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A CLASSIFICATION OF σ-IDEALS IN POLISH GROUPS

In this paper we consider σ-ideals of subsets of a Polish dense-in-itself abelian group X. The type of a σ-ideal is defined and investigated. Under the assumption that X is connected, locally connected and locally compact, we construct, in the main theorem, σ-ideals of all possible types.

Throughout the paper, we shall assume that X is a Polish dense-in-itself abelian group with the group operation + and the neutral element e. Polish group means a topological group whose topology is generated by a metric which makes X a complete and separable space.

This paper is essentially an improvement of article [1] in which X is equal to the real line $\mathbb{R}$ and condition (v) given below is not required in the definition of the σ-ideal.

The power set of X will be denoted by $\mathcal{P}(X)$. Let ω and ω_1 denote the first infinite and uncountable ordinal numbers, respectively.

A nonempty family $\mathcal{J} \subseteq \mathcal{P}(X)$ is called a σ-ideal if and only if it fulfils the conditions:

(i) if $A \in \mathcal{J}$ and $B \subseteq A$, then $B \in \mathcal{J}$;

(ii) if $A_n \in \mathcal{J}$, $n<\omega$, then $\bigcup_{n<\omega} A_n \in \mathcal{J}$.

Moreover, we shall always assume that $\mathcal{J}$ is nontrivial, i.e.

(iii) $\{\emptyset\} \neq \mathcal{J} \neq \mathcal{P}(X)$,

and that the following conditions concerning the group operations are valid:

(iv) if $A \in \mathcal{J}$ and $x \in X$, then $A + x \in \mathcal{J}$;
(v) if $A \in \mathcal{J}$, then $-A \in \mathcal{J}$,
where $A + x = \{a + x : a \in A\}$, $-A = \{-a : a \in A\}$.

Observe that $\mathcal{J}$ obviously contains all singletons $\{x\}$ and does not contain any nonempty open set (indeed, if such a set U existed, then, in virtue of (iv), the sets $U + x$, $x \in X$, would cover X; thus, by taking a countable subcovering $U + x_n$, $n < \omega$, we would obtain $X \in \mathcal{J}$, which is impossible).

In the sequel, $\mathcal{J}$ will always denote a σ-ideal.

For any $\mathcal{F} \subseteq \mathcal{P}(X)$, we define

$$I(\mathcal{J}, \mathcal{F}) = \{A : A \subseteq B \text{ for some } B \in \mathcal{J} \cap \mathcal{F}\} .$$

We shall consider $\mathcal{F}$ equal to the family $\mathcal{B}$ of all Borel subsets of X or to any of its subfamilies F_α, G_α, $\alpha < \omega_1$, (see [5], pp.251-252). We define (comp. [1]) the type of $\mathcal{J}$ as a pair $(\alpha; \beta)$ such that α (resp. β) is the first ordinal $\leq \omega_1$ for which $I(\mathcal{J}, \mathcal{B}) = I(\mathcal{J}, F_\alpha)$ (resp. $I(\mathcal{J}, \mathcal{B}) = I(\mathcal{J}, G_\beta)$). Here $F_{\omega_1} = G_{\omega_1} = \mathcal{B}$.

We shall say that $\mathcal{J}$ is a Borel (resp. non-Borel) σ-ideal if and only if $\mathcal{J} = I(\mathcal{J}, \mathcal{B})$ (resp. $\mathcal{J} \neq I(\mathcal{J}, \mathcal{B})$).

P r o p o s i t i o n 1. If $(\alpha; \beta)$ is the type of a σ-ideal, then

$$(*) \quad 1 < \alpha = \beta \leq \omega_1 \text{ or } 1 < \alpha = \beta + 1 < \omega_1 \text{ or } 1 < \beta = \alpha + 1 < \omega_1 .$$

The proof is analogous to that in [1].

Denote by $\mathcal{K}$ the σ-ideal of all meager subsets of X. It is easily observed that $\mathcal{K}$ is a Borel σ-ideal of type (1; 2).

Write $I(\mathcal{J}, F_1)$ briefly as $\mathcal{J}_1$ and observe that $\mathcal{J}_1$ is a Borel σ-ideal of type (1;2).

