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SOME PROPERTIES OF TENSOR PRODUCTS OF SEMIGROUPS

1. Introduction

Let $\mathcal{J}$ be the variety of all semigroups, $\mathcal{JC}$ be the variety of all commutative semigroups and $\mathcal{G}$ be the variety of all groups.

The tensor product in the class $\mathcal{JC}$ have been defined by T. Head [12]. The first properties of this tensor product have been investigated in a series of papers [3,6,12].

The tensor product in the class $\mathcal{J}$ have been defined by P.A.Grillet [5] in 1969. The tensor product in a variety of algebras have been defined by G.A.Fraser [4] in 1976.

In this paper we will interested in relation between the tensor products in $\mathcal{JC}$, $\mathcal{G}$ and $\mathcal{J}$ and similarities and differences of properties of tensor products in $\mathcal{J}$ and $\mathcal{JC}$.

D e f i n i t i o n 1. Let $A,B \in \mathcal{T}$ and $\mathcal{T}$ be a variety of algebras. Let $A \times B$ be the Cartesian product of A and B. An algebra $S \in \mathcal{T}$ is called a tensor product of A and B in $\mathcal{T}$ if there is a bihomomorphism $\omega: A \times B \longrightarrow S$, such that S is generated $\omega(A \times B)$ and for any $C \in \mathcal{T}$ and a bihomomorphism $\beta: A \times B \longrightarrow C$ there exists a homomorphism $\alpha: S \longrightarrow C$, such that $\beta = \omega\alpha$.

If $\mathcal{T} = \mathcal{J}$ then the tensor product A and B is denoted by $A \otimes B$ and $\omega(a,b) = a \otimes b$.

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The tensor product of A in B in $\mathcal{JC}$ we will call c-tensor product and we will denote it $A \otimes^c B$. The image of $(a,b) \in A \times B$ under $\omega : A \times B \longrightarrow A \otimes^c B$ we will write as $a \otimes^c b$.

The tensor product of groups A and B in $\mathcal{G}$ we will denote as $A \otimes^g B$.

By the Definition 1 follows next theorem in $\mathcal{J}$:

T h e o r e m 1 ([5]): For any $A,B \in \mathcal{J}$ the semigroup $A \otimes B \cong (A \times B)^*/\tau$, where $(A \times B)^*$ is a free semigroup on $A \times B$ and τ is the smallest congruence over the relation τ_0, which is defined on $(A \times B)^*$ in this way:

For any $a,c \in A$, $b,d \in B$ the conditions:

$(ac,b)\ \tau_0\ (a,b)(c,b)$

$(a,bd)\ \tau_0\ (a,b)(a,d)$ hold.

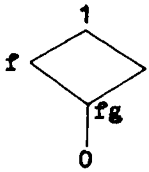
The congruence τ will be called the tensor congruence (on $(A \times B)^*$). The class of τ which contains the element $(a_1,b_1)\ldots(a_n,b_n) \in (A \times B)^*$ is $(a_1 \otimes b_1)\ldots(a_n \otimes b_n) \in A \otimes B$.

The c-tensor product is given analogously. If $A,B \in \mathcal{JC}$, then $A \otimes^c B \cong (A \times B)^{*c}/\tau$, where $(A \times B)^{*c}$ is the free commutative semigroup on $A \times B$. The congruence τ and the element $(a_1 \otimes^c b_1)\ldots(a_n \otimes^c b_n) \in A \otimes^c B$ are given in the some way.

E x a m p l e 1. Let G be the two-elements semigroup $\{0,1\}$ with a zero element 0 and a unique element 1.

By [5] the tensor product $G \otimes G \cong \{0,1\} \cup C$, where C is the semigroup generating by elements f,g with identities $ff = f$, $gg = g$ and 0 is a zero element of $G \otimes G$ and 1 is a unique element of $G \otimes G$.

The c-tensor product $G \otimes^c G \cong D$, where $D = \{0,1,f,g,fg\}$ is the next lower semilattice



D e f i n i t i o n 2. A semigroup A is called N-semigroup, if for any $x,y \in A$ and any natural number $n \in N$ the condition $(xy)^n = x^n y^n$ holds.

Let A be a semigroup. Then we will denote:

C(A) - the greatest commutative homomorphic image of A

E(A) - the greatest idempotent homomorphic image of A

N(A) - the greatest homomorphic image of A which is an N-semigroup

E - a one element semigroup

F - a free semigroup with one generator

G - the semigroup by Example 1

X^* - a free semigroup on a set X

X^{*c} - a free commutative semigroup on a set X.

