DEMONSTRATIO MATHEMATICA
Vol XX No 1-2 1987

Antoni Chronowski

EXTENSIONS OF INVERSE LOOPS

The paper consists of two parts. The first one contains necessary and sufficient conditions for extensions of inverse loops. The second one deals with mappings generating extensions of inverse loops.

The terms a left inverse loop and a right inverse loop are used by Belousov [1]. The terminology concerning these notions is not uniform, some authors use different terminology (cf. [4]).

Notions as quasigroup, loop, subloop, normal subloop, coset, quotient loop are used according to Bruck [2].

1. We start with the following definition.

Definition 1 (cf.[3]). A loop Σ is said to be an extension of a loop K by a loop L if the following conditions hold:

(i) K is a normal subloop of the loop Σ,

(ii) the quotient loop Σ/K and the loop L are isomorphic.

Let $(\Sigma, \circ)$ be a loop. We define a mapping $I : \Sigma \to \Sigma$ in the following way

$$x \circ I(x) = 1 \tag{1}$$

for $x \in \Sigma$.

We shall often use the same letter I for a mapping of type (1) in different loops. The mapping I is a bijection and $I(1) = 1$.

In the set Σ we define an operation ${}^{-1}\circ$ by the formula

$$x \,{}^{-1}\circ\, y = z \Leftrightarrow z \circ y = x$$

for $x,y,z \in \Sigma$.

An algebraic structure $(\Sigma, {}^{-1}\circ)$ is a quasigroup but it is not generally a loop (cf. [1]).

In the set Σ we define an operation / by the formula

$$x/y = z \Leftrightarrow x \,{}^{-1}\circ\, I(y) = z$$

for $x,y,z \in \Sigma$.

An algebraic structure $(\Sigma, /)$ is a loop isotopic to the quasigroup $(\Sigma, {}^{-1}\circ)$ (cf. [1]). A triple $(id_\Sigma, I, id_\Sigma)$ is an isotopy of $(\Sigma, /)$ upon $(\Sigma, {}^{-1}\circ)$. An identity of the loop $(\Sigma, /)$ is an identity of the loop $(\Sigma, \circ)$. If I and I_1 are mappings of type (1) in the loops $(\Sigma, \circ)$ and $(\Sigma, /)$, respectively, then $I_1 = I^{-1}$.

Let us notice that

$$x / y = z \Leftrightarrow x \,{}^{-1}\circ\, I(y) = z \Leftrightarrow z \circ I(y) = x$$

for $x,y,z \in \Sigma$.

Further on we shall often omit the sign $\circ$ in the loop $(\Sigma, \circ)$ writing xy and Σ instead of $x \circ y$ and $(\Sigma, \circ)$. Thus,

$$x / y = z \Leftrightarrow z\, I(y) = x$$

for $x,y,z \in \Sigma$.

D e f i n i t i o n 2 (cf. [1]). A loop $(\Sigma, /)$ is called a left inverse loop of a loop Σ.

L e m m a 1 (cf. [1]). If $(\Sigma, /)$ is a left inverse loop of a loop Σ, then Σ is a left inverse loop of the loop $(\Sigma, /)$.

L e m m a 2. Let K and Σ be loops. K is a normal subloop of the loop Σ if and only if the left inverse loop $(K, /)$ is a normal subloop of the left inverse loop $(\Sigma, /)$.

P r o o f . (i) Let K be a normal subloop of the loop $\sum$. The mapping I of type (1) is a bijection on the set K. It is easy to check that (K, /) is a subloop of the loop ($\sum$, /). To show that (K, /) is a normal subloop of the loop ($\sum$, /) we are going to prove the following conditions:

(a) $x / K = K / x$,

(b) $x / (y / K) = (x / y) / K$,

(c) $(K / x) / y = K / (x / y)$,

for $x, y \in \sum$.

(a)
$$z \in x / K \Leftrightarrow \bigvee_{k \in K} z = x / k \Leftrightarrow \bigvee_{k \in K} z\, I(k) = x \Leftrightarrow K z = $$
$$= K x \Leftrightarrow (K z) I(x) = (K x) I(x) \Leftrightarrow K(z\, I(x)) = K \Leftrightarrow \bigvee_{k_1 \in K} z I(x) = $$
$$= k_1 \Leftrightarrow \bigvee_{k_1 \in K} z = k_1 / x \Leftrightarrow z \in K / x.$$

(b) Suppose that $z \in x / (y / K)$ then $z = x / (y / k)$, where $k \in K$. Let us denote $u = y / k$.

