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ON COMPARISONS OF ALGEBRAS BY USING THE ENRICHMENTAL THEORIES AND CLONING SYSTEMS

In universal algebra there are problems concerning the enrichments of algebras. For those problems the notion of abstract algebra as an algebra of a given type is not comfortable. We study in this paper algebras without types. We introduce a category AL of algebras. The poset category Enr(AL) = $\langle$Ob AL, $\leq_r\rangle$ is a subcategory of AL admiting only the enrichmental morphisms. The enrichmental theories of algebras are considered as the monads of the category Enr(AL). We also study a notion of a clone. We remark that the general notion of a clone depends on a given cloning closure operator or a cloning system of algebras. We introduce a notion of a cloning closure operator as a special quasi-functor from a covering poset category to the Set category. A quasi-functor from a category K to a category K′ is a pair $F = \langle F_0, F_1\rangle$ of mappings F_0 : Ob K $\to$ Ob K′ and F_1:Mor K $\to$ Mor K′ having the properties: $F_1(A \xrightarrow{h} B) = F_0(A) \xrightarrow{F_1(h)} F_0(B)$ and $F_1(h' \circ h) = F_1(h') \circ F_1(h)$ for all h and h′ in Mor K. If a quasi-functor $F:K \to K'$ fulfils the equality $F_1(1_A) = 1_{F_0(A)}$ for all objects A in K, then F is a functor. If F is a quasi-functor, then we write F(A) and F(h) instead of $F_0(A)$ and $F_1(h)$. We

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give for every set M the interpolational $I^{(M)}$ and localizational $L^{(M)}$ cloning closure operators over M. We have for every algebra A the algebraic al(A), interpolation-algebraic Int(A) and localization-algebraic Loc(A) cloning closure operators induced by A. Every topological algebra A determines a smooth cloning closure operator D(A). The operator D(A) for the usual topological algebra A of all real numbers with all ∞-differentiable operations is used in differential geometry.

A notion of a cloning system of algebras as a special functor from the category Enr(AL) to a poset category CLO_r of all cloning closure operators is considered. Each cloning closure operator E determines a category Sp(E) of invariant spaces over E. A cloning system Q of algebras defines a comparison $\underset{Q}{\equiv}$ of algebras such that: $A \underset{Q}{\equiv} A'$ if and only if $Sp(Q(A)) = Sp(Q(A'))$. If $\underset{Q}{\equiv}$ is a kernel of an enrichmental theory H of algebras (i.e. $A \underset{Q}{\equiv} A'$ iff $H(A) = H(A')$), then H is said to be an enrichmental hull of Q. We prove that the cloning systems of algebras al, Int, Loc, D and theirs subsystems have the enrichmental hulls. Moreover, the ways of enrichment of algebras to clones over cloning systems are given. For concepts used in this paper without definitions see [5,8].

1. The enrichmental theories and comparisons of algebras

A n-ary operation over a set X is a function $f: X^n \to X$. If $f: X^n \to X$ is a n-ary operation over X and $Y \subseteq X$, then Y is said to be closed under f provided $f(Y^n) \subseteq Y$. If $Y = \emptyset$ is closed under f, then f is non-constant. An abstract algebra or briefly an algebra is a pair $A = \langle A_0, A_1 \rangle$ such that A_0 is a non-empty set and A_1 is a function that relates to each natural number n a set $A_{1,n}$ of some n-ary operations over A_0. It may be assumed (without loss of generality) that for all $n \neq \emptyset$ the projections $pr_j^n: A_0^n \to A_0$ belong to $A_{1,n}$, for $j = 1,\ldots,n$. Let A be an algebra. Then A_0 is called the universe of A and for all n, $A_{1,n}$ is said to be the set of all n-ary fundamental operations of A. A subset $Y \subseteq A_0$ is closed in A if for all n and all f in $A_{1,n}$, Y is closed under f. An algebra A' is said to be a subalgebra of A provided A'_0 is a closed subset in A and for all n $A'_{1,n}$ is the set of all restricted functions $f|A'_0$, where $f \in A_{1,n}$. If $Y \subseteq A_0$ is any non-empty subset, then there is the least subalgebra A' of A with $Y \subseteq A'_0$, it is denoted by $[Y]_A$ and called generated by Y. For each non-empty set M we have the M-power algebra A^M of A. A^M is an algebra such that $(A^M)_0 = A_0^M$ is the set of all functions from M to A_0 and $(A^M)_1$ is a function that relates to each natural number n a set $(A^M)_{1,n}$ of all functions $\bar{f} : (A_0{}^M)^n \longrightarrow A_0{}^M$ induced by functions $f \in A_{1,n}$, i.e. $\bar{f} = f^M$ is defined by the formula

$$\bar{f}(\varphi_1,\varphi_2,\ldots,\varphi_n) = f \circ \langle \varphi_1,\varphi_2,\ldots,\varphi_n \rangle .$$

