1 \ldots x_n f \ u) f.$$

Now if $a, a_1, \ldots, a_n, b_1, \ldots, b_n$ are in $A$ and $a_1 \ldots a_n f \ \varphi \ b_1 \ldots b_n f$, then by $(b_i)$, (2.1) and (c),

$$\begin{aligned} a_1 \ldots a_n f &= a_1 \ldots a_n u_i \ \ldots \ a_1 \ldots a_n u_i \ f \\ &= (a_1 \ldots a_n u_2 \ \ldots \ a_1 \ldots a_n u_2 \ u_{i-2}) \ldots \\ &\quad (a_1 \ldots a_n u_2 \ \ldots \ a_1 \ldots a_n u_2 \ u_{i-2}) f \\ &= (((a_1 \ldots a_n f \ \ldots \ a_1 \ldots a_n f \ u) \ldots \\ &\quad (a_1 \ldots a_n f \ \ldots \ a_1 \ldots a_n f \ u) f) \ldots \\ &\quad ((a_1 \ldots a_n f \ \ldots \ a_1 \ldots a_n f \ u) \ldots \\ &\quad (a_1 \ldots a_n f \ \ldots \ a_1 \ldots a_n f \ u) f) u_{i-2} \ldots \\ &\quad ((a_1 \ldots a_n f \ \ldots \ a_1 \ldots a_n f \ u) \ldots \\ &\quad (a_1 \ldots a_n f) \ \ldots \ a_1 \ldots a_n f \ u) f) \ldots \\ &\quad ((a_1 \ldots a_n f \ \ldots \ a_1 \ldots a_n f \ u) \ldots \\ &\quad (a_1 \ldots a_n f \ \ldots \ a_1 \ldots a_n f \ u) f) u_{i-2}) f \end{aligned}$$

$$= (((b_1 \ldots b_n f \ \ldots \ b_1 \ldots b_n f \ u) \ \ldots$$

$$(b_1 \ldots b_n f \ \ldots \ b_1 \ldots b_n f \ u) f) \ \ldots$$

$$((b_1 \ldots b_n f \ \ldots \ b_1 \ldots b_n f \ u) \ \ldots$$

$$(b_1 \ldots b_n f \ \ldots \ b_1 \ldots b_n f \ u) f) u_{i-2} \ \ldots$$

$$((b_1 \ldots b_n f \ \ldots \ b_1 \ldots b_n f \ u) \ \ldots$$

$$(b_1 \ldots b_n f \ \ldots \ b_1 \ldots b_n f \ u) f) \ \ldots$$

$$((b_1 \ldots b_n f \ \ldots \ b_1 \ldots b_n f \ u) \ \ldots$$

$$(b_1 \ldots b_n f \ \ldots \ b_1 \ldots b_n f \ u) f) u_{i-2}) f$$

$$= (b_1 \ldots b_n u_2 \ \ldots \ b_1 \ldots b_n u_2 \ u_{i-2}) \ \ldots$$

$$(b_1 \ldots b_n u_2 \ \ldots \ b_1 \ldots b_n u_2 \ u_{i-2}) f$$

$$= b_1 \ldots b_n u_i \ \ldots \ b_1 \ldots b_n u_i \ f$$

$$= b_1 \ldots b_n f$$

It follows that the condition (2.2ii) is satisfied.

Now if the identity

$$x_1 \ldots x_i y_1 \ldots y_j w = x_1 \ldots x_i z_1 \ldots z_k v$$

belongs to B then the identity

$$w \ldots w u = v \ldots v u$$

holds in $\underline{W}$. Hence for $a_1, \ldots, a_i$, $b_1, \ldots, b_j$, $c_1, \ldots, c_k$ in A,

$$a_1 \ldots a_i b_1 \ldots b_j \bar{w} \ \varphi \ a_1 \ldots a_i c_1 \ldots c_k \bar{v}$$

which implies that $(A,F)/\varphi$ belongs to $\underline{V}$. Consequently (A,F) is in the class $\tilde{\underline{V}}$.

B. $\tilde{\underline{V}} \subseteq \bar{\underline{V}}$. At first we show that for an arbitrary F-word $x_1 \ldots x_k w$ that is not just a variable, and for $a_1, \ldots, a_k$ in A, (2.5) $$a_1 \dots a_k \overline{w} = O_f(a_1 \dots a_k \overline{w}^{\varphi}),$$

where $w = w_1 \dots w_n f$. If $w = x_1 \dots x_n f$ for some $f$ in F, then (2.5) coincides with (2.3). Now suppose that

$$w = x_{11} \dots x_{1i_1} w_1 \;\dots\; x_{n1} \dots x_{ni_n} w_n \; f$$

for some n-ary $f$ in F and F-words $w_1, \dots, w_n$. Then for $a_{11}, \dots, a_{1i_1}, \dots, a_{n1}, \dots, a_{ni_n}$ in A, (2.3) implies that

$$\begin{aligned} a_{11} \dots a_{ni_n} \overline{w} &= a_{11} \dots a_{1i_1} \overline{w}_1 \;\dots\; a_{n1} \dots a_{ni_n} \overline{w}_n \; f = \\ &= O_f(a_{11} \dots a_{1i_1} \overline{w}_1 \;\dots\; a_{n1} \dots a_{ni_n} \overline{w}_n \; f^{\varphi}) = \\ &= O_f(a_{11} \dots a_{ni_n} \overline{w}^{\varphi}). \end{aligned}$$