We get the following equivalences:

$y / k = u \Leftrightarrow u\, I(k) = y$ and $x / u = z \Leftrightarrow z\, I(u) = x$. Thus
$$u K = K y \Rightarrow u(K\, I(y)) = u(K\, I(u)) \Rightarrow K\, I(y) = K\, I(u) \Rightarrow$$
$$\Rightarrow (z\, I(y))K = (z\, I(u))K \Rightarrow (z\, K)I(y) = x\, K \Rightarrow \bigvee_{k' \in K} (zk')I(y) = $$
$$= x \Rightarrow \bigvee_{k' \in K} x / y = zk' \Rightarrow zK = (x / y)K \Rightarrow \bigvee_{k_1 \in K} zI(k_1) = $$
$$= x / y \Rightarrow \bigvee_{k_1 \in K} z = (x / y) / k_1 \Rightarrow z \in (x / y) / K.$$

Suppose that $z \in (x / y) / K$ then $z = (x / y) / k$ for some $k \in K$. Let us denote $u = x / y$.

We get the following equivalences:

$x / y = u \Leftrightarrow uI(y) = x$ and $u / k = z \Leftrightarrow z\, I(k) = u$. Thus
$$Kz = Ku \Rightarrow K(zI(y)) = K(uI(y)) = Kx \Rightarrow z(KI(y)) = Kx \Rightarrow$$
$$\Rightarrow \bigvee_{k' \in K} z(k' I(y)) = x.$$
There exists a $w \in \sum$ such that $I(w) = k' I(y)$ and so $zI(w) = x$ that is $z = x / w$. Moreover,
$$w(K\, I(w)) = (yK)\, I(y) \Rightarrow w[K(k' I(y))] = (yK)\, I(y) \Rightarrow (wK)I(y) = $$
$$= (yK)I(y) \Rightarrow wK = yK \Rightarrow \bigvee_{k_1 \in K} w\, I(k_1) = y \Rightarrow \bigvee_{k_1 \in K} y / k_1 = w.$$

Since $z = x / w$ and so $z = x / (y / k_1)$ for some $k_1 \in K$, that is $z \in x / (y / K)$.

The proof of (c) is similar.

(ii) Let $(K, /)$ be a normal subloop of the left inverse loop $(\Sigma, /)$.

Using part (i) of this proof and Lemma 1 we obtain the second part of the proof.

L e m m a 3. Let K be a normal subloop of a loop Σ. Then the left inverse loop $(\Sigma/K, /)$ of the quotient loop Σ/K modulo K and the quotient loop of the left inverse loop $(\Sigma, /)$ modulo $(K, /)$ are identical.

P r o o f . At first, we shall show that $x K = x/K$ for an arbitrary $x \in \Sigma$. Indeed, $z \in x K \Leftrightarrow z K = x K \Leftrightarrow$ $\Leftrightarrow \bigvee_{k \in K} z I(k) = x \Leftrightarrow \bigvee_{k \in K} z = x / k \Leftrightarrow z \in x / K$. Notice that $I(x K) = I(x)K$ for $x K \in \Sigma/K$. In fact, $(x K)(I(x)K) =$ $= (x I(x))K = K$.

In the quotient loop $(\Sigma/K, \phi)$ of the loop $(\Sigma, /)$ modulo $(K, /)$ is defined an operation ϕ in the following way

$$(x K) \phi (y K) = (x / y)K$$

for arbitrary $x K, y K \in \Sigma/K$.

In the quotient loop Σ/K of the loop Σ modulo K is defined an operation by the formula

$$(x K)(y K) = (x y) K$$

for arbitrary $x K, y K \in \Sigma/K$.

We shall prove that the quotient loop $(\Sigma/K, \phi)$ and the left inverse loop $(\Sigma/K, /)$ of the quotient loop Σ/K are identical. Indeed, $(x K) / (y K) = z K \Leftrightarrow (zK)I(yK) = x K \Leftrightarrow$ $\Leftrightarrow (z K)(I(y)K) = x K \Leftrightarrow (zI(y))K = x K \Leftrightarrow \bigvee_{k \in K} zI(y) =$ $= x k \Leftrightarrow \bigvee_{k \in K} (x k) / y = z \Leftrightarrow (x K) / (y K) = z K \Leftrightarrow$ $\Leftrightarrow (x / y) K = z K$ for $x K, y K, z K \in \Sigma/K$.

