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BIREGULAR AND UNIFORM IDENTITIES OF BISEMILATTICES

In this paper we study varieties of algebras with two binary operations + and • defined by some special types of identities introduced in [11] under the names of uniform, biregular, S-uniform and S-biregular. (See section 0 for the definitions).

In section 1 we give a representation theorem for algebras in the variety defined by all uniform identities with two operation symbols + and • and we find a (finite) equational base of it.

Section 2 is devoted to varieties defined by all uniform (all biregular, all + - uniform) identities satisfied in a given variety of bisemilattices. For these varieties we obtain similar and related results.

0. Preliminaries

Let $\tau : F \longrightarrow N$ be a type of algebras i.e. F is a set of fundamental operation symbols and N is the set of non-negative integers. If φ is a term of type τ then we denote by var(φ) the set of variables occuring in φ and by F(φ) the set of fundamental operation symbols occuring in φ. An identity $\varphi = \psi$ is called regular if

$$\mathrm{var}(\varphi) = \mathrm{var}(\psi) \quad \text{(see [9])}.$$

Such identities were considered e.g. in [4-10]. The following types of identities were defined in [11].

Definition 1. An identity $\varphi = \psi$ of type τ is called biregular if it is regular and $F(\varphi) = F(\psi)$.

Definition 2. An identity $\varphi = \psi$ of type τ is called uniform if $F(\varphi) = F(\psi)$ and if $F(\varphi) \neq F$ then $\varphi = \psi$ is regular.

Sometimes we write $\varphi =_U \psi$ to stress that the identity $\varphi = \psi$ is uniform.

Now let $S \subseteq F$.

Definition 3. An identity $\varphi = \psi$ of type τ is called S-biregular if it is regular and $F(\varphi) \cap S = F(\psi) \cap S$.

Definition 4. An identity $\varphi = \psi$ of type τ is called S-uniform if $F(\varphi) \cap S = F(\psi) \cap S$ and if $F(\varphi) \cap S \neq S$ then it is regular.

In [11] varieties of algebras defined by the identities described in Definitions 1 - 4 were investigated.

If E is a set of identities of type τ we denote by V(E) the variety of type τ defined by E and we denote by C(E) the set of all identities of type τ provable from E by Birkhoff's deriviation rules. For a variety K of type τ we denote by E(K) the set of all identities satisfied in K.

We say that a property p of identities is preserved by consequences if p satisfies the following condition:

if E is a set of identities having the property p then every identity from C(E) has this property, too.

As it was noticed in [11]:

(i) each of the properties "to be regular", "to be biregular", "to be uniform", "to be S-biregular", "to be S-uniform" is preserved by consequences.

Let us denote by R(K), B(K), U(K), $B_S(K)$, $U_S(K)$ the sets of all regular, biregular, uniform, S-biregular, S-uniform identities satisfied in K, respectively. It follows from (i) that every of the sets of identities mentioned above is an equational theory (see [14]).

If φ is a term of type τ and $\mathrm{var}(\varphi) = \{x_{i_1}, \ldots, x_{i_n}\}$ then we sometimes write $\varphi(x_{i_1}, \ldots, x_{i_n})$ instead of φ.

For two varieties K_1 and K_2 of the same type we denote by $K_1 \vee K_2$ the join of K_1 and K_2. For varieties $K_1,\dots,K_n$ of the same type we denote by $K_1 \otimes \dots \otimes K_n$ the class of all algebras isomorphic to a subdirect product of algebras $\mathfrak{A}_1,\dots,\mathfrak{A}_n$ where $\mathfrak{A}_i \in K_i$, $i = 1,\dots,n$.

1. Uniform identities in algebras with two binary operations

From now on we consider only algebras of type $\tau_0 : F_0 \to N$ where $F_0 = \{+, \cdot\}$, $\tau_0(+) = \tau_0(\cdot) = 2$. We denote by $U(\tau_0)$ the set of all uniform identities of type τ_0.

Let U^* be the set of the following identities:

(1.1) $$x_1 + x_2 = x_2 + x_1,$$

(1.2) $$x_1 \cdot x_2 = x_2 \cdot x_1,$$

(1.3) $$(x_1 + x_2) + x_3 = x_1 + (x_2 + x_3),$$

(1.4) $$(x_1 \cdot x_2) \cdot x_3 = x_1 \cdot (x_2 \cdot x_3),$$

(1.5) $$x_1 + x_1 + x_2 = x_1 + x_2,$$

(1.6) $$x_1 \cdot x_1 \cdot x_2 = x_1 \cdot x_2,$$

(1.7) $$(x_1 + x_2) \cdot x_3 = (x_4 \cdot x_5) + x_6.$$

The following theorem explains the role of the identities U^* for the variety $V(U(\tau_0))$.