If $n = 0$, then $\tilde{f}$ is the constant function with value f. The M-power A^M for $M = A_0{}^n$ will be denoted by $F_n(A)$. If $M = \emptyset$, then A^M is a one-element algebra.

An equivalence relation $\sim$ over a set X is said to be a congruence of a n-ary operation $f:X^n \longrightarrow X$ provided for all $a_i, a'_i \in X$ if $a_i \sim a'_i$ for $i = 1,2,\ldots,n$, then $f(a_1,a_2,\ldots,a_n) \sim$ $\sim f(a'_1,a'_2,\ldots,a'_n)$. If an equivalence relation $\sim$ over A_0 is a congruence relation of all fundamental operations of A, then $\sim$ is called a congruence of algebra A. A homomorphism from an algebra A to an algebra A' is a pair $h = \langle h_0,h_1 \rangle$ such that $h_0:A_0 \longrightarrow A'_0$, h_1 is a function that relates to each natural number n a function $h_{1,n}:A_{1,n} \longrightarrow A'_{1,n}$ having the following property: for all $f \in A_{1,n}$ and all $x_1,\ldots,x_n$ in A_0, $h_0(f(x_1,\ldots,x_n)) = h_{1,n}(f)(h_0(x_1),\ldots,h_0(x_n))$. The composition of homomorphisms $h:A \longrightarrow A'$ and $h:A' \longrightarrow A''$ is given by the formulas:

$$(h' \circ h)_0 = h'_0 \circ h_0, \ (h' \circ h)_{1,n} = h'_{1,n} \circ h_{1,n}$$

and it is a homomorphism $h' \circ h : A \to A''$. Hence we obtain a category AL of all algebras regarding algebras as objects and homomorphisms as morphisms. If A and A′ are algebras, then A′ is said to be an enrichment of A if $A_0 = A'_0$ and for all n $A_{1,n} \subseteq A'_{1,n}$. Hence A′ is an enrichment of A iff there is a homomorphism $r: A \to A'$ of the form $r = (r_0, r_1)$, where $r_0 = 1_{A_0}$ and for all n $r_{1,n}: A_{1,n} \hookrightarrow A'_{1,n}$ is the inclusion map. The homomorphisms of the form r are called enrichmental. The composition of enrichmental homomorphisms is an enrichmental homomorphism. If A′ is an enrichment of A, then we write $A \leq_r A'$. The subcategory of AL defined by all enrichmental homomorphisms will be denoted by Enr(AL) and called enrichmental category of algebras. Hence Enr(AL) is the poset category $\langle \mathrm{Ob}\ \mathrm{Al}, \leq_r \rangle$. Now let us consider the endofunctors of the category Enr(AL), i.e. the functors of the form $H : \mathrm{Enr(AL)} \to \mathrm{Enr(AL)}$. H is called an enrichmental endofunctor of algebras if for all algebras A we have $A \leq_r H(A)$. The monads of the category Enr(AL) are said to be the enrichmental theories of algebras. Hence H is an enrichmental theory of algebras if and only if H is an enrichmental endofunctor of algebras and $H^2 = H$. We give examples of enrichmental theories of algebras.

E x a m p l e 1. For each algebra A we relate an algebra O(A) such that $O(A)_0 = A_0$ and $O(A)_{1,n}$ is the universe of the subalgebra of $F_n(A)$ generated by the set pr^n of all projections $pr^n_j : A_0{}^n \to A_0$. The elements of $O(A)_{1,n}$ are called n-ary algebraic operations of A.

E x a m p l e 2. For each algebra A we relate an algebra p(A) such that $p(A)_0 = A_0$ and $p(A)_{1,n}$ is the universe of the subalgebra of $F_n(A)$ generated by the set $pr^n \cup \{c^n_a : a \in A_0\}$, where c^n_a is the constant function with value a. The elements in $p(A)_{1,n}$ are called n-ary polynomials of A.