Then, $(x K) / (y K) = (x / y)K = (x K) \phi (y K)$ for arbitrary $x K, y K \in \Sigma/K$.

L e m m a 4. Let Σ_1 and Σ_2 be loops. A mapping $f : \Sigma_1 \to \Sigma_2$ is an isomorphism of the loops Σ_1 and Σ_2 if and only if the mapping f is an isomorphism of the left inverse loops $(\Sigma_1, /)$ and $(\Sigma_2, /)$.

P r o o f . (i) Let $f : \Sigma_1 \to \Sigma_2$ be an isomorphism of the loops Σ_1 and Σ_2.

Since $f(x\ I(x)) = 1$ then $f(x)f(I(x)) = 1$, hence $f(I(x)) = I(f(x))$ for $x \in \Sigma_1$. We have $f(x / y) = f(z) \Leftrightarrow x / y = z \Leftrightarrow z\ I(y) = x \Leftrightarrow f(z\ I(y)) = f(x) \Leftrightarrow f(z)f(I(y)) = f(x) \Leftrightarrow f(z)I(f(y)) = f(x) \Leftrightarrow f(x) / f(y) = f(z)$ for $x,y,z \in \Sigma_1$. Thus, $f(x / y) = f(x) / f(y)$ for arbitrary $x,y \in \Sigma_1$.

(ii) Let $f : \Sigma_1 \to \Sigma_2$ be an isomorphism of the loops $(\Sigma_1, /)$ and $(\Sigma_2, /)$.

Using part (i) of this proof and Lemma 1 we get the second part of the proof.

T h e o r e m 1. Let Σ, K and L be loops. The loop Σ is an extension of the loop K by the loop L if and only if the left inverse loop $(\Sigma, /)$ is an extension of the left inverse loop (K, /) by the left inverse loop (L, /).

P r o o f . (i) Let the loop Σ be an extension of the loop K by the loop L.

It follows from Lemma 2 that the loop (K, /) is a normal subloop of the loop $(\Sigma, /)$. In virtue of Lemma 3 the quotient loop of the loop $(\Sigma, /)$ modulo (K, /) and the left inverse loop $(\Sigma/K, /)$ of the loop Σ/K are identical. Since the loops L and Σ/K are isomorphic, then by Lemma 4 we get that the loops (L, /) and $(\Sigma/K, /)$ are isomorphic.

Thus, the loop $(\Sigma, /)$ is an extension of the loop (K, /) by the loop (L, /).

(ii) Let the left inverse loop $(\Sigma, /)$ be an extension of the left inverse loop (K, /) by the left inverse loop (L, /). Using part (i) of this proof and Lemma 1 we obtain the second part of the proof.

We shall define a right inverse loop of a loop $(\Sigma, \circ)$.

In the set Σ we define an operation $\circ^{-1}$ by the formula

$$x \circ^{-1} y = z \Leftrightarrow x \circ z = y$$

for $x,y,z \in \Sigma$.

An algebraic structure $(\Sigma, \circ^{-1})$ is a quasigroup but it is not generally a loop.

In the set Σ we define an operation $\backslash$ by the following formula

$$x \backslash y = z \Leftrightarrow I^{-1}(x) \circ^{-1} y = z$$

for $x,y,z \in \Sigma$.

An algebraic structure $(\Sigma, \backslash)$ is a loop isotopic to the quasigroup $(\Sigma, \circ^{-1})$ (cf. [1]). A triple $(I^{-1}, id_{\Sigma}, id_{\Sigma})$ is an isotopy of $(\Sigma, \backslash)$ upon $(\Sigma, \circ^{-1})$. An identity of the loop $(\Sigma, \backslash)$ is an identity of the loop $(\Sigma, \circ)$.

Omitting the sign of the operation $\circ$ in the loop $(\Sigma, \circ)$ we can write

$$x \backslash y = z \Leftrightarrow I^{-1}(x)\, z = y$$

for $x,y,z \in \Sigma$.

D e f i n i t i o n 3 (cf. [1]). A loop $(\Sigma, \backslash)$ is called a right inverse loop of a loop Σ.

L e m m a 5 (cf. [1]). If $(\Sigma, \backslash)$ is a right inverse loop of a loop Σ, then Σ is a right inverse loop of the loop $(\Sigma, \backslash)$.

