DEMONSTRATIO MATHEMATICA
Vol. XX No 1-2 1987

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THEORIES OF DEDUCTIVE SYSTEMS EQUIVALENT TO $\underline{T}_{\underline{W}}$

This note is to present some recent improvements in a sentential calculus named $\underline{W}$, and, mainly, in a theory of deductive systems, strongly adequate to it named $\underline{T}_{\underline{W}}$. Therefore, only indispensable definitions and theorems of earlier papers are quoted here; a full survey of previous results may be found in [2]. There is a comprehensive bibliography also there. Finally, present notation and terminology are those of [2] as well.

By the system $\underline{W}$ we mean the following sentential calculus given axiomatically:

its primitive terms are functors $\Rightarrow$, $\wedge$, and $\sim$, standing for implication, conjunction, and negation, respectively;

its axioms - the expressions denoted with the functors and sentential variables p, q, r:

A1. $(p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \Rightarrow r))$,

A2. $p \Rightarrow (q \Rightarrow p)$,

A3. $((p \Rightarrow q) \Rightarrow p) \Rightarrow p$,

A4. $p \wedge q \Rightarrow p$,

A5. $p \wedge q \Rightarrow q$,

A6. $p \Rightarrow (q \Rightarrow p \wedge q)$,

A7. $p \Rightarrow (\sim p \Rightarrow q)$,

The theory $\underline{T}_{\underline{W}}$ has been presented in [1]. The results dealt here with were announced first at the Conference on Universal Algebra held at the Pedagogical University in Opole (Jarmołtówek), May 23-27, 1984.

A8. $\sim(\sim p) \Rightarrow p$,

A9. $p \Rightarrow \sim(\sim p)$,

A10. $p \Rightarrow ((q \Rightarrow \sim q) \Rightarrow \sim(p \Rightarrow q))$,

A11. $\sim(p \wedge q) \Rightarrow \sim(q \wedge p)$,

A12. $\sim(p \wedge q) \Rightarrow ((p \Rightarrow \sim p) \Rightarrow \sim p)$,

A13. $p \wedge \sim q \Rightarrow \sim(p \wedge q)$,

A14. $\sim p \wedge \sim q \Rightarrow \sim(p \wedge q)$;

and its rules of inference - the modus ponens for $\Rightarrow$ and the substitution rule.

Subsequently, the notions of formula of $\underline{W}$ (or $\underline{W}$ - formula), theorem of $\underline{W}$, provability, and derivability in $\underline{W}$, are defined in the usual way.

a. The system $\underline{W}$ is equivalent to the sentential calculus determined by the three-valued matrix:

$$\underline{M}_3 = \langle \{1,0,1/2\}, \{1\}, \Rightarrow, \wedge, \sim \rangle,$$

with functions $\Rightarrow$, $\wedge$, and $\sim$ defined by the truth-tables:

$\Rightarrow$	1	0	1/2
1	1	0	0
0	1	1	1
1/2	1	1	1

$\wedge$	1	0	1/2
1	1	0	1/2
0	0	0	1/2
1/2	1/2	1/2	1/2

	$\sim$
1	0
0	1
1/2	1/2

In previous considerations on $\underline{W}$ the following notion of essential variable of a formula of $\underline{W}$ has been an important one:

if the formula is a sentential variable, this variable is its only essential one;

if the formula is of the form $\sim\varphi$, then its all and only essential variables are these of φ;

if the formula is of the form $\varphi \wedge \psi$, then its all and only essential variables are both these of φ and of ψ;

no variable is an essential one of $\varphi \Rightarrow \psi$.

This notion has been used in constructing a suppositional system of $\underline{W}$, a theory of deductive systems, strongly adequate to it, and in describing classes of these two-valued tautologies, which are also $\underline{M}_3$-tautologies.