A subuniverse of an algebra A is any closed subset in A and moreover the empty subset provided every algebraical operation of A is a non-constant function over A_0. The set of all subuniverses of A is denoted by Su(A). Su(A) is an intersection structure on the set A_0 and therefore it defines a closure operator on A_0. For every $Y \subseteq A_0$ there is the least subuniverse U of A such that $Y \subseteq U$, it will be denoted by $Sg_A(Y)$ and called generated by Y. If $Sg_A(\emptyset) \neq \emptyset$, then $X = Sg_A(\emptyset)$ is a closed subset in A and thus X defines a subalgebra A' of A with $A'_0 = X$ which is called generated by $\emptyset$ and we write $A' = [\emptyset]_A$.

E x a m p l e 3. For any algebra A we relate an algebra Cs(A) such that $Cs(A)_0 = A_0$ and for all n $Cs(A)_{1,n}$ is the set of all functions $f: A_0^n \to A_0$ having the property: for every non-empty set M

(k) if Z is a subuniverse of A^M and $\varphi_1, \varphi_2, \dots, \varphi_n \in Z$, then

$$f \circ \langle \varphi_1, \varphi_2, \dots, \varphi_n \rangle \in Z.$$

The elements in $Cs(A)_{1,n}$ are n-ary operations of A compatible with power subuniverses. In analogical way using (k) only for a fixed M we obtain the algebra $Cs^{(M)}(A)$.

E x a m p l e 4. For any algebra A we relate an algebra Co(A) such that $Co(A)_0 = A_0$ and $Co(A)_{1,n}$ is the set of all n-ary operations f over A_0 such that if $\sim$ is a congruence of A, then $\sim$ is also a congruence of f. The elements in $Co(A)_{1,n}$ are the n-ary operations of A compatible with congruences.

From above we obtain

(1.1) The mappings O, p, Cs, $Cs^{(M)}$ and Co defined in examples 1-4 are enrichmental theories of algebras such that for all algebras A: $A \leqslant_r O(A) \leqslant_r p(A) \leqslant_r Co(A)$.

The identity endofunctor id of algebras (id(A) = A) is the least enrichmental theory of algebras. The complete endofunctor cm such that $cm(A)_0 = A_0$ and $cm(A)_{1,n} = A_0^{A_0^n}$ is the greatest enrichmental theory of algebras. If H is an enrichmental endofunctor of algebras, then H is said to be closed under an algebra A provided for all n $H(A)_{1,n}$ is a subuniverse of $F_n(A)$. If this condition holds only for a fixed n, then H is n-closed under A. If the identity endofunctor id is closed under an algebra A and $pr^n \subseteq A_{1,n}$ for all $n \neq 0$, then A is called an algebraic clone. The enrichmental theories O, p, Cs and Cc are closed i.e. closed under all algebras and enrichments under those theories are algebraic clones. H-algebras under an enrichmental theory H are algebras A such that $H(A) = A$. O-algebras are algebras in the sense of Marczewski [9]. An equivalence $\equiv$ between algebras is said to be an enrichmental comparison of algebras provided there is an enrichmental theory H of algebras such that for any algebras A and A' we have

(c) $A \equiv A'$ if and only if $H(A) = H(A')$.

If $\equiv$ is an enrichmental comparison of algebras, then H from (c) is called an enrichmental hull of $\equiv$. Hence the enrichmental comparisons of algebras are the kernels of enrichmental theories. For an example let us observe that J. Schmidt [10] has proved that the equivalence: $A \equiv_1 A'$ if and only if $Su(A) = Su(A')$ has an enrichmental hull $H = Cs^1$, where $1 = \{\emptyset\}$ (see above Ex. 3) and thus $\equiv_1$ is an enrichmental comparison of algebras. In the sequel we give other enrichmental comparisons by using the cloning systems.

R e m a r k . If G is a fixed function that relates to each natural number n a set G_n of n-ary symbols with $G_n \cap G_m = \emptyset$ for $n \neq m$, then an algebra of the type G is an algebra A together with a family $\varphi = \{\varphi_n\}$ of surjective functions $\varphi_n : G_n \to A_{1,n}$. If A or $\langle A, \varphi \rangle$ is an algebra of the type G, then each n-ary symbol $g \in G_n$ determines the n-ary fundamental operation $\varphi_n(g)$ of A which is denoted by g_A. The category G-AL of algebras of the type G is a subcategory of AL with morphisms $h: A \to A'$ such that $h_{1,n}(g_A) = g_{A'}$ for all n and all $g \in G_n$. For the theory of algebras of a type G see the papers [3,5,11].