By analogy with the case of left inverse loops we get the following

T h e o r e m 2. Let Σ, K and L be loops. The loop Σ is an extension of the loop K by the loop L if and only if the right inverse loop $(\Sigma, \backslash)$ is an extension of the right inverse loop $(K, \backslash)$ by the right inverse loop $(L, \backslash)$.

Let $(\Sigma, /^*)$ be a left inverse loop of a loop $(\Sigma, \backslash)$ and $(\Sigma, \backslash^*)$ be a right inverse loop of a loop $(\Sigma, /)$. Let $(\Sigma, \circ^*)$ be a right inverse loop of a loop $(\Sigma, /^*)$.

We get the following loops:

$(\Sigma, \circ)$, $(\Sigma, /)$, $(\Sigma, \backslash)$, $(\Sigma, /^*)$, $(\Sigma, \backslash^*)$, $(\Sigma, \circ^*)$.

One can prove that continuing the process of the inversion of loops we get, up to isomorphism, the only six above-mentioned loops (cf. [1]).

To simplify the formulation of the next theorem we denote loops with the operations $\circ, /, \backslash, /^*, \backslash^*, \circ^*$ by capital letters with indices 0, 1, 2, 3, 4, 5, respectively.

In virtue of Theorems 1 and 2 we have the following

T h e o r e m 3. A loop Σ_i is an extension of a loop K_i by a loop L_i if and only if a loop Σ_j is an extension of a loop K_j by a loop L_j for arbitrary $i,j \in \{0,1,2,3,4,5\}$.

2. Let K and L be loops.

Let a mapping $\varphi : L \times K \times L \times K \longrightarrow K$ satisfy the following conditions:

1° $\varphi(l, k, 1, 1) = (1, 1, l, k) = k$,

2° $\varphi(1, k_1, 1, k_2) = k_1 k_2$,

3° the mapping $\varphi(l_1, k_1, l_2, \cdot) : K \longrightarrow K$ is a bijection,

4° the mapping $\varphi(l_1, \cdot, l_2, k_2) : K \longrightarrow K$ is a bijection,

for $l, l_1, l_2 \in L$ and $k, k_1, k_2 \in K$.

It follows from the considerations in [3] the following corollary.

By means of an arbitrary mapping φ satisfying conditions 1° - 4° we get, up to isomorphism, an extension of the loop K by the loop L. An arbitrary extension of the loop K by the loop L one can obtain, up to isomorphism, by means of a mapping satisfying conditions 1° - 4°.

D e f i n i t i o n 4. A mapping $\varphi : L \times K \times L \times K \longrightarrow K$ satisfying conditions 1° - 4° is said to be a mapping generating an extension of a loop K by a loop L.

L e m m a 6. If φ is a mapping generating an extension of a loop K by a loop L, then the following condition:

(2) $$\bigwedge_{l_1,l_2\in L}\ \bigwedge_{k,k_1,k_2\in K} \varphi_1(l_1,k_1,l_2,k_2) = k \Leftrightarrow \bigvee_{k_0\in K}\Big(\varphi(l_2,k_2,$$

$$I(l_2),k_0) = 1 \wedge \varphi(l_1/l_2,k,I(l_2),k_0) = k_1\Big),$$

where I is a mapping of type (1) in the loop L, defines a mapping generating an extension of the left inverse loop (K, /) by the left inverse loop (L, /).

P r o o f . Applying conditions 3^0 and 4^0 to the mapping φ, it is easy to see that condition (2) defines a mapping $\varphi_1 : L \times K \times L \times K \longrightarrow K$. We shall prove that the mapping φ_1 satisfies conditions 1^0 - 4^0 for the loops (K, /) and (L, /).

Notice that $\varphi_1(l_1,k_1,1,1) = k \Leftrightarrow \bigvee_{k_0\in K} (\varphi(1,1,1,k_0) =$

$$= 1 \wedge \varphi(l_1,k,1,k_0) = k_1) \Leftrightarrow \bigvee_{k_0\in K} (k_0 = 1 \wedge k = k_1) \Leftrightarrow k = k_1$$

for $l_1 \in L$ and $k,k_1 \in K$. Similarly, $\varphi_1(1,1,l_2,k_2) = k \Leftrightarrow$

$$\Leftrightarrow \bigvee_{k_0\in K} (\varphi(l_2,k_2,I(l_2),k_0) = 1 \wedge \varphi(l_2,k,I(l_2),k_0) = 1) \Leftrightarrow$$

$\Leftrightarrow k = k_2$ for $l_2 \in L$ and $k,k_2 \in K$. Then the mapping φ_1 satisfies condition 1^0.

