DEMONSTRATIO MATHEMATICA
Vol XX No 1-2 1987

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ON BISEMILATTICES WITH GENERALIZED ABSORPTION LAWS, I

0. The aim of this paper is to present some results concerning powers of bisemilattices with generalized absorption laws ([4]).

An algebra $\mathfrak{A} = (A,+,\cdot)$ of type (2.2) is called a bisemilattice if it satisfies the following conditions ([8], [9]):

$$x+x = x \qquad xx = x;$$
$$x+y = y+x, \qquad xy = yx;$$
$$(x+y)+z = x+(y+z), \qquad (xy)z = x(yz).$$

For a bisemilattice $\mathfrak{A}$ we can define two partial orders on A ([1], [10]): $x \leqslant_+ y$ iff $x+y = y$; $x \leqslant_\cdot y$ iff $xy = x$. The partial order $\leqslant_+$ satisfies the least upper bound condition for finite subsets of A; the partial order $\leqslant_\cdot$ satisfies the greatest lower bound condition for finite subsets of A. On a set with two partial orders which satisfy the above conditions we can define a bisemilattice structure.

For a polynomial f over a bisemilattice we define the dual polynomial $\hat{f}$ as follows: $\hat{x} = x$, $\hat{y} = y$, $\widehat{x+y} = \hat{x}\hat{y}$, $\widehat{xy} = \hat{x}+\hat{y}$ and the following polynomials are obtained by induction.

As in [4] we define the following sequence of binary polynomials:

$$f_0(x,y) = x+y, \quad f_{n+1}(x,y) = f_n(x,y)(n)y,$$

where (n) is $\cdot$ if n is even and + if n is odd.

The following identities are called the n-generalized absorption laws ([4]):

(a_n) $\qquad f_n(x,y) = y,$

$(\hat{a}_n)$ $\qquad \hat{f}_n(x,y) = y.$

Let Bsl denote the variety of all bisemilattices, $Bsl(a_n)$ denote the subvariety of the variety Bsl which is defined by the identity (a_n). Analogously we define the subvariety $Bsl(\hat{a}_n)$. Let L_n denote the subvariety of the variety Bsl which is defined by the identities (a_n) and $(\hat{a}_n)$ ([4]).

By Lemma 2.2 in [4] we have that for every natural number n, $Bsl(a_n) \cup Bsl(\hat{a}_n) \subseteq L_{n+1}$. It is easily checked that for the elements a, a′ of the bisemilattice $\mathfrak{A}_{n+1}$ defined in [4], we have $f_n(a',a) \neq a$ and $\hat{f}_n(a',a) \neq a$ (cf. [4] item 9 of the proof of Lemma 2.4). It is also shown in [4] that $\mathfrak{A}_{n+1}$ is an element of L_{n+1}. Then we have the following inclusions:

$$(1) \qquad L_n \begin{matrix} \subsetneq Bsl(a_n) \subsetneq \\ \subsetneq Bsl(\hat{a}_n) \subsetneq \end{matrix} Bsl(a_n) \cup Bsl(\hat{a}_n) \subsetneq L_{n+1}.$$

1. Proposition 1.1. Let $\mathfrak{A} \in Bsl(a_{n+1})$ for a natural number n. Then $\mathfrak{A} \in Bsl(a_{n+1}) \setminus Bsl(a_n)$ iff there exist two elements $a, b \in A$ such that the elements $f_0(a,b), \ldots, f_n(a,b), f_{n+1}(a,b) = b$ are all distinct.

The dual version of this proposition is also true.

Proof. $\Leftarrow$ By the assumption $f_n(a,b) \neq b$. Thus $\mathfrak{A} \notin Bsl(a_n)$.