2. The cloning closure operators

A covering space is a pair $C = \langle C_0, C_1 \rangle$ such that C_0 is a set and C_1 is a subset of 2^{C_0} with $C_0 = \cup C_1$. If C is a covering space, then C_0 is called the support of C and C_1 the coverring family of C. If C and C' are covering spaces, then a morphism from C to C' is a function $h: C_0 \to C'_0$ such that for all $X \in C'_1$ $h^{-1}(X) \in C_1$. The composition of the morphisms is a morphism. Hence we obtain a category Cov of all covering spaces. A covering space C is called topological if the following conditions hold:

(tp 1) $\emptyset, C_0 \in C_1$,

(tp 2) if $X, Y \in C_1$, then $X \cap Y \in C_1$,

(tp 3) for all $\alpha \subseteq C_1$ $\cup\alpha \in C_1$.

A covering space C is said to be open provided

(op) for all $p \in C_0, X, Y \in C_1$, if $p \in X \cap Y$, then there is $U \in C_1$ with $p \in U \subseteq X \cap Y$. Every open covering space C determines a unique topological space $\bar{C}$ with $C_0 = \bar{C}_0$ and $C_1 \subseteq \bar{C}_1$ such that C_1 is a base of $\bar{C}$. The full subcategory of Cov defined by all topological spaces is denoted by Top.

A morphism $h: C \to C'$ in Cov is said to be minimal provided $C_1 = \{h^{-1}(X) : X \in C'_1\}$. The subcategory of Cov defined by the minimal morphisms is denoted by Cov_m. Moreover, we shall denote by Cov_s the subcategory of Cov_m admitting only the inclusion map $h(x) = x$ for $x \in C_0$. If $h: C \hookrightarrow C'$ is a morphism in Cov_s, then $C_1 = \{X \cap C_0 : X \in C'_1\}$, $C_0 \subseteq C'_0$ and we write $C \leqslant_s C'$ or $C = C' | C_0$ and moreover, C is called a subspace of C'. For every subset $X \subseteq C'_0$ there is exactly one subspace $C \leqslant_s C'$ with $C_0 = X$ and we write $C = C'|X$. The set of all subspace of C' will be denoted by $\bar{s}(C')$. Hence the category Cov_s may be considered as the poset category $\langle \mathrm{Ob\ Cov}, \leqslant_s \rangle$. A covering poset category is a full subcategory of Cov_s defined by a class of covering spaces closed under the subspace operator $\bar{s}$. Hence a covering poset category is a poset category $\langle V, \leqslant_s \rangle$, where $V \subseteq \mathrm{Ob\ Cov}$ and $\bar{s}(V) = V$. The covering poset category $\langle V, \leqslant_s \rangle$ will be briefly denoted by V and the opposite category $\langle V, {}_s\geqslant \rangle$ by V^{op}. Now we give a definition.

(2.1) D e f i n i t i o n . A cloning closure operator is a triple $E = \langle E_0, E_1, E_2 \rangle$ such that E_0 is a non-empty set, E_1 is a covering poset category and E_2 is a quasi-functor from E_1^{op} to Set having the following properties:

(c0) for all objects C in E_1 $E_2(C)$ is the set of all subsets of $E_0^{C_0}$,

(c1) for every morphism $C \leqslant C'$ in E_1 the function $E_2(C \leqslant C')$ preserves the inclusion relation and for every $Z \in E_2(C')$ and $X \in C_1$ we have $Z|X \subseteq E_2(C|X \leqslant C')(Z)$, where $Z|X$ is the set of all restricted functions $f|X$ with $f \in Z$,

(c2) for every morphism $C \leqslant C'$ in E_1, set $Z \in E_2(C')$ and $f \in E_0^{C_0}$ we have $f \in E_2(C \leqslant C')(Z)$ if and only if for each $X \in C_1$ the function $f|X$ belongs to $E_2(C|X \leqslant C')(Z)$.

If E is a cloning closure operator and U is a full subcategory of E_1 closed under the operator $\bar{s}$, then the triple $E|U = \langle E_0, U, E_2|U^{op} \rangle$ is also a cloning closure operator and we call it the suboperator of E induced by U. Now we prove the following fact.