Example 2. ([5]). Let A be a semigroup, then $A \otimes E \cong E(A)$ and $A \otimes F \cong N(A)$ hold.

Which is a relation between $\otimes^c$, $\otimes^g$ and $\otimes$?

Theorem 2 ([6]): Let A,B be commutative semigroups, then $A \otimes^c B \cong C(A \otimes B)$ holds.

Theorem 3 ([5]): Let A,B be groups, then $A \otimes^g B \cong A \otimes B$ holds.

Theorem 3′ ([6]): Let A,B be commutative groups, then $A \otimes^c B$ is the same that the tensor product commutative groups A and B.

Let us remark, that it is not know when $A \otimes^{\mathcal{V}} B \cong A \otimes^{\mathcal{T}} B$, if $\mathcal{V}$ is a subvariety of $\mathcal{T}$ and $A,B \in \mathcal{V}$.

Example 3: Let A be a commutative semigroup, then

$$A \otimes^c E \cong C(A \otimes E) \cong C(E(A)) \cong E(A)$$

$$A \otimes^c F \cong C(A \otimes F) \cong C(N(A)) \cong C(A) \cong A.$$

2. Some properties of $\otimes$ and $\otimes^c$

The functors $\otimes^c$ on $\mathcal{JC}$ and $\otimes$ on $\mathcal{J}$ have interesting properties. We will mention to some of their. Let us remark that in most of papers about tensor products of semigroups have been studied the c-tensor product. The tensor product in $\mathcal{J}$ have been studied only in [5,9,10,11,25,26]. In the References is given the most of papers about tensor products of semigroups.

T h e o r e m 4 ([5]): a) For every $A,B \in \mathcal{J}$ the commutative low $A \otimes B \cong B \otimes A$ holds.

b) There exist semigroups $A,B,C \in \mathcal{J}$, such that $(A \otimes B) \otimes C \ncong A \otimes (B \otimes C)$.

For example, if $A = B = G$ and $C = E$, then by Example 2 we have

$$(G \otimes G) \otimes E \cong E(G \otimes G)$$

$$G \otimes (G \otimes E) \cong G \otimes E(G) \cong G \otimes G$$

and by Example 1 $G \otimes G$ is not a idempotent semigroup. That means $(G \otimes G) \otimes E \ncong G \otimes (G \otimes E)$.

There are example where the associative low holds. For example $S \otimes (S \otimes S) \cong (S \otimes S) \otimes S$ for any semigroup S by Theorem 4a.

T h e o r e m 4'. ([6]): If $A,B,C \in \mathcal{JC}$, then $A \otimes^c B \cong B \otimes^c A$ and $(A \otimes^c B) \otimes^c C \cong A \otimes^c (B \otimes^c C)$ hold.

By Example 1, the tensor product $\otimes$ of finite semigroups need not be a finite semigroup and the tensor product $\otimes$ of idempotent semigroups need not be a idempotent semigroup. It is not known any condition for preservation of these properties with the tensor product.

T h e o r e m 6 ([11]): If $A,B \in \mathcal{J}$ and $card(A-A^2) > 1$ and $card(B-B^2) > 1$, then $A \otimes B$ is infinite non-commutative semigroup.

It is the following situation for $\otimes^c$:

T h e o r e m 6': a) If $A,B \in \mathcal{JC}$, A is a finite semigroup and B is a finite generated semigroup, then $A \otimes^c B$ is finite.

b) If $A,B \in \mathcal{JC}$ and A is a idempotent semigroup, then $A \otimes^c B$ is a idempotent semigroup.

Theorem 6'a is proved in [6]. Theorem 6'b follows from commutativity of $A \otimes^c B$ and from the fact $(a \otimes^c b)^2 = a^2 \otimes^c b = a \otimes^c b$ for $a \in A$, $b \in B$.

Now we are interested in codomains of $\otimes$ and $\otimes^c$.

T h e o r e m 7 ([10]): Let S be a semigroup, $card(S-S^2)>1$ and there exists for every $x,y \in S-S^2$, $x \neq y$, a natural number $n \in N$, $n>1$, such that $(xy)^n \neq x^n y^n$. Then S is not a tensor product of semigroups.

By [10] from Theorem 7 follows that there exist infinitely many semigroups, finite and infinite, which are not tensor products ⊗ of semigroups. For example, any free semigroup X^*, where $cardX>1$, is not a tensor product of semigroups.