It appears that the property "to be an essential variable of a formula", though obviously external of $\underline{W}$, may be, in some sense, expressed in it. Before the appropriate statement is given, let us quote theorems from [2] used later in its proof:

b. For every formula φ of $\underline{W}$ and every valuation h in $\underline{M}_3$, if $h(\varphi) = 1/2$, then there exists an essential variable p of φ such that $h(p) = 1/2$.

c. A variable p is an essential one of φ iff for every valuation h in $\underline{M}_3$ $h(p) = 1/2$ implies $h(\varphi) = 1/2$.

The following remark is also helpful for the proof:

If p is a sentential variable and h - a valuation in $\underline{M}_3$, then

$$1. \qquad h(\sim p \Rightarrow p) = \begin{cases} 1 & \text{iff} \quad h(p) \neq 0 \\ 0 & \text{iff} \quad h(p) = 0, \end{cases}$$

and

$$2. \qquad h((\sim p \Rightarrow p) \Rightarrow p) = \begin{cases} 1 & \text{iff} \quad h(p) \neq 1/2 \\ 0 & \text{iff} \quad h(p) = 1/2. \end{cases}$$

Given $\varphi_1, \ldots, \varphi_k$ formulas of an axiom systems L "$\varphi_1, \ldots, \varphi_k \vdash_L$" denotes the predicate of derivability from the set $\{\varphi_1, \ldots, \varphi_k\}$ in L and "$\vdash_L$" denotes the predicate of provability in it, in what follows.

T h e o r e m 1. If φ and ψ are arbitrary formulas of $\underline{W}$, then every essential variable of ψ is an essential one of φ iff

$$\vdash_{\underline{W}} ((\sim \varphi \Rightarrow \varphi) \Rightarrow \varphi) \Rightarrow ((\sim \psi \Rightarrow \psi) \Rightarrow \psi).$$

P r o o f . Given φ and ψ formulas of $\underline{W}$, it is enough, by a., to prove the following:

every essential variable of ψ is this of φ iff

$((\sim \varphi \Rightarrow \varphi) \Rightarrow \varphi) \Rightarrow ((- \psi \Rightarrow \psi) \Rightarrow \psi)$ is $\underline{M}_3$-tautology.

By way of contradiction, assume first that every essential variable of ψ is an essential one of φ, and that there is a valuation h in $\underline{M}_3$ such that

$$h(((\sim\varphi\Rightarrow\varphi)\Rightarrow\varphi)\Rightarrow((\sim\psi\Rightarrow\psi)\Rightarrow\psi)) \neq 1.$$

Then by the truth-table of $\Rightarrow$

$$h((\sim\varphi\Rightarrow\varphi)\Rightarrow\varphi) = h(\sim\psi\Rightarrow\psi) = 1$$

and $h(\psi) \neq 1$, hence, by 1. and 2.,

3. $h(\varphi) \neq 1/2$ and $h(\psi) = 1/2$.

Now, use b. to get such an essential variable p of ψ that $h(p) = 1/2$; by assumption, p is also an essential variable of φ, so by c. $h(\varphi) = 1/2$, which contradicts 3.

Suppose now that there exists p, an essential variable of ψ not being an essential one of φ. Then by b. and c. for h, a valuation in $\underline{M}_3$ that takes the value 1/2 for p only

$$h(\psi) = 1/2 \quad \text{and} \quad h(\varphi) \neq 1/2.$$

Thus 2. and the table of $\Rightarrow$ yield

$$h(((\sim\varphi\Rightarrow\varphi)\Rightarrow\varphi)\Rightarrow((\sim\psi\Rightarrow\psi)\Rightarrow\psi)) = 0,$$

which ends the proof.

We conclude this section with some consequences of the above theorem.