(2.2) For every cloning closure operator E we have:

(i) $E_2(C \leqslant C') = E_2(1_C) \circ E_2(C \leqslant C') = E_2(C \leqslant C') \circ E_2(1_{C'}) = E_2(1_C) \circ E_2(C \leqslant C') \circ E_1(1_{C'})$,

(ii) $Z|C_0 \subseteq E_2(C \leqslant C')(Z)$,

(iii) for every object C in E_1 $E_C = E_2(1_C)$ is a closure operator on the set $E_0^{C_0}$.

P r o o f . (i) holds since E_2 is a quasi-functor. (ii) holds by (c1) and (c2). By (c1) E_C preserves the inclusion relation. By (ii) we have $Z \subseteq E_C(Z)$ and $E_C^2 = E_C$ since $E_2(1_C) = E_2(1_C) \circ E_2(1_C)$ by (i). Hence E_C is a closure operator on the set $E_0^{C_0}$ and (iii) holds.

An invariant space over a cloning closure operator E is a pair $\langle C, Z \rangle$ such that C is an object in E_1 and $E_C(Z) = Z$. If $\langle C, Z \rangle$ and $\langle C', Z' \rangle$ are invariant spaces over E, then a morphism from $\langle C, Z \rangle$ to $\langle C', Z' \rangle$ is any function $h: C_0 \to C_0'$ such that for all $f \in Z'$ the composition $f \circ h \in Z$. Hence we have a category Sp(E) of all invariant spaces over a cloning closure operator E. Every invariant space $\langle C,Z\rangle$ over E and any subset $X\subseteq C_0$ determines by (2.2) a invariant space $\langle C|X,E_2(C)X\leqslant C)(Z)\rangle$ over E which is called a subspace of $\langle C,Z\rangle$ induced by X. A selection over a cloning closure operator E is a function α that relates to each natural number n an object $\alpha(n)$ in E_1 such that $\alpha(n)_0 = E_0^n$. Every selection α over E defines a function $E^{(\alpha)}$ that relates to each algebra A with $A_0 = E_0$ an algebra $A' = E^{(\alpha)}(A)$ such that $A'_0 = E_0$ and for all n $A'_{1,n} = E_{\alpha(n)}(A_{1,n})$. From (2.2) we obtain:

(2.3) For every selection α over a cloning closure operator E the function $E^{(\alpha)}$ is a monad of the poset category $\mathrm{Enr}_{E_0}(AL)$ being the full subcategory of Enr(AL) defined by all algebras A with $A_0 = E_0$.

An algebra A with $pr^n\subseteq A_{1,n}$ for all $n\neq 0$ is said to be a clone over E provided there is a selection α over E and $A = E^{(\alpha)}(A)$. Since E does not depend on A the algebra $A' = E^{(\alpha)}(A)$ with $pr^n\subseteq A'_{1,n}$ is always a clone for all A and all selections α over E. If A is a clone over E and $A = E^{(\alpha)}(A)$, then A is called an α-clone over E.

A covering space C with a one-element covering family $C_1 = \{C_0\}$ may be considered as the set C_0 and the covering poset category defined by all those covering spaces may be considered as the poset category $\mathrm{Set}_s = \langle \mathrm{Ob\,Set},\subseteq\rangle$. By easy verification we obtain

(2.4) Every algebra A defines a cloning closure operator al(A) such that $al(A)_0 = A_0$, $al(A)_1 = \mathrm{Set}_s$ and $Al(A)_2(M\subseteq M')(Z)$ is the set $Sg_A M'(Z)|M$, where $A^{M'}$ is the M'-power algebra of A. The operator al(A) admits only one selection α with $\alpha(n) = A_0^n$.

An algebra A' with $A'_0 = A_0$ is a clone over al(A) if and only if for all n $A'_{1,n}$ is a subuniverse of $F_n(A)$ and $pr^n\subseteq A'_{1,n}$ for all $n\neq 0$. Hence A is an algebraic clone if and only if A is a clone over al(A).

A cloning closure operator E preserves an algebra A provided $E_0 = A_0$ and for each morphism $C\leqslant C'$ in E_1 if Z is a subuniverse of the algebra $A^{C'_0}$, then $E_2(C\leqslant C')(Z)$ is a subuniverse of A^{C_0}. Obviously al(A) preserves A. Now we introduce the interpolational cloning closure operators.