Finite semigroups, which are not a tensor product of semigroups, we can construct in this way:

Let X^* be a free semigroup, $1<cardX<\infty$ and J be a ideal of all words of X^* the lenght of which is greater then a fixed natural number k, $k \geqq 4$. Then the Rees factor semigroup X^*/J is not a tensor product of semigroups.

There exist classes of semigroups which are a tensor product of semigroups.

T h e o r e m 8 ([10]): Every N-semigroup is a tensor product of some semigroups.

Because any commutative semigroup is an N-semigroup, then the class of all commutative semigroups is obtain in the codomain of ⊗.

It is not known any necessary and sufficient condition for the fact, if a semigroup is an element of the codomain of ⊗.

In this point of view the situation is simply for c-tensor product of commutative semigroups.

T h e o r e m 7′ ([10]): Every commutative semigroup is a c-tensor product of some commutative semigroups.

3. Classes of the tensor congruence

Some investigations of tensor products of semigroups given the question, when are two elements of a tensor product of semigroups different. Some results of this problem in $\mathcal{J}$ and $\mathcal{J}\mathcal{C}$ is given i [9,10].

T h e o r e m 9 ([10]): Let $A,B \in \mathcal{J}$ and $a_1,\dots,a_n \in A-A^2$, $b_1,\dots,b_n \in B-B^2$, where $a_i \neq a_{i+1}$, $b_i \neq b_{i+1}$ for

$i=1,\dots,n-1$. The class $(a_1 \otimes b_1)\dots(a_n \otimes b_n)$ of the tensor congruence on $(A \times B)^*$ contains only unique element, namely $(a_1,b_1)\dots(a_n,b_n)$.

R e m a r k . In special case $n=2$, if we have indecomposable elements a,b,c,d and $a,c \in A$; $b,d \in B$ then $a \otimes b \neq c \otimes d$ iff $(a,b) \neq (c,d)$ in $(A \times B)^*$.

T h e o r e m 9′ ([10]): Let $A,B \in \mathcal{IC}$ and $a,c \in A$, $b,d \in B$ are indecomposable elements. Then $a \otimes^c b \neq c \otimes^c d$ iff $(a,b) \neq (c,d)$ in $(A \times B)^{*c}$.

T h e o r e m 10 ([9]): If A,B are idempotent semigroups and $a,c \in A$, $b,d \in B$, then $a \otimes b \neq c \otimes d$ iff $(a,b) \neq (c,d)$ in $(A \times B)^*$.

Theorem 10 is a corollary of the next theorem:

T h e o r e m 11 ([9]): Let $H(A)$ be an idempotent homomorphic image of $A \in \mathcal{I}$ and δ be that homomorphism. If $\delta(a_1\dots a_n) \neq \delta(c_1,\dots,c_m)$ for $a_1,\dots,a_n,c_1,\dots,c_m \in A$, then for all elements $b_1,\dots,b_n,d_1,\dots,d_m \in B$, $B \in \mathcal{I}$, is

$$(a_1 \otimes b_1)\dots(a_n \otimes b_n) \neq (c_1 \otimes d_1)\dots(c_m \otimes d_m).$$

It is clear, that the interesting case is $H(A) = E(A)$. By [5, Example 2.3] Theorem 11 is given only sufficient condition, not necessary.

T h e o r e m 11′ ([9]): Let $A,B \in \mathcal{IC}$, $H(A)$ be an idempotent homomorphic image of A and δ be this homomorphism. If $\delta(a_1\dots a_n) \neq \delta(c_1\dots c_m)$ for $a_1,\dots,a_n,c_1,\dots,c_m \in A$, then for all elements $b_1,\dots,b_n,d_1,\dots,d_m \in B$ are

$$(a_1 \otimes^c b_1)\dots(a_n \otimes^c b_n) \neq (c_1 \otimes^c d_1)\dots(c_m \otimes^c d_m) \quad \text{in} \quad A \otimes^c B.$$

If in Theorem 11 we have $H(A) = E(A)$, then we have some interesting corollaries:

T h e o r e m 12. Let A be an idempotent semigroup and $B,C \in \mathcal{I}$. Then

a) $A \otimes B \cong A \otimes E(B)$,

b) $A \otimes B \cong A \otimes C$, if $E(B) \cong E(C)$.

Theorem 13. Let A be finite idempotent semigroup. Then a semigroup X is the solution of a equation $A \otimes X \cong A$ iff $\mathrm{card} E(X) = 1$.

In the special case we can take X equal of any group or a bicyclic semigroup.

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