T h e o r e m 1. The set U^* is an equational base for the variety $V(U(\tau_0))$.

For to prove Theorem 1 we need some lemmas.

L e m m a 1.

(d_1) If $\varphi(x_i)$ is a term of type τ_0 and $F(\varphi(x_i)) = \{+\}$ then the identity $\varphi(x_i) = x_i + x_i$ belongs to $C(U^*)$.

(d_2) If $\varphi(x_i)$ is a term of type τ_0 and $F(\varphi(x_i)) = \{\cdot\}$ then the identity $\varphi(x_i) = x_i \cdot x_i$ belongs to $C(U^*)$.

(d_3) If $\varphi(x_{i_1},\dots,x_{i_n})$ is a term of type τ_0 such that $n > 1$ and $F(\varphi(x_{i_1},\dots,x_{i_n})) = \{+\}$ then the identity

$\varphi(x_{i_1},\dots,x_{i_n}) = x_{j_1} + x_{j_2} + \dots + x_{j_n}$ belongs to $C(U^*)$ where $\{i_1,\dots,i_n\} = \{j_1,\dots,j_n\}$ and $j_1 < j_2 < \dots < j_n$.

(d_4) If $\varphi(x_{i_1},\dots,x_{i_n})$ is a term of type τ_0 such that $n>1$ and $F(\varphi(x_{i_1},\dots,x_{i_n})) = \{\cdot\}$ then the identity $\varphi(x_{i_1},\dots,x_{i_n}) = x_{j_1}\cdot x_{j_2}\cdot \dots \cdot x_{j_n}$ belongs to $C(U^*)$ where $\{i_1,\dots,i_n\} = \{j_1,\dots,j_n\}$ and $j_1 < j_2 < \dots < j_n$.

P r o o f . It is obvious and follows directly from the identities (1.1)-(1.6).

In the sequel we write $x_i|\psi$, whenever we substitute a term ψ for a variable x_i in a term φ.

L e m m a 2. If φ is a term of type τ_0 and $F(\varphi) = \{+,\cdot\}$ then the identity $\varphi = x_1\cdot x_2 + x_3$ belongs to $C(U^*)$.

P r o o f . First we prove that the following two identities belong to $C(U^*)$

$$x_1\cdot x_2 + x_3 = x_4\cdot x_5 + x_6, \tag{1.8}$$

$$(x_1+x_2)\cdot x_3 = x_1\cdot x_3 + x_2\cdot x_3. \tag{1.9}$$

Indeed we get (1.8) by the following substitutions $x_4|x_1$, $x_5|x_2$, $x_6|x_3$ in (1.7) and by using (1.7) again. Similarly, we get (1.9) substituting $x_4|x_1$, $x_5|x_3$, $x_6|x_2\cdot x_3$ in (1.7).

Now using the distributivity (1.9) we infer that one of the following identities (1.10) and (1.11) belongs to $C(U^*)$

$$\varphi = p_1 + p_2 + \dots + p_r, \tag{1.10}$$

where $r>1$ and any p_i is a product of at least two variables,

$$\varphi = p_1 + \dots + p_s + x_{i_1} + \dots + x_{i_t}, \tag{1.11}$$

where $s,t \geq 1$ and any p_i is a product of at least two variables.

Assume that $p_1 = x_k\cdot q$, where q is a variable or product of variables and substitute $x_4|x_k$, $x_5|q$, $x_6|p_2 + \dots + p_r$ in (1.8). Then (1.10) implies that the identity $p_1 + p_2 + \dots + p_r = x_1\cdot x_2 + x_3$ belongs to $C(U^*)$ and we get the statement of our lemma. Analogously substituting $x_4|x_k$, $x_5|q$, $x_6|x_{i_1} + x_{i_2} + \dots + x_{i_t} + p_2 + \dots + p_s$ in (1.8) and using (1.11) we get the same statement. Q.E.D.