C o r o l l a r y 1. For every formula φ of $\underline{W}$ and every sentential variable p

i. p is an essential variable of φ iff

$$\vdash_{\underline{W}} ((\sim\varphi\Rightarrow\varphi)\Rightarrow\varphi)\Rightarrow(\sim p\Rightarrow p)\Rightarrow p);$$

ii. no variable is an essential one of φ iff

$$\vdash_{\underline{W}} (\sim\varphi\Rightarrow\varphi)\Rightarrow\varphi.$$

C o r o l l a r y 2. For $\varphi_1,\varphi_2,\dots,\varphi_n,\psi$, arbitrary formulas of W, every essential variable of ψ is an essential one of at least one of $\varphi_1,\varphi_2,\dots,\varphi_n$ iff

$$\vdash_{\underline{W}} ((\sim\varphi_1 \Rightarrow \varphi_1) \Rightarrow \varphi_1) \Rightarrow$$

$$\Rightarrow (\ldots \Rightarrow (((\sim\varphi_n \Rightarrow \varphi_n) \Rightarrow \varphi_n) \Rightarrow ((\sim\psi \Rightarrow \psi) \Rightarrow \psi))\ldots).$$

A suppositional system of W, as described in [2], may be now modified and, in fact, simplified with Corollary 2, but we are not going into this here.

The rest of the paper concerns the theory of deductive systems $\underline{T}_{\underline{W}}$ presented in [1]. This theory is an extension of Zermelo-Fraenkel set theory therefore only the new terms and axioms are discussed below.

Primitive terms of $\underline{T}_{\underline{W}}$ are

$$S,\ Cn,\ \Rightarrow,\ \wedge,\ \sim,$$

with S - a set, Cn - a function taking the power set P(S) into itself, and $\Rightarrow$, $\wedge$, $\sim$ - names for the corresponding primitive terms of $\underline{W}$. (The use of the same notation for those and their names does not, however, lead to any clashes).

To formulate axioms of $\underline{T}_{\underline{W}}$, and of theories related to it, some more notation and notions are needed:

the lower-case x,y,z,... denote, from now on, elements of S while the upper-case X,Y,Z,... - its subsets;

as a $\underline{T}_{\underline{W}}$-name of a $\underline{W}$-formula φ its S-substitution is taken; where by an S-substitution of a $\underline{W}$-formula we mean any expression resulting from φ after all its sentential variables are replaced with some of the variables x,y,z,..., provided that the same-shaped sentential variables are replaced with the same-shaped from among x,y,z,...

essential variable of an S-substitution of φ is defined analogously to that of φ itself, i.e.

a variable x is an essential one of an S-substitution of a $\underline{W}$-formula φ iff a sentential variable p replaced in φ by x is an essential one of φ.

The following expressions are the axioms of $\underline{T}_{\underline{W}}$:

T1. $\bar{\bar{S}} = \aleph_0$,

T2. $X \subseteq CnX \subseteq S$,

T3. $CnCnX = CnX$,

T4. if $X \subseteq Y$, then $CnX \subseteq CnY$,

T5. if $x \in CnX$, then there exists Y, a finite subset of X such that $x \in CnY$,

T6. $x \Rightarrow y,\ x \wedge y,\ \sim x \in S$,

T7. $x \Rightarrow y \in CnX$ iff $y \in Cn(X \cup \{x\})$,

T8. $Cn\{x,y\} = Cn\{x \wedge y\}$,

T9. $Cn\{x, \sim x\} = S$,

T10. $x \Rightarrow y \in CnX$ iff $Cn(X \cup \{x \wedge (y \Rightarrow \sim y)\}) = S$,

T11. if $Cn\{\varphi \wedge \sim\psi\} = S$, then $\varphi \Rightarrow \psi \in Cn\Lambda$,

T12. if $Cn\{\chi, \varphi \wedge \sim \psi\} = S$, then $\chi \Rightarrow (\varphi \Rightarrow \psi) \in Cn\Lambda$,

where φ, ψ, and χ are S-substitutions of $\underline{W}$-formulas such that in the scheme T11 every essential variable of ψ is this of φ, and in T12 every essential variable of ψ is this of χ.

It is known (see [1]) that $\underline{T}_{\underline{W}}$ is strongly adequate (see [3]) to $\underline{W}$, i.e. in $\underline{T}_{\underline{W}}$ one may prove that all S-substitutions of the axioms of $\underline{W}$ are in $Cn\Lambda$ and one may also prove every expression obtained from any of the axioms T1-T12 by exchanging every occurence of Cn with $Cn_{\underline{W}}$, where $Cn_{\underline{W}}$ is the consequence function determined by $\underline{W}$.