Now, we shall study some properties of σ-ideals of type (2;1). We shall next give a few examples of such σ-ideals.

P r o p o s i t i o n 2. If $\mathcal{J}$ is of type (2;1), then there is a nowhere dense set of type G_δ belonging to $\mathcal{J} \setminus \mathcal{J}_1$.

P r o o f . At first, we shall choose a closed nowhere dense set $A \in \mathcal{J}$. Consider a countable set A_1 dense in X and a set A_2 of type G_δ such that $A_1 \subseteq A_2 \in \mathcal{J}$. Then the set $X \setminus A_2$ is of type F_σ and belongs to $\mathcal{K} \setminus \mathcal{J}$; hence it contains a set A which has the desired properties. Let B denote the set of all points x such that each open set containing x intersects A in a set which does not belong to $\mathcal{J}$. It is easy to verify that B is a perfect nowhere dense set and $A \setminus B \in \mathcal{J}$. Thus $B \notin \mathcal{J}$. Let D be a countable subset of B dense in B and let E be a superset of D, of type G_δ, belonging to $\mathcal{J}$ and contained in B. Clearly, E is residual in B. If E were included in a set $H \in \mathcal{J}$, of type F_σ, then both sets $B \cap H$, $B \setminus H$ would be meager in B, which contradicts the Baire Category Theorem. Thus, we can easily see that the set E fulfils the assertion.

C o r o l l a r y 1. If $\mathcal{J}$ is of type (2;1), then $\mathcal{J}_1 \subsetneq \mathcal{J} \cap \mathcal{K}$.

P r o o f . The inclusion $\mathcal{J}_1 \subseteq \mathcal{J} \cap \mathcal{K}$ is self-evident. It follows from Proposition 2 that equality does not hold here.

C o r o l l a r y 2. If $\mathcal{J}$ is of type (2;1), then $\mathcal{J} \cap \mathcal{K}$ is of type (2;2).

The proof is analogous to that in [1], prop. 1.

Below, in Examples 1, 2, the Cantor set C will be considered as a countable product (equipped with the Tychonoff topology) of identical discrete topological groups $\{0,1\}$ with the operator of addition modulo 2. Note that C forms a dense-in-itself Polish abelian group.

E x a m p l e 1. Let X be locally compact. Then there exists a Haar measure m on X ([4]). Observe that the family $\mathcal{L}$ of all sets of measure m zero forms a σ-ideal. Since m is regular, $\mathcal{L}$ is of type (2;1). Notice that if $X = \mathbb{R}$, we may treat m as Lebesgue measure, and if X = C, we may treat it as the product measure generated by measure μ on $\{0,1\}$ for which $\mu(\{0\}) = \mu(\{1\}) = 1/2$.

E x a m p l e 2. Let X = C. Mycielski, using notions of game theory, defined in [10] a σ-ideal $\mathcal{M} \subseteq \mathcal{P}(C)$ which is of type (2;1) (condition (v) is not required in [10] for σ-ideals, but one can easily check that $\mathcal{M}$ fulfils it).

A subset of X will be called totally imperfect (comp.[5]), p.421) if and only if it does not contain any nonempty perfect set.

We shall now give some examples of non-Borel σ-ideals.

The following lemma is a simple consequence of the Alexandroff-Hausdorff theorem ([5], p.355):

L e m m a 0. Each σ-ideal which contains an uncountable set and consists of totally imperfect sets is non-Borel.

E x a m p l e 3. We say (see [11]) that a set A has the property (s_0) if and only if each nonempty perfect set contains a nonempty perfect set disjoint from A. Observe that this property is preserved by homeomorphisms. Let $\mathcal{J}_0$ be the family of all subsets of X with property (s_0). Then $\mathcal{J}_0$ forms a σ-ideal since condition (ii) was verified in [11] and the other conditions are self-evident. Of course, $\mathcal{J}_0$ consists of totally imperfect sets. Notice that $\mathcal{J}_0$ contains a set of power $2^{\aleph_0}$. Indeed, consider the Cantor set C (as a metric space; here we do not require C to be a group). We can find a set of power $2^{\aleph_0}$ having the property (s_0) with respect to $C \times C$ (analogously as in [9], th.5.10). Thus, for the space C, this is also possible since $C \times C$ and C are homoeomorphic (comp. [5], p.235). At last, choose a subset $\tilde{C}$ of X homeomorphic to C (by the Alexandroff-Hausdorff theorem). Then $\tilde{C}$ contains a set A of power $2^{\aleph_0}$ having the property (s_0) with respect to X, as well. Hence, by Lemma 0, $\mathcal{J}_0$ is a non-Borel σ-ideal.