P r o o f of Theorem 1. We have to prove that $C(U^*) = U(\tau_0)$. By (i) we have $C(U(\tau_0)) = U(\tau_0)$. Since each of the identities (1.1) - (1.7) is uniform it follows that $U^* \subseteq U(\tau_0)$, and consequently $C(U^*) \subseteq C(U(\tau_0)) = U(\tau_0)$. Now we prove that $U(\tau_0) \subseteq C(U^*)$. Let

$$\varphi = \psi \tag{1.12}$$

be a uniform identity of type τ_0. If (1.12) is of the form $x_i = x_i$ then obviously it belongs to $C(U^*)$. If var $\varphi =$ var $\psi = \{x_i\}$ and $F(\varphi) = F(\psi) = \{+\}$ then by (d_1) the identities $\varphi = x_i + x_i$ and $\psi = x_i + x_i$ belongs to $C(U^*)$ and consequently (1.12) does. If var φ = var $\psi = \{x_i\}$ and $F(\varphi) = F(\psi) = \{\cdot\}$ then by (d_2) the argument is analogous. If var φ = var $\psi = \{x_{i_1},\dots,x_{i_n}\}$, $n>1$, and $F(\varphi) = F(\psi) = \{+\}$ then by (d_3) each of the identities $\varphi = x_{j_1} + \dots + x_{j_n}$ and $\psi = x_{j_1} + \dots + x_{j_n}$ belongs to $C(U^*)$, whence (1.12) belongs to $C(U^*)$. The case $\mathrm{var}(\varphi) = \mathrm{var}(\psi) = \{x_{i_1},\dots,x_{i_n}\}$, $n>1$, and $F(\varphi) = F(\psi) = \{\cdot\}$ is similar. Finally if $F(\varphi) = F(\psi) = \{+,\cdot\}$ then by Lemma 2 the identity (1.12) belongs to $C(U^*)$ as well. Q.E.D.

The next theorem explains the structure of algebras in $V(U(\tau_0))$. To formulate it we need some definitions.

We define three varieties K_1, K_2, K_3 of type τ_0 as follows.

K_1 is defined by the identities

$$\begin{gathered} x_1 + x_1 = x_1, \\ x_1 + x_2 = x_2 + x_1, \\ (x_1 + x_2) + x_3 = x_1 + (x_2 + x_3), \end{gathered} \tag{1.13}$$

(1.14)
$$x_1 \cdot x_2 = x_3 \cdot x_4,$$
$$x_1 + x_1 \cdot x_2 = x_1 \cdot x_2;$$

K_2 is defined by the identities

(1.15)
$$x_1 \cdot x_1 = x_1,$$
$$x_1 \cdot x_2 = x_2 \cdot x_1,$$
$$(x_1 \cdot x_2) \cdot x_3 = x_1 \cdot (x_2 \cdot x_3),$$

(1.16)
$$x_1 + x_2 = x_3 + x_4,$$
$$x_1 \cdot (x_1 + x_2) = x_1 + x_2;$$

K_3 is defined by the identity

(1.17)
$$x_1 + x_2 = x_3 \cdot x_4.$$

Obviously each algebra from K_1 has a join-semilattice structure where $x_1 \cdot x_2$ is the unit element. Analogously each algebra from K_2 has a meet-semilattice structure where $x_1 + x_2$ is the zero element. In algebras of the variety K_3 both operations are equal to the same constant.

T h e o r e m 2. $V(U(\tau_0)) = K_1 \vee K_2 \vee K_3 = K_1 \otimes K_2 \otimes K_3$.

For to prove Theorem 2 we need some definitions and lemmas. Let $\mathfrak{A} = (A; +, \cdot)$ be an algebra belonging to $V(U(\tau_0))$. We define in $\mathfrak{A}$ three relations R_1, R_2, R_3 as follows. For $a, b \in A$,

$$a\ R_1\ b \Longleftrightarrow a + a = b + b,$$
$$a\ R_2\ b \Longleftrightarrow a \cdot a = b \cdot b$$

$a\ R_3\ b$ iff one of the following conditions holds

(e_1) $a = b$,
(e_2) $a + a = a$ and $b + b = b$,
(e_3) $a \cdot a = a$ and $b \cdot b = b$,
(e_4) $a + a = a$ and $b \cdot b = b$,
(e_5) $a \cdot a = a$ and $b + b = b$.

In the sequal we write $q_+(x)$ for $x + x$ and $q_\cdot(x)$ for $x \cdot x$.

