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EXTERNALLY COMPATIBLE IDENTITIES IN PSEUDOCOMPLEMENTED DISTRIBUTIVE LATTICES

We deal with generalisations of pseudocomplemented distributive lattices in connection with externally compatible identities introduced by W. Chromik in [1].

Let K be a variety of type $\tau: T \to N \cup \{0\}$, where T is a nonempty set and N denotes the set of all positive integers. By identities of type τ we mean expressions of the form $p = q$, where p,q are n-ary polynomial symbols of type τ for some $n \in N \cup \{0\}$ (see [2]).

An identity $p = q$ is called externally compatible if it is of the form $x = x$ or of the form $f_t(p_1,\ldots,p_{\tau(t)}) = f_t(q_1,\ldots,q_{\tau(t)})$ for some polynomial symbols $p_1,\ldots,p_{\tau(t)}$, $q_1,\ldots,q_{\tau(t)}$ and some fundamental operation symbol f_t (see [1]).

If K is a variety of algebras of type τ, then E(K) (Ex(K)) denotes the set of all identities (all externally compatible identities) satisfied in K.

If S is a set of identities of type τ, then V(S) denotes the variety defined by S.

In [1] representation theorems for algebras of V(Ex(K)) are given for some varieties K, namely, when K is an idempotent variety or K is a variety with unary operation symbol f such that $f(f(x)) = x$ belongs to E(K) and any operation symbol g different from f is idempotent. Examples of such classes are the class of all distributive lattices and the class of all Boolean algebras. Moreover, the finite equational base for V(Ex(K)) in these both cases is given.

The aim of this note is to prove a representation theorem for algebras of V(Ex(K)) and construct a finite equational base for V(Ex(K)) in the case K is the class of all pseudo-complemented distributive lattices.

From now on we shall consider only algebras with two binary fundamental opérations + and • and one unary fundamental operation ′.

Let us denote by Σ the system of the following identities:

(1) $(x+y)+z = x+(y+z)$,

(2) $(x\cdot y)\cdot z = x\cdot(y\cdot z)$,

(3) $x+y = y+x$,

(4) $x\cdot y = y\cdot x$,

(5) $x+y = (x+x)+y$,

(6) $x\cdot y = (x+x)\cdot y$,

(7) $x\cdot y = (x\cdot x)\cdot x$,

(8) $(x+x)' = x'$,

(9) $x''' = x'$.

T h e o r e m 1. Let K be a variety of algebras with two binary operations + and • and one unary ′. Let $\Sigma \subset E(K)$. An algebra $\mathfrak{A} = (A;+,\cdot,')$ belongs to V(Ex(K)) iff there exists a congruence $\sim$ on $\mathfrak{A}$ satisfying the following:

(i) $\mathfrak{A}/\sim \in K$.

(ii) For each $a \in A$ there exists exactly one element $0_+([a]) \in [a]$ such that

$$b+c = 0_+([b]+[c]) \quad \text{for all} \quad b,c \in A.$$

(iii) For each $a \in A$ there exists exactly one element $0_\cdot([a]) \in [a]$ such that

$$b\cdot c = 0_\cdot([b]\cdot[c]) \quad \text{for all} \quad b,c \in A.$$

(iv) For each $a \in A$ there exists exactly one element $0_{'}([a']) \in [a']$ such that for $b \in A$, $[b'] = [a']$ imply $b' = 0_{'}([a'])$.

P r o o f . ($\Longrightarrow$) On the set A we define the relation $\sim$ by setting (see [3]):

$$a \sim b \text{ iff } a+a = b+b \text{ for all } a,b \in A.$$

Obviously $\sim$ is an equivalence relation on $\mathfrak{A}$. Let $a \sim b$ and $c \sim d$ for $a,b,c,d \in A$. Then $(a+c)+(a+c) = (a+a)+(c+c) = (b+b) + (d+d) = (b+d) + (b+d)$ by (1) and (3). Similarly, using (2) and (4) we get $a \cdot c \sim b \cdot d$. Further, by (8) we have $a'+a' = (a+a)' + (a+a)' = (b+b)' + (b+b)' = b'+b'$ what proves that $\sim$ is a congruence on $\mathfrak{A}$.

Let $p = p(x_1,\dots,x_m)$, $q = q(y_1,\dots,y_n)$ and $(p=q) \in E(K)$. Then $(p+p = q+q) \in Ex(K)$. Hence for $a_1,\dots,a_m,b_1,\dots,b_n$ in A we have $[p(a_1,\dots,a_m)] = [q(b_1,\dots,b_n)]$. Thus $\mathfrak{A}/\sim \in K$.

Let $a \in A$ and $b,c,d,e \in [a]$. Then by (5), $b+c = (b+b)+(c+c) = (d+d)+(e+e) = d+e$. Hence $b+c$ is a constant in $[a]$ and we can define $O_+([a]) = b+c$ for any $b,c \in [a]$. Since $a+a \sim a$ it follows that $O_+([a]) \in [a]$. Analogously, using (7), we prove that $b \cdot c$ is a constant in $[a]$ and we can define $O_\cdot([a]) = b \cdot c$ for any $b,c \in [a]$.