E x a m p l e 4. For any families $\mathcal{F}^{(1)}, \mathcal{F}^{(2)} \subseteq \mathcal{P}(X)$, let $\mathcal{F}^{(1)} \oplus \mathcal{F}^{(2)} = \{A_1 \cup A_2 : A_i \in \mathcal{F}^{(i)},\ i = 1,2\}$. Denote by $\mathcal{H}$ the family of all subsets of X of power less than $2^{\aleph_0}$. Analogously as in [2], where $X = \mathbb{R}$, we obtain a set $E \subseteq X$ such that $D \cap E \neq \emptyset$, $D \setminus E \neq \emptyset$ for each nonempty perfect set D, and that $E \triangle (-E) \in \mathcal{H}$, $E \triangle (E + x) \in \mathcal{H}$ for each $x \in X$ (here $\triangle$ denotes the operation of the symmetric difference of sets). Put $\mathcal{X}(E) = \mathcal{P}(E) \oplus \mathcal{H}$. It can be easily shown that $\mathcal{X}(E)$ is a σ-ideal, consists of totally imperfect sets and contains the set E which is of power $2^{\aleph_0}$. Thus, by Lemma 0, $\mathcal{K}(E)$ is a non-Borel σ-ideal.

E x a m p l e 5. Assume that $\mathcal{J}$ is a Borel σ-ideal. Let $\hat{\mathcal{J}} = \mathcal{K}(E) \oplus \mathcal{J}$, where $\mathcal{K}(E)$ is the σ-ideal described in Example 4. Similarly as in [1] we prove that $\hat{\mathcal{J}}$ is a non-Borel σ-ideal and $I(\hat{\mathcal{J}}, \mathcal{B}) = \mathcal{J}$, so the types of $\mathcal{J}$ and $\hat{\mathcal{J}}$ are identical.

The question arises whether condition (*) formulated in Proposition 1 is sufficient for a pair $(\alpha; \beta)$ to be the type of a σ-ideal. In Theorem 1 we shall give an affirmative answer under the assumption that X is connected, locally connected and locally compact. In the proof we use methods described in [8] and our technique from [1] in the modified version. Observe that our result can be applied, for example, to the Hilbert cube and to the space of all continuous functions $f: [0,1] \to \mathbb{R}$ with the uniform convergence topology, both considered with the standard addition operations. However, we do not know whether Theorem 1 would be valid without assuming the connectivity and the local connectivity of X (for instance if $X = C$).

T h e o r e m 1. Assume that X is connected, locally connected and locally compact. If a pair $(\alpha; \beta)$ fulfils (*), then there exist a Borel σ-ideal $\mathcal{J}(\alpha,\beta)$ and a non-Borel σ-ideal $\hat{\mathcal{J}}(\alpha,\beta)$ both of type $(\alpha; \beta)$.

The proof will be based on a few lemmas. Let X fulfil the assumptions of Theorem 1.

Recall ([6], p.296) that a subset Y of X is said to be independent, provided that if $y_1,\dots,y_n$ are distinct elements of Y and $k_1,\dots,k_n$ are integers such that $k_1y_1 + \dots + k_ny_n = e$, then $k_1 = k_2 = \dots = k_n = 0$.

From [8] (th. 0 and remark following it) we obtain

L e m m a 1. There exists an independent, compact, perfect, zero-dimensional subset P of X.

For any $A \subseteq X$, let $\langle A \rangle$ denote the subgroup of X generated by A (we assume $\langle \emptyset \rangle = \emptyset$).

L e m m a 2. Let $A, D \subseteq P$. For arbitrary $x, y \in X$, we have:

(a) if A is closed, then $\langle A \rangle$ is meager of type F_σ;

(b) if $A \cap D = \emptyset$, then $(\langle A \rangle + x) \cap (\langle D \rangle + y)$ has at most one element;

(c) if $D \subseteq A$, then $(\langle D \rangle + x) \cap A$ is equal to D for $x \in \langle D \rangle$ and has at most one element for $x \notin \langle D \rangle$.