**Next let**

(2.6) $$x_1 \dots x_i y_1 \dots y_j w_1 \dots w_n f = x_1 \dots x_i z_1 \dots z_k v_1 \dots v_n f$$

be an arbitrary identity holding in $\overline{\underline{V}}$. Then the first part of the proof and the fact that $(A,F)/\varphi$ is in $\underline{V}$ imply that for

$$a_1, \dots, a_i, \; b_1, \dots, b_j, \; c_1, \dots, c_k \quad \text{in} \quad A,$$

$$a_1 \dots a_i b_1 \dots b_j \overline{w_1 \dots w_n f} = O_f(a_1 \dots a_i b_1 \dots b_j \overline{w_1 \dots w_n f}^{\varphi}) =$$

$$= O_f(a_1^{\varphi} \dots a_i^{\varphi} b_1^{\varphi} \dots b_j^{\varphi} \overline{w_1 \dots w_n f}) = O_f(a_1^{\varphi} \dots a_i^{\varphi} c_1^{\varphi} \dots c_k^{\varphi} \overline{v_1 \dots v_n f}) =$$

$$= O_f(a_1 \dots a_i c_1 \dots c_k \overline{v_1 \dots v_n f}^{\varphi}) = a_1 \dots a_i c_1 \dots c_k \overline{v_1 \dots v_n f}.$$

Consequently, (2.6) is satisfied in (A,F).

C. $\overline{\underline{V}} \subseteq \underline{W}$. Obviously all the identities of $\overline{B}$ and hence all the identities satisfied in $\underline{W}$ are externally compatible. Hence $\overline{\underline{V}} \subseteq \underline{W}$. □

Let us note that if $a^{\varphi}$ is a $\varphi$-class of an algebra (A,F) in $\overline{\underline{V}}$, the elements $a_1, \dots, a_n$, $b_1, \dots, b_n$ are in $a^{\varphi}$ and $a^{\varphi}$ is a subalgebra of (A,F), then

$$a_1 \ldots a_n f = b_1 \ldots b_n f.$$

It follows that the algebras in $\overline{\underline{V}}$ may be characterized as members of the product class $\underline{C}(F) \cdot \underline{V}$ satisfying the condition (2.2.ii).

2.7. Corollary. Let $\underline{V}$ be a variety of F-algebras satisfying the assumptions of Theorem 2.4 and let B be a basis for the identities satisfied in $\underline{V}$. Then the identities of $\overline{B}$ form a basis for the identities satisfied in $\overline{\underline{V}}$. In particular, if $\underline{V}$ has a finite basis for its identities, then so does the variety $\overline{\underline{V}}$. □

2.8. Corollary. The characterization given in Theorem 2.4 holds for varieties $\underline{V}$ satisfying one of the following conditions:

(i) For each n-ary f in F the identity

$$x \ldots xf = x$$

holds in $\underline{V}$.

(ii) There are unary and non-unary operators in F. For each n-ary f in F with $n \geq 2$,

$$x \ldots xf = x;$$

and for each unary f in F,

$$xf = xfff$$

hold in $\underline{V}$.

(iii) For each n-ary f in F the identity

$$x \ldots xf \ \ldots \ x \ldots xf \ f = x \tag{2.8}$$

holds in $\underline{V}$.

(iv) There are unary and non-unary operators in F. For each n-ary f in F with $n \geq 2$,

$$x \ldots xf = x;$$

and for each unary f in F

$$xff = x$$

hold in $\underline{V}$. □

Case (i) gives Chromik's Theorem 1.1. Case (iv) was also considered by Chromik [3].

3. Examples

One may apply Theorem 2.4 to all varieties $\underline{V}$ of algebras with idempotent fundamental operations, for example lattices, (F-) semilattices (in particular bisemilattices), modes and modals as defined in [9], and bands (idempotent semigroups). Let us give one more detailed example here. In the case that $\underline{V}$ is the variety $\underline{S\ell}$ of all semilattices, the variety $\overline{\underline{S\ell}}$ consists of commutative semigroups satisfying the identity $x\cdot y = (x\cdot y)\cdot(x\cdot y)$ and Theorem 2.4 gives a structural characterisation of such semi-groups. More generally, Theorem 2.4 can be applied to any variety of semigroups satisfying the identities

(a′) $$x\cdot y = x^k\cdot y^k;$$

($b_i'$) $$x\cdot y = (x\cdot y)^{2^i}$$

for some natural numbers i, k and any not externally compatible identity that in the case of semigroups has the form

$$x_1\ldots x_n w = x.$$

If we consider groupoids instead of semigroups, the identities (a) and ($b_i$) take the form

(a″) $$x\cdot y = (x\ldots xu)\cdot(y\ldots yu),$$

($b_i''$) $$x\cdot y = (xyu_i)\cdot(xyu_i),$$

and each externally compatible identity has the form $x_1\ldots x_n w = x$.

Finally, according to Corollary 2.8 Theorem 2.4 may be applied to all varieties of algebras with idempotent n-ary F-operations with $n\geq 2$ and nonidempotent unary operations as for example Boolean algebras, p-algebras, double p-algebras, de Morgan algebras (as defined in [1]), then to pM-algebras [8], to ortholattices [6] and more general polarity lattices [10]

and to closure algebras [2]. (In all these algebras with lattice structure, the F-word u is taken to be x + y).
The case of distributive p-algebras was considered independently in [5]. Theorem 2.4 do not apply to algebras having a group structure. For varieties V of such algebras the problem of characterising algebras in $\bar{V}$ remains open.

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