Now, let $\underline{T}'_{\underline{W}}$ be the theory resulting from $\underline{T}_{\underline{W}}$ by replacing the axioms T9 and T10, with the following single one:

T9'. $x \in CnX$ iff $Cn(X \cup \{x \Rightarrow \sim x\}) = S$,

and by omitting the scheme T12.

T h e o r e m 2. The theories $\underline{T}_{\underline{W}}$ and $\underline{T}'_{\underline{W}}$ are equivalent to each other.

The proof consists of three lemmas on $\underline{T}$, the theory obtained from $\underline{T}_{\underline{W}}$ by omitting the axioms T9, T10, and the schemas T11, T12.

L e m m a 1.

i. $\vdash_{T} x \in CnX$ iff $Cn(X \cup \{x\}) = CnX$,

ii. $\vdash_{T}$ if $x \in CnX$ and $x \Rightarrow y \in CnX$, then $y \in CnX$,

iii. $\vdash_{T} Cn\{x, \sim x\} = Cn\{x, x \Rightarrow \sim x\}$.

This is an easy consequence of T2, T3, T4, and T7.

L e m m a 2.

$\vdash_{T}(Cn\{x,\sim x\} = S$ and $(x \Rightarrow y \in CnX$ iff $Cn(X \cup \{x \wedge (y \Rightarrow \sim y)\}) = S))$
iff $(x \in CnX$ iff $Cn(X \cup \{x \Rightarrow \sim x\}) = S)$.

In other words, $\vdash_{T}$ ((T9 and T10) iff T9$'$).

P r o o f . First, suppose

$$Cn\{x, \sim x\} = S$$

and

$$x \Rightarrow y \in CnX \quad \text{iff} \quad Cn(X \cup \{x \wedge (y \Rightarrow \sim y)\} = S,$$

and let $x \in CnX$. Then obviously

$$x \in Cn(X \cup \{x \Rightarrow \sim x\}),$$

hence by Lemma 1 ii.

$$\sim x \in Cn(X \cup \{x \Rightarrow \sim x\}).$$

Thus

$$S = Cn\{x, \sim x\} \subseteq Cn(X \cup \{x \Rightarrow \sim x\}),$$

and

$$Cn(X \cup \{x \Rightarrow \sim x\}) = S$$

follows. This gives the "only if" part of T9$'$.

To prove the "if" part of it let

$$Cn(X \cup \{x \Rightarrow \sim x\}) = S.$$

Then

$$x \in Cn(X \cup \{x \Rightarrow \sim x\}),$$

so T7 yields that

4. $$(x \Rightarrow \sim x) \Rightarrow x \in CnX.$$

Since by T8 and Lemma 1 ii.

$$x, \sim x \in Cn\{((x \Rightarrow \sim x) \Rightarrow x) \wedge (x \Rightarrow \sim x)\}$$

holds, thus

$$Cn\{((x \Rightarrow \sim x) \Rightarrow x) \wedge (x \Rightarrow \sim x)\} = S.$$

Whence by T10

$$((x \Rightarrow \sim x) \Rightarrow x) \Rightarrow x \in Cn\,\Lambda,$$

and all the more

$$((x \Rightarrow \sim x) \Rightarrow x) \Rightarrow x \in CnX.$$

Now, use 4. and Lemma 1 ii. to get $x \in CnX$.

To end the proof, suppose

$$x \in CnX \quad \text{iff} \quad Cn(X \cup \{x \Rightarrow \sim x\}) = S.$$

Since $x \in Cn\{x\}$, so by assumption

$$Cn\{x, x \Rightarrow \sim x\} = S.$$

This, together with Lemma 1 iii., gives

$$Cn\{x, \sim x\} = S.$$

Finally, since by assumption again,

$$y \in Cn(X \cup \{x\}) \quad \text{iff} \quad Cn(X \cup \{x, y \Rightarrow \sim y\}) = S$$

thus T7 and T8 yield

$$x \Rightarrow y \in CnX \quad \text{iff} \quad Cn(X \cup \{x \wedge (y \Rightarrow \sim y)\}) = S,$$

and we are done.