P r o o f (comp. [8]). Observe that the set $\tilde{P} = P \cup (-P)$ is zero-dimensional ([3], p.438). It follows from [7] (comp. also [3], p.457) that there exists a linear ordering $<$ on $\tilde{P}$ such that all sets

$$\{x \in \tilde{P} : a < x < b\},\ \{x \in \tilde{P} : x < a\},\ \{x \in \tilde{P} : a < x\}$$

for $a, b \in \tilde{P}$ form a basis of the topology of $\tilde{P}$ generated from X. It is not difficult to check that the set $\{(x,y) \in \tilde{P}^2 : x \leq y\}$ is then closed. Consider arbitrary $n < \omega$ and $A \subseteq P$. Let

$$W_n = \{(x_1, \dots, x_n) \in \tilde{P}^n : x_1 \leq x_2 \leq \dots \leq x_n\},$$

$$T_n = \{(x_1, \dots, x_n) \in \tilde{P}^n : x_i \neq -x_j \quad \text{for} \quad 1 \leq i \leq j \leq n\},$$

$$S_n(A) = W_n \cap T_n \cap (A \cup (-A))^n,$$

$$f_n : X^n \to X \quad \text{and} \quad f_n(x_1, \dots, x_n) = x_1 + \dots + x_n,$$

$$Z_n(A) = f_n(S_n(A)).$$

Notice that:

(1) if A is closed, then $S_n(A)$ is of type F_σ (since W_n is closed and T_n is open with respect to $\tilde{P}^n$);

(2) each element $x \in \langle A \rangle \setminus \{e\}$ has a unique representation $x = x_1 + \dots + x_n$ for some $n < \omega$ and $(x_1, \dots, x_n) \in S_n(A)$ (since $A \subseteq P$ and P is independent);

(3) $f_n | S_n(A)$ is a homeomorphism of $S_n(A)$ onto $Z_n(A)$ (since f_n is an open continuous mapping and $f_n | S_n(A)$ is one-to-one).

Now, we shall prove (a). Since P is zero-dimensional, therefore the sets $\tilde{P}^n$, $S_n(A)$, $n < \omega$, are, such, too ([3], pp.445-446). Then, for any $n < \omega$, the set $Z_n(A)$ is zero-dimensional. Next, observe that it has no interior points. Indeed, if there existed an interior point x of $Z_n(A)$, then, by the local connectivity of X, we would find a connected neighbourhood U of x included in $Z_n(A)$. Since $Z_n(A)$ is zero-dimensional, we have $U = \{x\}$ (comp. [3], p.443) which contradicts the fact that X is dense-in-itself. So, $Z_n(A)$ has no interior points and, thus, is meager since, by (1), (3), it is of type F_σ. In virtue of (2), we have

$$\bigcup_{n<\omega} Z_n(A) = \langle A\rangle \setminus \{e\},$$

thus $\langle A\rangle$ is meager of type F_σ.

To prove (b), observe that it is enough to show that, for each $x\in X$, the set $(\langle A\rangle + x)\cap\langle D\rangle$ has at most one element. At first, assume that $x \in \langle A\cup D\rangle$. If $x = e$, then, by (2), we easily obtain $(\langle A\rangle + x)\cap\langle D\rangle = \langle A\rangle\cap\langle D\rangle = \{e\}$. Let $x \neq e$. We have $x = a + d$ for some $a\in\langle A\rangle$, $d\in\langle D\rangle$. There is a unique representation of that form, which follows from (2) and the assumption $A\cap D = \emptyset$. Suppose that $z\in(\langle A\rangle + x)\cap\langle D\rangle$. Then $z\in\langle D\rangle$ and $z = a' + x$ for some $a'\in\langle A\rangle$. Hence $x = a + d = -a' + z$, which implies that $z = d$. So, $(\langle A\rangle + x)\cap\langle D\rangle$ has at most one element d. Finally, let $x\notin\langle A\cup D\rangle$. Then

$$(\langle A\rangle + x)\cap\langle D\rangle\subseteq(\langle A\cup D\rangle + x)\cap\langle A\cup D\rangle = \emptyset.$$