L e m m a 3. The relation R_1 is a congruence on $\mathfrak{U}$ such that $\mathfrak{U}/R_1 \in K_1$.

P r o o f . Recall that $\mathfrak{U}$ belongs to $V(U(\tau_0))$ and all uniform identities of type τ_0 are satisfied in $V(U(\tau_0))$.

Obviously R_1 is an equivalence relation on $\mathfrak{U}$. If $a\ R_1\ c$ and $b\ R_1\ d$ then $(a+b) + (a+b) =_U (a+a) + (b+b) = (c+c) + (d+d) =_U (c+d) + (c+d)$ and similarly $(a\cdot b) + (a\cdot b) =_U (a+a)\cdot(b+b) = (c+c)\cdot(d+d) =_U (c\cdot d) + (c\cdot d)$. Hence R_1 has the substitution property.

Now for $x_1,x_2,x_3,x_4 \in A$ we have $q_+(x_1\cdot x_2) =_U q_+(x_3\cdot x_4)$, whence $(x_1\cdot x_2)\ R_1\ (x_3\cdot x_4)$ and $\mathfrak{U}/R_1$ satisfies $x_1\cdot x_2 = x_3\cdot x_4$. Further $q_+(x_1+x_1\cdot x_2) =_U q_+(x_1\cdot x_2)$ and $q_+(x_1+x_1) =_U q_+(x_1)$. Consequently $\mathfrak{U}/R_1$ satisfies (1.14) and the first identity of (1.13). The remaining identities of (1.13) are satisfied in $\mathfrak{U}/R_1$ since they are satisfied in $\mathfrak{U}$. Q.E.D.

L e m m a 4. The relation R_2 is a congruence on $\mathfrak{U}$ and $\mathfrak{U}/R_2 \in K_2$.

The proof is dual to that of Lemma 3.

L e m m a 5. The relation R_3 is a congruence on $\mathfrak{U}$ such that $\mathfrak{U}/R_3 \in K_3$.

P r o o f . Obviously R_3 is an equivalence. For $a,b,c,d \in A$ we have

$$q_+(a + b) =_U a + b \quad \text{and}$$

$$q_\cdot(c\cdot d) =_U c\cdot d.$$

Hence R_3 has the substitution property and $\mathfrak{U}/R_3$ satisfies (1.17). Q.E.D.

L e m m a 6. $R_1 \cap R_2 \cap R_3 = \omega$, where ω is the equality relation.

P r o o f . Let $a,b \in A$ and $a(R_1 \cap R_2 \cap R_3)b$. We consider five cases.

(f_i) We assume that $q_+(a) = q_+(b)$, $q_\cdot(a) = q_\cdot(b)$ and (e_i) hold, where $i = 1,\dots,5$.

Obviously in the cases (f_1)-(f_3), $a = b$. In the case (f_4) we have

$$a = a + a = b + b = b\cdot b + b\cdot b = a\cdot a + a\cdot a$$

$$=_U (a+a)\cdot(a+a) = a\cdot a = b\cdot b = b.$$

The case (f_5) is similar. Q.E.D.

P r o o f of Theorem 2. Obviously $K_1 \otimes K_2 \otimes K_3 \subseteq \subseteq K_1 \vee K_2 \vee K_3$. By theorem 1 it is easy to check that $K_i \subseteq \subseteq V(U(\tau_0))$ for $i = 1,2,3$ since axioms of K_i imply (1.1)-(1.7). Hence $K_1 \vee K_2 \vee K_3 \subseteq V(U(\tau_0))$. To complete the proof it is enough to show that if $\mathfrak{A} \in V(U(\tau_0))$ then $\mathfrak{A}$ is decomposable into a subdirect product of algebras $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3$ where $\mathfrak{A}_i \in K_i$. But this follows directly from Lemmas 3-6 and Birkhoff's theorem (see [5] pp.123). Q.E.D.

2. Uniform and biregular identities in bisemilattices

An algebra $\mathfrak{A}$ of type τ_0 is called a bisemilattice (see [8]) if it satisfies the following identities

(2.1) $$x + x = x\cdot x = x,$$

(2.2) $$x + y = y + x, \quad x\cdot y = y\cdot x,$$

(2.3) $$(x + y) + z = x + (y + z), \quad (x\cdot y)\cdot z = x\cdot(y\cdot z).$$

In the sequel we write $q(x)$ for $q_+(q_\cdot(x))$.