The mapping φ_1 satisfies condition 2^0. Indeed,

$$\varphi_1(1,k_1,1,k_2) = k \Leftrightarrow \bigvee_{k_0\in K} (\varphi(1,k_2,1,k_0) = 1 \wedge \varphi(1,k,1,k_0) = k_1) \Leftrightarrow$$

$$\Leftrightarrow \bigvee_{k_0\in K} (k_2k_0 = 1 \wedge k\,k_0 = k_1) \Leftrightarrow \bigvee_{k_0\in K} (k_0 = I(k_2) \wedge$$

$$\wedge\, k\,I(k_2) = k_1) \Leftrightarrow \bigvee_{k_0\in K} (k_0 = I(k_2) \wedge k_1 / k_2 = k) \Leftrightarrow$$

$\Leftrightarrow k_1 / k_2 = k$ for $k_1,k_2 \in K$.

Let us take arbitrary fixed elements $l_1,l_2 \in L$ and $k_1 \in K$. We shall prove that the mapping $\varphi_1(l_1,k_1,l_2,\cdot) : K \longrightarrow K$ is a bijection. Suppose that $\varphi_1(l_1,k_1,l_2,k_2) = k$ and $\varphi_1(l_1,k_1,l_2,k_2') = k$ and for $k_2,k_2' \in K$. It is easy to check that there exists $k_0 \in K$ such that $\varphi(l_2,k_2,I(l_2),k_0) = 1$ and $\varphi(l_2,k_2',I(l_2),k_0) = 1$, hence $k_2 = k_2'$. If $k \in K$ then there exist $k_0, k_2 \in K$ such that $\varphi(l_1/l_2, k, I(l_2), k_0) = k_1$ and $\varphi(l_2, k_2, I(l_2), k_0) = 1$ that is $\varphi_1(l_1, k_1, l_2, k_2) = k$. Thus, the mapping φ_1 satisfies condition 3^o.

By analogy with condition 3^o we can prove that the mapping φ_1 satisfies condition 4^o.

Then φ_1 is a mapping generating an extension of the loop (K, /) by the loop (L, /).

Let F_0 be a family of all mappings generating extensions of a loop K by a loop L. Let F_1 be a family of all mappings defined by condition (2) for any mapping $\varphi \in F_0$.

T h e o r e m 4 . The family F_1 is a family of all mappings generating extensions of the left inverse loop (K, /) by the left inverse loop (L, /).

P r o o f . It follows from Lemma 6 that every mapping $\varphi_1 \in F_1$ is a mapping generating an extension of the loop (K, /) by the loop (L, /). Let φ_1 be an arbitrary mapping generating an extension of the loop (K, /) by the loop (L, /).

We define a mapping $\varphi : L \times K \times L \times K \to K$ as follows:

$$\bigwedge_{l_1, l_2 \in L} \ \bigwedge_{k, k_1, k_2 \in K} \Big[\varphi(l_1, k_1, l_2, k_2) = k \Leftrightarrow \bigvee_{k_0 \in K} (\varphi_1(l_2, k_2, I_1(l_2), k_0) = 1 \wedge \varphi_1(l_1 l_2, k, I_1(l_2), k_0) = k_1\Big],$$

where I_1 is a mapping of type (1) in the loop (L, /). It follows from Lemmas 1 and 6 that $\varphi \in F_0$.

We shall prove that the mappings φ and φ_1 satisfy condition (2). Indeed,

$$\bigvee_{k_0 \in K} \Big[\varphi(l_2, k_2, I(l_2), k_0) =$$

$$= 1 \wedge \varphi(l_1 / l_2, k, I(l_2), k_0) = k_1\Big] \Leftrightarrow$$

$$\Leftrightarrow \bigvee_{k_0 \in K} \Big[\bigvee_{k_0' \in K} (\varphi_1(I(l_2), k_0, l_2, k_0') =$$

$$= 1 \wedge \varphi(l_2 I(l_2), 1, l_2, k_0') = k_2) \wedge \bigvee_{k_0'' \in K} (\varphi_1 (I(l_2), k_0, l_2, k_0'') =$$