L e m m a 3.

i. T11 $\vdash_{\mathrm{T}}$ T12,

ii. T12 $\vdash_{\mathrm{T}}$ T11.

P r o o f . i. Let φ, ψ, and χ be such S-substitutions of formulas of $\underline{W}$ that the antecedent of T12 holds, i.e. that every essential variable of ψ is also this of χ, and that

$$Cn\{\chi, \varphi \wedge \sim\psi\} = S.$$

Then by T8

$$Cn\{(\chi \wedge \varphi) \wedge \sim \psi\} = S.$$

Moreover, the definition of essential variable of an S-substitution of a $\underline{W}$-formula, and assumption on these of ψ imply that every essential variable of ψ is an essential one of $\chi \wedge \varphi$. Thus T11 may be applied with $\chi \wedge \varphi$ instead of φ to receive

$$\chi \wedge \varphi \Longrightarrow \psi \in Cn\Lambda,$$

which is, equivalent in $\underline{T}$ to

$$\chi \Longrightarrow (\varphi \Longrightarrow \psi) \in Cn\Lambda.$$

So, the proof of i. is completed.

ii. Now, let φ and ψ be such that the antecedent of T11 holds. Since

$$\varphi \Longrightarrow (\varphi \Rightarrow \psi) \in Cn\Lambda \quad \text{iff} \quad (\varphi \Rightarrow \psi) \in Cn\Lambda$$

and

$$Cn\{\varphi, \varphi \wedge \sim \psi\} = Cn\{\varphi \wedge \sim \psi\}$$

are provable in $\underline{T}$, thus to get

$$\varphi \Longrightarrow \psi \in Cn\Lambda$$

it suffices to take φ for χ in T12. Details are left to the reader.

Now, it is easy to see that Theorem 2 is an immediate consequence of the above lemmas. So, the theory $\underline{T}'_{\underline{W}}$, being simpler than $\underline{T}_{\underline{W}}$, is still strongly adequate to $\underline{W}$. Lemma 3 makes it sure that the same holds for the theory resulting from $\underline{T}'_{\underline{W}}$ after the scheme T11 is replaced by T12 in it.

Since in both theories, $T_{\underline{W}}$ and $T''_{\underline{W}}$, the notion of essential variable, external of these, is equally important, the question arises if it is possible to define a theory of deductive systems, strongly adequate to $\underline{W}$ not using that notion. In what follows, the positive answer to this question is given.

Let $T''_{\underline{W}}$ be the theory obtained from $T'_{\underline{W}}$ by replacing the scheme T11 with the following three axioms:

T10'. $Cn\{(\sim x \Rightarrow x) \Rightarrow x,\ (\sim y \Rightarrow y) \Rightarrow y\} = Cn\{(\sim (x \wedge y) \Rightarrow (x \wedge y)) \Rightarrow x \wedge y\}$,

T11'. $Cn\{\sim(x \Rightarrow y) \Rightarrow (x \Rightarrow y)\} = Cn\{x \Rightarrow y\}$,

T12'. $Cn\{\sim(\sim x)\} = Cn\{x\}$.

In this theory one may prove that all S-substitutions of the axioms of $\underline{W}$ are elements of $Cn\Lambda$. If, moreover, $T''_{\underline{W}}$ is extended by adding a definition of $Cn_{\underline{W}}$, the consequence function determined by the derivability in $\underline{W}$, then in such a theory one may prove all the expressions obtained from the axioms T1 - T8 and T9' - T12' by replacing Cn with $Cn_{\underline{W}}$. Easy proofs of these remarks are omitted.

The above proves the following

T h e o r e m 3. The theory $T''_{\underline{W}}$ is strongly adequate to $\underline{W}$.

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Received November 11, 1985.