Now, we shall prove (c). Observe that $\langle D\rangle\cap A = D$ since we obviously have $D\subseteq\langle D\rangle\cap A$ and, by (2), it is impossible to find $d\in(\langle D\rangle\cap A)\setminus D$. Hence, if $x\in\langle D\rangle$, then $\langle D\rangle + x = \langle D\rangle$ and $(\langle D\rangle + x)\cap A = D$. Next, assume that $x\in\langle A\rangle\setminus\langle D\rangle$. Then $x = x' + d$ where $x'\in\langle A\setminus D\rangle$, $d\in\langle D\rangle$ and, by (2), x has a unique representation of this form. Suppose that $z\in(\langle D\rangle + x)\cap A$. Then $z\in A$ and $z = d' + x$ for some $d'\in\langle D\rangle$. Since $z\in A$ and $z\notin D$ (if $z\notin D$, then we would have $x = z - d'\in\langle D\rangle$, a contradiction), therefore $z\in\langle A\setminus D\rangle$. Consequently, $x = x' + d = z - d'$ implies that $z = x'$. Hence $(\langle D\rangle + x)\cap A$ has at most one element x'. Finally, let $x\notin\langle A\rangle$. Then

$$(\langle D\rangle + x) \cap A \subseteq (\langle A\rangle + x) \cap \langle A\rangle = \emptyset.$$

The proof is completed.

Choose pairwise disjoint perfect sets A_α, B_β, $\alpha, \beta<\omega_1$, contained in P. For $\alpha = 0,1$, let $D_\alpha = E_\alpha = \emptyset$ and, for each α, $2\leqslant\alpha<\omega_1$, let D_α, E_α be such that $D_\alpha \subseteq A_\alpha$, $E_\alpha \subseteq B_\alpha$, $D_\alpha \in F_\alpha \setminus G_\alpha$, $E_\alpha \in G_\alpha \setminus F_\alpha$ (comp. [5], p.275).

By repeating the argument from [8], we obtain

L e m m a 3. For all α, $2\leqslant\alpha<\omega_1$, we have $\langle D_\alpha\rangle \in F_\alpha \setminus G_\alpha$, $\langle E_\alpha\rangle \in G_\alpha \setminus F_\alpha$.

The next part of the proof is similar to that in [1]. However we shall provide it for the reader's convenience.

For each α, $0<\alpha<\omega_1$, we denote by $T(\alpha)$ the family of all double sequences $\{t_{n\gamma}\}_{n<\omega,\ \gamma<\alpha}$ with terms from X. For any $t \in T(\alpha)$, $t = \{t_{n\gamma}\}_{n<\omega,\ \gamma<\alpha}$, let us denote

$$D(\alpha,t) = \bigcup_{\gamma<\alpha}\ \bigcup_{n<\omega} (\langle D_\gamma\rangle + t_{n\gamma}),$$

$$E(\alpha,t) = \bigcup_{\gamma<\alpha}\ \bigcup_{n<\omega} (\langle E_\gamma\rangle + t_{n\gamma}).$$

R e m a r k . Note that in [1] we used simply D_γ, E_γ in the definition of $D(\alpha,t)$, $E(\alpha,t)$. Here we use $\langle D_\gamma\rangle$, $\langle E_\gamma\rangle$ because we want to construct a suitable σ-ideal fulfilling condition (v).

L e m m a 4. Let $2\leqslant\alpha<\omega_1$. Then $D(\alpha,t) \in F_{\alpha-1}$, $E(\alpha, t) \in G_{\alpha-1}$ when $\alpha-1$ exists, and $D(\alpha, t)$, $E(\alpha, t) \in F_\alpha \cap G_\alpha$ when α is a limit number.