Let K be a variety of bisemilattices. Let G be an equational base of the variety K. We define the set G^* of identities of type τ_0 as follows.

(g_1) The identities (1.1) - (1.6) belong to G^*;

(g_2) the identities $q(x_1+x_2) = q(x_1) + q(x_2)$ and $q(x_1\cdot x_2) = q(x_1)\cdot q(x_2)$ belong to G^*;

(g_3) if an identity $\varphi = \psi$ belongs to G then the identity $q(\varphi) = q(\psi) = q(\psi)$ belongs to G^*;

(g_4) the identities $q(x_1+x_2) = x_1 + q(x_2)$ and $q(x_1\cdot x_2) = x_1\cdot q(x_2)$ belong to G^*;

(g_5) the identities $q((x_1+x_2)\cdot x_3) = (x_1+x_2)\cdot x_3$ and $q(x_1\cdot x_2+x_3) = x_1\cdot x_2+x_3$ belong to G^*;

(g_6) the set G^* contains exactly the identities mentioned in (g_1)-(g_5).

T h e o r e m 3. For any variety K of bisemilattices $V(U(K)) = K \vee V(U(\tau_0)) = K \otimes V(U(\tau_0))$. Moreover if G is an equational base of K then G^* is an equational base of $V(U(K))$.

First we prove some lemmas. Let $\mathcal{U} = (A; +, \cdot)$ be an algebra belonging to $V(G^*)$. We define in $\mathcal{U}$ two relations Q_1 and Q_2 as follows

$a Q_1 b \Leftrightarrow q(a) = q(b)$;

$a Q_2 b \Leftrightarrow a = b$ or $q(a) = a$ and $q(b) = b$.

L e m m a 7. The relation Q_1 is a congruence on $\mathcal{U}$ such that $\mathcal{U}/Q_1 \in K$.

P r o o f . Obviously Q_1 is an equivalence. Q_1 has the substitution property by (g_2) and $\mathcal{U}/Q_1 \in K$ by (g_3). Q.E.D.

L e m m a 8. The relation Q_2 is a congruence on $\mathcal{U}$ such that $\mathcal{U}/Q_2 \in V(U(\tau_0))$.

P r o o f . Obviously Q_2 is an equivalence. We check the substitution property. Let $a Q_2 c$ and $b Q_2 d$. If $a = c$ and $b = d$ then $a + b = c + d$ and $a \cdot b = c \cdot d$ and we are done. If $a = c$, $q(b) = b$ and $q(d) = d$ then by (g_4) we have $q(a+b) = a + q(b) = a + b$ and similarly $q(c + d) = c + q(d) = c + d$ and so $(a + b) Q_2 (c + d)$. Analogously we check the substitution property for $\cdot$. If $q(a) = a$, $q(b) = b$, $q(c) = c$, $q(d) = d$, then the substitution property is satisfied by (g_2).

Now by (g_5) the identities (1.7) hold in $\mathcal{U}/Q_2$. The identities (1.1) - (1.6) are satisfied in $\mathcal{U}/Q_2$ since by (g_1) they are satisfied in $\mathcal{U}$. Thus $\mathcal{U}/Q_2$ satisfies (1.1) - (1.7) and by Theorem 1 $\mathcal{U}/Q_2 \in V(U(\tau_0))$. Q.E.D.

L e m m a 9. $Q_1 \cap Q_2 = \omega$.

The proof is obvious.

P r o o f of Theorem 3. Obviously $K \otimes V(U(\tau_0)) \subseteq K \vee V(U(\tau_0))$. Since $U(K) = B(K) \cap U(\tau_0)$ it follows that $K \vee V(U(\tau_0)) = V(U(K))$. Since each of the identities from G^* is satisfied both in K and in $V(U(\tau_0))$, it follows that $V(U(K)) \subseteq V(G^*)$. Recall that operations + and $\cdot$ are idempotent, commutative and associative. To complete the proof we have to show that if $\mathcal{U} \in V(G^*)$ then $\mathcal{U}$ is decomposable into a subdirect product of algebras $\mathcal{U}_1$ and $\mathcal{U}_2$ where $\mathcal{U}_1 \in K$ and

$\mathfrak{U}_2 \in V(U(\tau_0))$. This however follows from Lemmas 7 - 9 and Birkhoff's theorem (see [5], p.123). Q.E.D.

From Theorem 3 we obtain some interesting corollaries. Let K be a variety of bisemilattices.