Now let $a \in A$ and $b,c \in [a]$. Then by (8), $b' = (b+b)' = (c+c)' = c'$. Hence for any $b \in [a]$, b' is constant in $[a']$ and we can define $O_{'}([a']) = a'$. If $b \in A$ and $[b'] = [a']$, then by (9) and (8), $b' = b''' = (b'+b')'' = (a'+a')'' = a''' = a' = O_{'}([a'])$.

Let $a,b \in A$. Then by (1), (3), (5) we have $(a+b)+(a+b) = a+b$, whence $a+b = O_+([a+b]) = O_+([a]+[b])$. Analogously by (2), (4), (7) $a \cdot b = O_\cdot([a]\cdot[b])$.

Let $(p=q) \in Ex(K)$, where $p=q$ is not of the form $x=x$. Then there exist some polynomial symbols p_1,p_2,q_1,q_2 such that $p = p_1+p_2$ and $q = q_1+q_2$ or $p = p_1p_2$ and $q = q_1q_2$ or $p = p_1'$ and $q = q_1'$.

Let $p = q$ be of the form $p_1+p_2 = q_1+q_2$. Since $Ex(K) \subset E(K)$, then the algebra $\mathfrak{A}/\sim$ satisfies $p = q$. Therefore, for $a_1,\dots,a_m,b_1,\dots,b_n$ in A we have $[p(a_1,\dots,a_m)] = [q(b_1,\dots,b_n)]$ and, by (ii) $p(a_1,\dots,a_m) = p_1(a_1,\dots,a_m) +$

$+ p_2(a_1,\ldots,a_m) = O_+([p_1(a_1,\ldots,a_m)] + [p_2(a_1,\ldots,a_m)]) =$
$= O_+([p(a_1,\ldots,a_m)]) = O_+([q(b_1,\ldots,b_n)]) = q(b_1,\ldots,b_n).$
Thus the identity $p = q$ is satisfied in $\mathfrak{A}$. The proof for $p = p_1 \cdot p_2$ and $q = q_1 \cdot q_2$ is analogous.

Let $p = p_1'$ and $q = q_1'$. Then by (iv) we have $p(a_1,\ldots,a_m) = p_1(a_1,\ldots,a_m)' = O_{\prime}([p_1(a_1,\ldots,a_m)]') = O_{\prime}([q_1(b_1,\ldots,b_n)]') = q_1(b_1,\ldots,b_n)' = q(b_1,\ldots,b_n)$ for all $a_1,\ldots,a_m,b_1,\ldots,b_n \in A$. Thus, the identity $p = q$ is satisfied in $\mathfrak{A}$.

Theorem 2. If K is a variety such that $\Sigma \subset E(K)$ and B is an equational base for K, then $\Sigma \cup B^*$, where $(p=q) \in B$ iff $(p+p = q+q) \in B^*$, form an equational base for $V(Ex(K))$.

Proof. We show that $V(B^* \cup \Sigma) = V(Ex(K))$. It is obvious that $B^* \cup \Sigma \subset Ex(K)$ and $V(Ex(K)) \subset V(B^* \cup \Sigma)$.

Let $\mathfrak{A} \in V(B^* \cup \Sigma)$. On the set A we define the relation $\sim$ by setting:

$$a \sim b \quad \text{iff} \quad a+a = b+b \quad \text{for all} \quad a,b \in A.$$

It is easy to check that $\sim$ satisfies conditions of Theorem 1. It follows that $\mathfrak{A} \in V(Ex(K))$.

Corollary 1. If B is finite, then $V(Ex(K))$ is finitely based.

It is known that the following identities form a base for the variety L of all pseudocomplemented distributive lattices (see [4]):

(L1) $(x+y)+z = x+(y+z)$,
(L2) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,
(L3) $x+y = y+x$,
(L4) $x \cdot y = y \cdot x$,
(L5) $x+x = x$,
(L6) $x \cdot x = x$,
(L7) $x+x \cdot y = x$,
(L8) $x \cdot (y+z) = x \cdot y+x \cdot z$,
(L9) $x \cdot x' = y \cdot y'$,
(L10) $x \cdot (x \cdot y)' = x \cdot y'$,

(L11) $x\cdot(x\cdot x')' = x$,

(L12) $(x\cdot x')'' = x\cdot x'$.

Note that pseudocomplemented distributive lattices satisfy identities (1) - (9) and $\Sigma \subset Ex(L)$.

Corollary 2. An algebra $\mathfrak{A} = (A;+,\cdot,')$ belongs to V(Ex(L)) iff the identities (1) - (9) and

(10) $x\cdot x' = y\cdot y'$,

(11) $x\cdot(x\cdot y)' = x\cdot y'$,

(12) $x\cdot(x\cdot x'+x\cdot(x\cdot x')') = x+x$,

(13) $(x\cdot x')''+(x\cdot x')'' = x\cdot x'+x\cdot x'$,

(14) $x\cdot x+y = x+y$,

(15) $x+x\cdot y = x+x$,

(16) $x\cdot(y+z)+x\cdot(y+z) = x\cdot y+x\cdot z$

are satisfied in $\mathfrak{A}$.

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