P r o o f . We shall demonstrate the assertion which deals with $D(\alpha,t)$; the proof concerning $E(\alpha,t)$ is analogous. Notice that $D(2,t) = \emptyset \in F_1$; therefore, in this case, the assertion holds. Now, let $\alpha > 2$. Let $t = \{t_{n\gamma}\}_{n<\omega,\ \gamma<\alpha}$. Denote

$$A_{n\gamma} = \langle A_\gamma\rangle + t_{n\gamma},\quad D_{n\gamma} = \langle D_\gamma\rangle + t_{n\gamma},\quad D'_{n\gamma} = A_{n\gamma} \setminus D_{n\gamma},$$

for $n<\omega$, $\gamma<\alpha$. Observe that, by Lemma 2(a), the sets $\langle A_\gamma\rangle$ and, consequently, $A_{n\gamma}$ are of type F_σ. By Lemma 3, we have

$D_{n\gamma} \in F_\gamma$, $D'_{n\gamma} \in G_\gamma$. If $\xi \neq \gamma$; $\xi, \gamma < \alpha$, then $A_\xi \cap D_\gamma = \emptyset$; so, by Lemma 2(b), for any $n, k < \omega$, the set $A_{k\xi} \cap D_{n\gamma}$ has at most one element. This implies that there is a countable set $H_{k\xi}$ included in $A_{k\xi}$ such that

$$A_{k\xi} \setminus D(\alpha,t) = D'_{k\xi} \setminus H_{k\xi}.$$

We then have

$$D(\alpha,t) = \bigcap_{\xi<\alpha} \bigcap_{k} \left(\bigcup_{\gamma<\alpha} \bigcup_{n} (A_{n\gamma} \setminus D'_{k\xi}) \cup H_{k\xi} \right),$$

which easily implies that $D(\alpha,t) \in \left(\bigcup_{\beta<\alpha} F_\beta \right)_\delta$. From the equality

$$D(\alpha,t) = \bigcup_{\gamma<\alpha} \bigcup_{n} D_{n\gamma},$$

it follows that $D(\alpha,t) \in \left(\bigcup_{\beta<\alpha} F_\beta \right)_\sigma$. Thus we have obtained

$$D(\alpha,t) \in \left(\bigcup_{\beta<\alpha} F_\beta \right)_\sigma \cap \left(\bigcup_{\beta<\alpha} F_\beta \right)_\delta .$$

Assume that $\alpha - 1$ exists. We have

$$\left(\bigcup_{\beta<\alpha} F_\beta \right)_\sigma = (F_{\alpha-1})_\sigma = F_{\alpha-1} \quad \text{when } \alpha \text{ is even},$$

$$\left(\bigcup_{\beta<\alpha} F_\beta \right)_\delta = (F_{\alpha-1})_\delta = F_{\alpha-1} \quad \text{when } \alpha \text{ is odd}.$$

Thus $D(\alpha,t) \in F_{\alpha-1}$. If α is a limit number, then

$$\left(\bigcup_{\beta<\alpha} F_\beta \right)_\delta = F_\alpha, \quad \left(\bigcup_{\beta<\alpha} F_\beta \right)_\sigma = \left(\bigcup_{\beta<\alpha} G_\beta \right)_\sigma = G_\alpha.$$

Hence $D(\alpha,t) \in F_\alpha \cap G_\alpha$. The Lemma has been proved.

L e m m a 5. If $3 \leq \alpha \leq \omega_1$, $3 \leq \beta < \omega_1$, $2 \leq \eta < \alpha$, $2 \leq \xi < \beta$, $s \in T(\alpha)$, $t \in T(\beta)$, then

(a) there is no set $M \in G_\xi$ such that

$$D_\xi \subseteq M \subseteq E(\alpha, s) \cup D(\beta, t);$$

(b) there is no set $M \in F_\rho$ such that

$$E_\rho \subseteq M \subseteq E(\alpha, s) \cup D(\beta, t).$$

Proof. We shall show (a); the proof of (b) is analogous. Suppose that there is a set $M \in G_\xi$ such that

$$D_\xi \subseteq M \subseteq E(\alpha, s) \cup D(\beta, t).$$

Then, obviously

$$D_\xi \subseteq A_\xi \cap M.$$

Let $s = \{s_{n\gamma}\}_{n<\omega,\, \gamma<\alpha}$, $t = \{t_{n\zeta}\}_{n<\omega,\, \zeta<\beta}$. By virtue of Lemma 2(c) each of the sets