C o r o l l a r y 1. If K is finitely based then V(U(K)) is finitely based.

Let L, L_m, L_d be the varieties of lattices, modular lattices and distributive lattices, respectively.

C o r o l l a r y 2. Each of the sets U(L), $U(L_m)$, $U(L_d)$ has a finite equational base.

C o r o l l a r y 3. If R(K) has a finite equational base then so does B(K).

P r o o f . It follows by Corollary 1 since $B(K) = R(K) \cap U(\tau_0)$ whence $V(B(K)) = V(U(V(R(K))))$. Q.E.D.

Let $B(\tau_0)$ denote the set of all biregular identities of type τ_0 and let $R(\tau_0)$ denote the set of all regular identities of type τ_0.

C o r o l l a r y 4. The set $B(\tau_0)$ has a finite equational base.

P r o o f . In fact $R(\tau_0)$ has a finite equational base defined by the identities (2.1) - (2.3) and $x_1 + x_2 = x_1 \cdot x_2$. Now the corollary follows by applying Corollary 3 to $K = V(R(\tau_0))$. Q.E.D.

C o r o l l a r y 5. Each of the sets B(L), $B(L_m)$, $B(L_d)$ has a finite equational base.

P r o o f . It was proved in [8] that any of the sets R(L), $R(L_m)$, $R(L_d)$ has a finite base. Now the corollary follows by Corollary 3. Q.E.D.

Other examples of finitely based varieties of bisemilattices can be found in [2], [3] and [12], [13].

From Theorems 2 and 3 we get a representation theorems for algebras from V(U(K)) and V(B(K)).

T h e o r e m 4. Let K be a variety of bisemilattices. Then an algebra $\mathfrak{U}$ belongs to V(U(K)) iff $\mathfrak{U}$ is isomorphic to a subdirect product of algebras $\mathfrak{U}_0$, $\mathfrak{U}_1$, $\mathfrak{U}_2$, $\mathfrak{U}_3$ where $\mathfrak{U}_0 \in K$ and $\mathfrak{U}_i \in K_i$, $i = 1,2,3$.

T h e o r e m 4'. Let K be a variety of bisemilattices. Then an algebra $\mathfrak{A}$ belongs to V(B(K)) iff $\mathfrak{A}$ is isomorphic to a subdirect product of algebras $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3$ where $\mathfrak{A}_0 \in V(R(K))$ and $\mathfrak{A}_i \in K_i$, $i = 1,2,3$.

In fact, if K is a variety of bisemilattices then V(R(K)) is a variety of bisemilattices too, since (2.1) - (2.3) are regular. Moreover B(K) = U(R(K)) so we can apply Theorem 4 to the variety V(R(K)).

Let us denote by $U_+(\tau_0)$ the set of all $\{+\}$-uniform identities of type τ_0. Studying the set of all $\{\cdot\}$-uniform identities of type τ_0 is a dual process. We denote by U_+^* the set of identities (1.15) and (1.16).

T h e o r e m 5. The set U_+^* is an equational base of the set $U_+(\tau_0)$.

P r o o f . By (i) we have $C(U_+(\tau_0)) = U_+(\tau_0)$. Each the identities (1.15) and (1.16) is $\{+\}$-uniform whence $U_+^* \subseteq U_+(\tau_0)$ and $C(U_+^*) \subseteq U_+(\tau_0)$.

To prove the converse inclusion recall that U_+^* is an equational base of K_2 and by Theorem 2, K_2 is a subvariety of $V(U(\tau_0))$. It follows that each of the identities (1.1) - (1.7) belongs to $C(U_+^*)$.

Now using Lemmas 1 and 2 and applying (1.15) and (1.16) we get the following.

1° If $\varphi(x_i)$ is a term of type τ_0 and $F(\varphi(x_i)) = \{\circ\}$ then the identity $\varphi(x_i) = x_i$ belongs to $C(U_+^*)$.

2° If $\varphi(x_{i_1},\dots,x_{i_n})$, $n>1$, is a term of type τ_0 and $F(\varphi(x_{i_1},\dots,x_{i_n})) = \{\cdot\}$ then the identity $\varphi(x_{i_1},\dots,x_{i_n}) = x_{j_1}\cdot x_{j_2}\cdot\ldots\cdot x_{j_n}$ belongs to $C(U_+^*)$ where $\{i_1,\dots,i_n\} = \{j_1,\dots,j_n\}$ and $j_1 < j_2 < \dots < j_n$.