$\Rightarrow$. Suppose that for any $a,b \in A$ there exists $r,s \leq n+1$ such that $f_r(a,b) = f_s(a,b)$ and $r \neq s$. Take $a,b \in A$ and $r,s \leq n+1$ satisfying the above assumption. Without loss of generality we can assume that $r < s$. Then $f_r(a,b)(s)b = f_s(a,b)(s)b = f_{s+1}(a,b)$, $f_r(a,b)(s)b(s+1)b = f_{s+1}(a,b)(s+1)b = f_{s+2}(a,b)$ and so on. Thus we obtain that $f_r(a,b)(s)b\ldots(n)b = f_{n+1}(a,b) = b$. By the definition of the operation (m) one has: $f_r(a,b)(s)b = f_r(a,b)$ or $f_r(a,b)(s)b = f_{r+1}(a,b)$; $f_r(a,b)(s)b(s+1)b = f_r(a,b)(s+1)b = f_{r+1}(a,b)$ or $f_r(a,b)(s)b(s+1)b = f_{r+1}(a,b)(s+1)b = f_{r+2}(a,b)$ and so on. Then $f_r(a,b)(s)b\dots(n)b = f_{r+n-s}(a,b)$ or $f_r(a,b)(s)b\dots(n)b = f_{r+n-s+1}(a,b)$. Hence $f_{r+n-s}(a,b) = b$ or $f_{r+n-s+1}(a,b) = b$. Analogously as in Lemma 2.2 of [4] we conclude that $f_n(a,b)=b$ (because $r+n-s \leq r+n-s+1 \leq n$). Then for all $a,b \in A$ $f_n(a,b) = b$, a contradiction with the assumption that $\mathfrak{A} \notin Bsl(a_n)$.

Analogously we prove the dual version of this proposition.

2. Let $\mathcal{L} = (C,\leq)$ be a nonempty chain. It is obvious that $(C,\leq,\leq)$ is a bisemilattice. For every natural number n let us consider two bisemilattices $\mathfrak{M}_{n,\mathcal{L}}$ and $\mathfrak{M}'_{n,\mathcal{L}}$ defined as follows: the underlying set of $\mathfrak{M}_{n,\mathcal{L}}$ is ordered as in Figure 1 if n is even and as in Figure 3 if n is odd, the underlying set of $\mathfrak{M}'_{n,\mathcal{L}}$ is ordered as in Figure 2 if n is even and as in Figure 4 if n is odd, where $\mathcal{L}$ is a subbisemilattice of $\mathfrak{M}_{n,\mathcal{L}}$ and $\mathfrak{M}'_{n,\mathcal{L}}$ for every n.

n even:

$\mathfrak{M}_{n,\mathcal{L}}$:

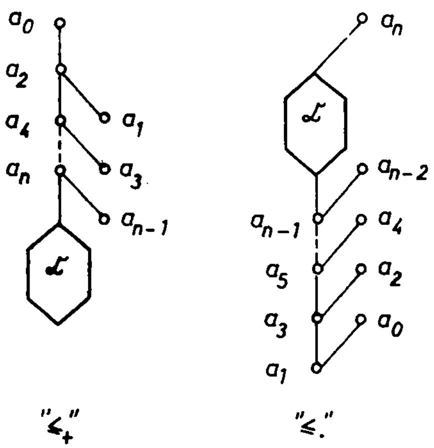

Fig.1

$\mathfrak{M}'_{n,\mathcal{L}}$:

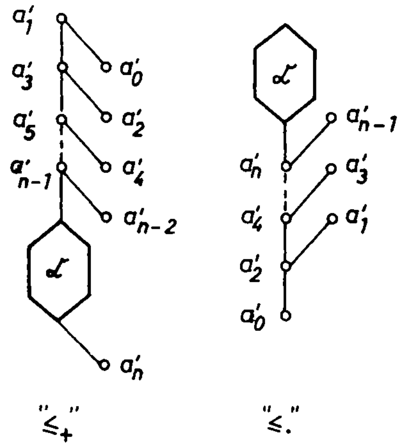

Fig.2

n odd:

$\mathfrak{M}_{n,\alpha}$: $\mathfrak{M}'_{n,\alpha}$:

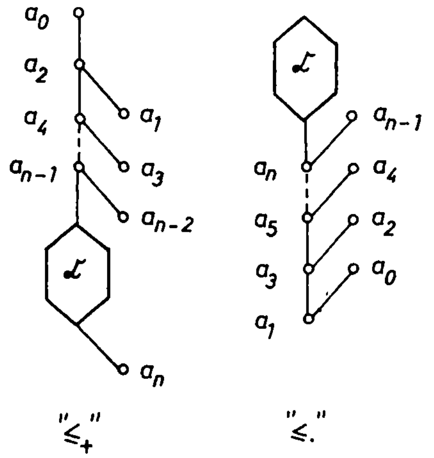

Fig.3

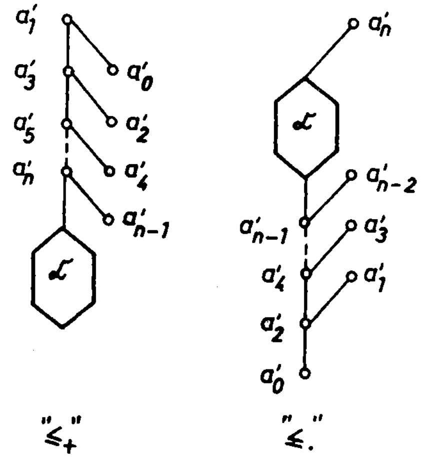

Fig.4

R e m a r k : We can easily see that $C \cap \{a_0,\dots,a_n\} = \emptyset$, $C \cap \{a'_0,\dots,a'_n\} = \emptyset$.

L e m m a 2.1. For every positive number n $\mathfrak{M}_{n,\alpha} \in \mathrm{Bsl}(\hat{a}_n) \setminus \mathrm{Bsl}(a_n)$, $\mathfrak{M}'_{n,\alpha} \in \mathrm{Bsl}(a_n) \setminus \mathrm{Bsl}(\hat{a}_n)$.

P r o o f . Let us take two elements a,b from $\mathfrak{M}_{n,\alpha}$ or from $\mathfrak{M}'_{n,\alpha}$. If $a,b \in C$ then $f_1(a,b) = \hat{f}_1(a,b) = b$ (for every n). Without loss of generality we can assume that $a \neq b$ and $a \notin C$. We proceed by induction on n. For $n = 1$ we have the following bisemilattices:

$\mathfrak{M}_{1,\alpha}$: $\mathfrak{M}'_{1,\alpha}$:

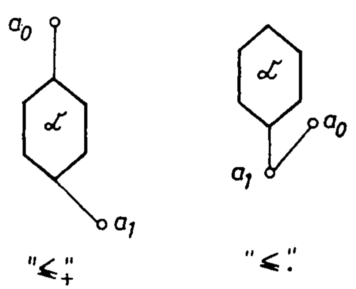

Fig.5

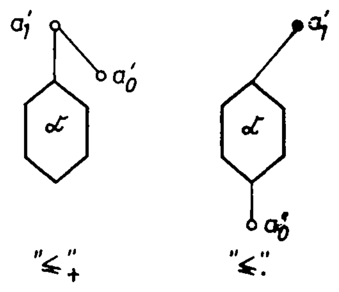

Fig.6

Let $a,b \in \mathfrak{M}_{1,\alpha}$. If $b \notin C$ or $a = a_1$ then the elements a,b form a two-element sublattice and $f_1(a,b) = \hat{f}_1(a,b) = b$, $f_1(b,a) = \hat{f}_1(b,a) = a$. Assume that $b \in C$, $a = a_0$. Then we have $f_1(a,b) = f_1(a_0,b) = (a_0+b)b = a_0b = a_1 \neq b$; $\hat{f}_1(a,b) = \hat{f}_1(a_0,b) = a_0b+b = a_1+b = b$, $\hat{f}_1(b,a) = \hat{f}_1(b,a_0) = ba_0+a_0 = a_1+a_0 = a_0$. Thus $\mathfrak{M}_{1,\alpha} \in Bsl(\hat{a}_1) \setminus Bsl(a_1)$. Analogously we prove that $\mathfrak{M}'_{1,\alpha} \in Bsl(a_1) \setminus Bsl(\hat{a}_1)$. Assume that the lemma is true for n-1. Moreover assume that n is even. (The case of n odd is similar). Let $a,b \in M_{n,\alpha}$. If $b \notin C$ then a,b belong to M_{n-1,α_n}, where α_n is the one-element chain (a_n). Then by induction hypothesis and by Lemma 2.2 of [4] we have $\hat{f}_n(a,b)=b$, $\hat{f}_n(b,a) = a$. If $b \in C$, $a \neq a_0$ then a,b belong to a subbisemilattice isomorphic to $\mathfrak{M}'_{n-1,\alpha}$. Then by the induction hypothesis and by Lemma 2.2 of [4] we have $\hat{f}_n(a,b) = b$, $\hat{f}_n(b,a) = a$. Let $b \in C$, $a = a_0$. Then $f_0(b,a) = f_0(b,a_0) = b+a_0 = a_0$, $f_n(b,a_0) = a_0 = a$; $f_0(a,b) = f_0(a_0,b) = a_0+b = a_0$, $f_1(a,b) = a_0b = a_1,\dots,f_n(a,b) = a_{n-1}+b = a_n \neq b$; $\hat{f}_0(b,a) = \hat{f}_0(b,a_0) = ba_0 = a_1$, $\hat{f}_1(b,a) = \hat{f}_1(b,a_0) = a_1+a_0 = a_0$, $\hat{f}_n(b,a) = \hat{f}_n(b,a_0) = a_0 = a$; $\hat{f}_0(a,b) = \hat{f}_0(a_0,b) = a_0b = a_1$, $\hat{f}_1(a,b) = a_1+b = a_2,\dots,\hat{f}_{n-1}(a,b) = a_{n-1}+b = a_n$, $\hat{f}_n(a,b) = a_nb = b$. Therefore $(\hat{a}_n)$ holds in $\mathfrak{M}_{n,\alpha}$ but (a_n) does not hold in $\mathfrak{M}_{n,\alpha}$. Thus $\mathfrak{M}_{n,\alpha} \in Bsl(\hat{a}_n) \setminus Bsl(a_n)$.

Analogously we prove that $\mathfrak{M}'_{n,\alpha} \in Bsl(a_n) \setminus Bsl(\hat{a}_n)$.

3. Theorem 3.1.

1) Let n be a natural number.

(i) If $\mathfrak{A} \in Bsl(a_{n+1}) \setminus Bsl(a_n)$ then $\mathrm{card} A \geq n+2$.

(ii) For every cardinal number $m \geq n+2$ there exists a bisemilattice $\mathfrak{A} \in Bsl(a_{n+1}) \setminus Bsl(a_n)$ such that $\mathrm{card} A = m$.

The dual versions of (i) and (ii) are also true.

2) Let n be a positive number.

(i) If $\mathfrak{A} \in Bsl(\hat{a}_n) \setminus Bsl(a_n)$ then $\mathrm{card} A \geq n+2$.

(ii) For every cardinal number $m \geq n+2$ there exists a bisemilattice $\mathfrak{A} \in Bsl(\hat{a}_n) \setminus Bsl(a_n)$ such that $\mathrm{card} A = m$.

The dual versions of (i) and (ii) are also true.

3) Let n be a natural number.

(i) If $\mathcal{A} \in L_{n+1} \setminus L_n$ then card$A \geqslant n+2$.

(ii) For every cardinal number $m \geqslant n+2$ there exists a bisemilattice $\mathcal{A} \in L_{n+1} \setminus L_n$ such that card$A = m$.

P r o o f . 1)(i) and its dual version are obtained as corollaries from Proposition 1.1.