$$= 1 \wedge \varphi_1((l_1 / l_2) I(l_2), k_1, l_2, k_0'') = k)\Big] \Leftrightarrow$$

$$\Leftrightarrow \bigvee_{k_0\in K}\Big[\bigvee_{k_0'\in K}(\varphi_1(I(l_2),k_0,l_2,k_0') =$$

$$= 1 \wedge \varphi_1(1,1,l_2,k_0') = k_2) \wedge$$

$$\bigvee_{k_0''\in K}(\varphi_1(I(l_2),k_0,l_2,k_0'') = 1 \wedge \varphi_1(l_1,k_1,l_2,k_0'') = k)\Big] \Leftrightarrow$$

$\Leftrightarrow \varphi_1(l_1,k_1,l_2,k_2) = k$ for $l_1,l_2 \in L$ and $k,k_1,k_2 \in K$.

The proof of the theorem is complete.

L e m m a 7. If φ is a mapping generating an extension of a loop K by a loop L, then the following condition:

(3) $$\bigwedge_{l_1,l_2\in L}\ \bigwedge_{k,k_1,k_2 \in K}\Big[\varphi_2(l_1,k_1,l_2,k_2) = k \Leftrightarrow$$

$$\bigvee_{k_0\in K}(\varphi(I^{-1}(l_1),k_0,l_1,k_1) =$$

$$= 1 \wedge \varphi(I^{-1}(l_1),k_0,l_1 \setminus l_2,k) = k_2)\Big],$$

where I is a mapping of type (1) in the loop L, defines a mapping generating an extension of the right inverse loop $(K,\setminus)$ by the right inverse loop $(L,\setminus)$.

The proof of this lemma is analogous to the proof of Lemma 6.

Let F_2 be a family of all mappings defined by condition (3) for any mapping $\varphi \in F_0$.

T h e o r e m 5. The family F_2 is a family of all mappings generating an extension the right inverse loop $(K,\setminus)$ by the right inverse loop $(L,\setminus)$.

P r o o f . It follows from Lemma 7 that an arbitrary mapping $\varphi_2 \in F_2$ is a mapping generating an extension of the loop $(K,\setminus)$ by the loop $(L,\setminus)$.

Let φ_2 be an arbitrary mapping generating an extension of the loop $(K, \backslash)$ by the loop $(L, \backslash)$. We define a mapping $\varphi : L \times K \times L \times K \longrightarrow K$ as follows

$$\bigwedge_{l_1, l_2 \in L} \bigwedge_{k, k_1, k_2 \in K} \Big[\varphi(l_1, k_1, l_2, k_2) = k \Leftrightarrow \bigvee_{k_0 \in K} (\varphi_2(I_2^{-1}(l_1), k_0, l_1, k_1) = 1 \wedge \varphi_2(I_2^{-1}(l_1), k_0, l_1 l_2, k) = k_2)\Big],$$

where I_2 is a mapping of type (1) in the loop $(L, \backslash)$.

From Lemmas 1 and 7 we deduce that $\varphi \in F_0$. Analogously like in the proof of Theorem 4 we can show that the mapping φ satisfies condition (3), what completes the proof.

Using the notations for inverse loops introduced for the formulation of Theorem 3 we obtain the following corollary.

C o r o l l a r y . If F_i is a family of all mappings generating extensions of a loop K_i by a loop L_i for some $i \in \{0,1,2,3,4,5\}$, then, by means of the formulas of types (2) and (3)', one can determine a family of all mappings generating extensions of the loop K_j by the loop L_j for each $j \in \{0,1,2,3,4,5\}$.

REFERENCES

[1] V.D. B e l o u s o v : Foundations of the theory of quasigroups and loops (Russian), Nauka, Moscow, 1967.

[2] R.H. B r u c k : A survey of binary systems, Springer, Berlin-Heidelberg-New York, 1966.

[3] A. C h r o n o w s k i : On Schreier extension of loops, Demonstratio Math., 15 (1982) 105-112.

[4] D. I l s e , I. L e h m a n n , W. S c h u l z : Gruppoide und Funktionalgleichungen, VEB Deutscher Verlag der Wissenschaften, Berlin, 1984.

INSTITUTE OF MATHEMATICS, PEDAGOGICAL UNIVERSITY, 30-084 KRAKÓW
Received October 24, 1985.