$$A_\xi \cap (\langle E_\gamma \rangle + s_{n\gamma}),\ A_\xi \cap (\langle D_\zeta \rangle + t_{n\zeta}),\ n<\omega,\ \gamma<\alpha,$$

has at most one element, except for the case when $\zeta = \xi$, $t_{n\zeta} \in \langle D_\zeta \rangle$ (then $A_\xi \cap (\langle D_\zeta \rangle + t_{n\zeta}) = D_\xi$). Hence

$$A_\xi \cap M \subseteq A_\xi \cap (E(\alpha, s) \cup D(\beta, t)) \subseteq D_\xi \cup H$$

where H is a countable set. We may assume that D_ξ and H are disjoint. Thus

$$D_\xi \subseteq A_\xi \cap M \subseteq D_\xi \cup H,$$

and so

$$D_\xi = (A_\xi \cap M) \setminus H.$$

Since D_ξ equals the difference of sets of types G_ξ and F_1, therefore it is of type G_ξ. This contradiction ends the proof.

For any $\mathcal{F} \subseteq \mathcal{P}(X)$, denote

$$\Gamma(\mathcal{F}) = \left\{A:\ A \subseteq \dot{\bigcup_{n<\omega}} (A_n + x_n) \quad \text{for some} \quad A_n \in \mathcal{F},\ x_n \in X\right\}.$$

It is easy to verify the following

L e m m a 6. If $\mathcal{F}\subseteq\mathcal{K}$ and $\mathcal{F}$ consists of subgroups of X, then $\Gamma(\mathcal{F})$ is the minimal (in the sense of inclusion) σ-ideal including $\mathcal{F}$. Moreover, if $\mathcal{F}\subseteq\mathcal{B}$, then $\Gamma(\mathcal{F})$ is a Borel σ-ideal.

P r o p o s i t i o n 3. For an arbitrary pair $(\alpha;\beta)$ of ordinal numbers such that

$$3\leqslant\alpha=\beta\leqslant\omega_1 \quad \text{or} \quad 3\leqslant\alpha+1=\beta<\omega_1 \quad \text{or} \quad 3\leqslant\beta+1=\alpha<\omega_1,$$

there is a Borel σ-ideal $\mathcal{J}(\alpha,\beta)$ of type $(\alpha;\beta)$ included in $\mathcal{K}$. Moreover, the σ-ideals $\mathcal{J}(\alpha,\beta)$ can be defined in such a way that if $\alpha\leqslant\alpha'$ and $\beta\leqslant\beta'$, then $\mathcal{J}(\alpha,\beta)\subseteq\mathcal{J}(\alpha',\beta')$.

P r o o f . We put

$$\mathcal{J}(\alpha,\beta)=\Gamma(\{\langle E_\gamma\rangle:\gamma<\alpha\}\cup\{\langle D_\gamma\rangle:\gamma<\beta\}).$$

Then $\mathcal{J}(\alpha,\beta)\subseteq\mathcal{K}$ by Lemma 2(a), $\mathcal{J}(\alpha,\beta)$ is a Borel σ-ideal by Lemma 6 and it is of type $(\alpha;\beta)$ by Lemmas 4, 5. The rest is obvious.

The proof of Theorem 1. Put $\mathcal{J}(1,2)=\mathcal{K}$, $\mathcal{J}(2,1)=\mathcal{L}$, $\mathcal{J}(2,2)=\mathcal{K}\cap\mathcal{L}$ (comp. Example 1 and Corollary 2). Let the remaining σ-ideals be identical with those from Proposition 3. Define

$$\hat{\mathcal{J}}(\alpha,\beta)=\mathcal{K}(\mathbb{R})\oplus\mathcal{J}(\alpha,\beta)$$

in the way described in Example 5. By this example, $\hat{\mathcal{J}}(\alpha,\beta)$ is a non-Borel σ-ideal type $(\alpha;\beta)$.

R e m a r k . As was observed in [8] each connected locally compact abelian group has an element of infinite order. Note that Lemma 1 will be valid (see [8], th. 0) if we assume, instead of the local compactness of X, that X has an element of infinite order. Similarly, Lemmas 2-6 and Proposition 3 will then hold, as well. In the proof of Theorem 1, the assumption that X is locally compact is needed to construct σ-ideals of types (2;1), (2;2).

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