2)(i). By Lemma 2.2 of [4] we get that $Bsl(\hat{a}_n) \setminus Bsl(a_n) \subseteq \subseteq Bsl(a_{n+1}) \setminus Bsl(a_n)$. Therefore by 1)(i) we obtain 2)(i). Analogously we obtain the dual version of 2)(i).

3)(i). Let $\mathcal{A} \in L_{n+1} \setminus L_n$ $(= L_{n+1} \setminus (Bsl(a_n) \cap Bsl(\hat{a}_n))$. Then $\mathcal{A} \in (L_{n+1} \setminus Bsl(a_n)) \cup (L_{n+1} \setminus Bsl(\hat{a}_n)) \subseteq (Bsl(a_{n+1}) \setminus Bsl(a_n)) \cup \cup (Bsl(\hat{a}_{n+1}) \setminus Bsl(\hat{a}_n))$. Therefore by 1)(i) we get card$A \geqslant n+2$.

2)(ii). By Lemma 2.1 $\mathcal{M}_{n,\mathcal{C}} \in Bsl(\hat{a}_n) \setminus Bsl(a_n)$ (for every positive n). If m is infinite, let us take a chain $\mathcal{C}$ such that card$C = m$. Then card$\mathcal{M}_{n,\mathcal{C}}$ = card$C = m$. If m is finite let us take a chain $\mathcal{C}$ such that card$C = m-n-1$. Then card$\mathcal{M}_{n,\mathcal{C}} =$ = card$C+n+1 = m$.

1)(ii). By Lemma 2.2 of [4] we have $Bsl(\hat{a}_n) \setminus Bsl(a_n) \subseteq \subseteq Bsl(a_{n+1}) \setminus Bsl(a_n)$. Then we obtain 1)(ii) as a consequence of 2)(ii).

Analogously we obtain the dual versions of 1)(ii) and 2)(ii).

3)(ii). Lemma 2.2 of [4] implies that $Bsl(\hat{a}_n) \setminus Bsl(a_n) \subseteq \subseteq Bsl(\hat{a}_n) \setminus L_n \subseteq L_{n+1} \setminus L_n$. Then by 2)(ii) we have 3)(ii).

As a corollary from Lemma 2.1 we have also a stronger form of the inclusions (1).

C o r o l l a r y 3.2. For every positive n we have the following inclusions:

(2) $$L_n \begin{matrix} \subsetneq Bsl(a_n) \subsetneq \\ \subsetneq Bsl(\hat{a}_n) \subsetneq \end{matrix} Bsl(a_n) \cup Bsl(\hat{a}_n) \subsetneq L_{n+1}.$$

REFERENCES

[1] R. B a l b e s : A representation theorem for distributive quasi-lattices, Fund. Math. 68 (1970) 207-214.
[2] G. B i r k h o f f : Lattice Theory, New York, 1948.
[3] J. D u d e k : On bisemilattices I, Colloq. Math. 47 (1982), 1-5.
[4] J. G a ł u s z k a : Generalized absorption laws in bisemilattices, Algebra Universalis, 19 (1984) 304-318.
[5] G. G r ä t z e r : General Lattice Theory, Berlin, 1978.
[6] G. G r ä t z e r : Universal Algebra, Springer Verlag, 1979.
[7] R. M c K e n z i e , A. R o m a n o w s k a : Varieties of .-distributive bisemilattices, in: Contributions to General Algebra, Proceedings of the Klagenfurt Conference, Klagenfurt, 1979, 213-218.
[8] R. P a d m a n a b h a n : Regular identities in lattices. Trans. Amer. Math. Soc. 158 (1971) 179-188.
[9] J. P ł o n k a : On distributive quasilattices, Fund. Math. 60 (1967) 191-200.
[10] A. R o m a n o w s k a : On bisemilattices with one distributive law, Algebra Universalis 10 (1980) 36-47.

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Received September 20, 1985.