3° If φ is a term of type τ_0 and $\{+\} \subseteq F(\varphi)$ then the identity $\varphi = x_1 + x_2$ belongs to $C(U_+^*)$.

Let the identity

$$\psi_1 = \psi_2 \tag{2.4}$$

belong to $U_+(\tau_0)$. If $F(\psi_1) = F(\psi_2) = \emptyset$ then (2.4) must be of the form $x_i = x_i$ and obviously belongs to $C(U_+^*)$. If $F(\psi_1) =$

$= F(\psi_2) = \{\cdot\}$ then (2.4) must be regular and by 1^o or 2^o belongs to $C(U_+^*)$. If $\{+\} \subseteq F(\psi_1) = F(\psi_2)$ then (2.4) belongs to $C(U_+^*)$ by 3^o. Q.E.D.

Corollary 6. $V(U_+(\tau_o)) = K_2$.

Let K be a variety of bisemilattices and let $U_+(K)$ denote the set of all $\{+\}$-uniform identities satisfied in K. The last theorem explains the structure of algebras in $V(U_+(K))$ and gives an equational base for it. At first we need some definitions and lemmas.

Let H be an equational base of K. We define the set H^* of identities of type τ_o as follows.

(h_1) The identities (1.15) belong to H^*;

(h_2) the identities $q_+(x_1 + x_2) = q_+(x_1) + q_+(x_2)$ and $q_+(x_1 \cdot x_2) = q(x_1) \cdot q(x_2)$ belong to H^*;

(h_3) for each identity $\varphi = \psi$ from H the identity $q_+(\varphi) = q_+(\psi)$ belongs to H^*;

(h_4) the identities $q_+(x_1 + x_2) = x_1 + q_+(x_2)$ and $q_+(x_1 \cdot x_2) = x_1 \cdot q_+(x_2)$ belong to H^*;

(h_5) the identities $q_+((x_1 + x_2) \cdot x_3) = (x_1 + x_2) \cdot x_3$ and $q_+(x_1 + x_2) = (x_1 + x_2)$ belong to $C(U_+^*)$;

(h_6) the set H^* contains exactly the identities mentioned in (h_1) - (h_5).

Let $\mathfrak{A} = (A; +, \cdot)$ be an algebra belonging to $V(H^*)$. We define on $\mathfrak{A}$ two relations T_1 and T_2 as follows.

$a\ T_1\ b \Leftrightarrow q_+(a) = q_+(b)$;

$a\ T_2\ b \Leftrightarrow a = b$ or $q_+(a) = a$ and $q_+(b) = b$.

Proofs of the following lemmas are similar to those of 7-9.

Lemma 10. The relation T_1 is a congruence on $\mathfrak{A}$ and $\mathfrak{A}/T_1 \in V(H)$.

Proof. It follows by (h_1) - (h_3).

Lemma 11. The relation T_2 is a congruence on $\mathfrak{A}$ and $\mathfrak{A}/T_2 \in V(U_+(\tau_o))$.

Proof. It follows by (h_4), (h_5) and Theorem 5.

Lemma 12. $T_1 \cap T_2 = \omega$.

T h e o r e m 6. $V(U_+(K)) = K \vee V(U_+(\tau_0)) = K \otimes V(U_+(\tau_0))$. Moreover if H is an equational base of K then H^* is an equational base of $V(U_+(K))$.

P r o o f . Obviously $K \otimes V(U_+(\tau_0)) \subseteq K \vee V(U_+(\tau_0)) = V(U_+(K))$, Further $V(U_+(K)) \subseteq V(H^*)$ since $H^* \subseteq U_+(K)$. Using Lemmas 10 - 12 we conclude that $V(H^*) \subseteq K \otimes V(U_+(\tau_0))$. Q.E.D.

Let $B_+(\tau_0)$ denote the set of all $\{+\}$-biregular identities of type τ_0 and for a variety K of bisemilattices let $B_+(K)$ denote the set of all $\{+\}$-biregular identities from E(K). The last theorem implies the following.

C o r o l l a r y 7. The corollaries 1-5 remain true if we write $U_+(U)$ instead of U(K), $B_+(K)$ instead of B(K), $B_+(\tau_0)$ instead of $B(\tau_0)$.

Problem. Let K be an arbitrary variety of algebras. Find a representation of algebras from V(U(K)) and from V(B(K)) by means of algebras from K.

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