(2.5) Definition. An interpolational operator is a triple $I = \langle I_0, I_1, I_2 \rangle$ such that I_0 is a non-empty set, I_1 is the covering poset category Cov_s and I_2 is a quasi-functor from I_1^{op} to Set such that for all objects C $I_2(C) = 2^{I_0^{C_0}}$ and for every morphism $C \leq C'$ in Cov_s and $Z \in I_2(C')$, $I_2(C \leq C')(Z)$ is the set of all functions $f: C_0 \to I_0$ such that for each $X \in C_1$ there is a function $h \in Z$ with $f|X = h|X$.

Now we prove

(2.6) Theorem. For every non-empty set Y there is only one interpolational operator I with $I_0 = Y$ and it is denoted by $I^{(Y)}$. Moreover, every interpolational operator I is a cloning closure operator preserving all algebras A with $A_0 = I_0$.

Proof. The mappings I_2 are uniquely determined by I_0. From the definition (2.5) it follows that I_2 is such a quasi-functor that it is a cloning closure operator. Let A be any algebra with $A_0 = I_0$ and let Z be a subuniverse of A^{C_0}. For all n and all $f \in A_{1,n}$ and all $\varphi_1, \varphi_2, \ldots, \varphi_n \in Z_1 = I_2(C \leq C')(Z)$, $\varphi = f^{C_0}(\varphi_1, \varphi_2, \ldots, \varphi_n) \in Z_1$ since for every $X \in C_1$ by (2.5) we have $\varphi|X = f^X(\varphi_1|X, \varphi_2|X, \ldots, \varphi_n|X) = f^X(h_1|X, h_2|X, \ldots, h_n(X) = f^{C'_0}(h_1, h_2, \ldots, h_n)|X$, where $h_i \in Z$ and $f^{C'_0}(h_1, h_2, \ldots, h_n) \in Z$, because Z is a subuniverse of $A^{C'_0}$. Hence Z_1 is a subuniverse of A^{C_0} and thus I preserves the algebra A.

A covering space C′ is called an enrichment of a covering space C if $C_0 = C'_0$ and $C_1 \subseteq C'_1$ and then it is written $C \vdash_r C'$. The covering poset category defined by all open covering spaces is denoted by $OCov_s$. Now we give a definition of a localizational operator.

(2.7) Definition. A localizational operator is a triple $L = \langle L_0, L_1, L_2 \rangle$ such that L_0 is a non-empty set, L_1 is the covering poset category $OCov_s$ and L_2 is a quasi-functor from L_1^{op} to Set such that for all objects C in L_1 $L_2(C) = 2^{L_0^{C_0}}$ and for every morphism $C \leq C'$ in L_1 and $Z \in L_2(C')$ we have

$$L_2(C \leq C')(Z) = \bigcup \left\{ I_2(C'' \mid C_0 \leq C'')(Z) \ : \ C'' \vdash_r C \right\},$$

where I is the unique interpolational operator with $I_0 = L_0$.

(2.8) Theorem. For every non-empty set Y there is only one localizational operator L with $L_0 = Y$ and it is denoted by $L^{(Y)}$. Moreover, every localizational operator L is a cloning closure operator preserving all algebras A with $A_0 = L_0$.

Proof. The mappings L_2 are uniquely determined by L_0 and by fixed category L_1. By (2.6) and (2.7) L is a cloning closure operator. Let A be any algebra with $A_0 = L_0$. Let us consider the set $Z_1 = L_2(C \leq C')(Z)$, where Z is a subuniverse of algebra $A^{C_0'}$. It is sufficient to prove that $\varphi = f^{C_0}(\varphi_1, \varphi_2, \ldots, \varphi_n) \in Z_1$ for all $n, f \in A_{1,n}$ and $\varphi_1, \varphi_2, \ldots, \varphi_n \in Z_1$. Since $\varphi_i \in Z_1$ there are $C_i'' \vdash_r C$ such that $\varphi_i \in I_2(C_i'' \mid C_0 \leq C_i'')(Z)$. Hence for $p \in C_0'$ there are $X_i^{(p)} \in C_{i1}'' \subseteq C_1'$ with $p \in X_i^{(p)}$ and $h_i \in Z$ with $\varphi_i | Y_i^{(p)} = h_i | Y_i^{(p)}$, where $Y_i^{(p)} = X_i^{(p)} \cap C_0$, $i=1,\ldots,n$. Since C' is open there is $U^{(p)} \in C_1'$ with $p \in U^{(p)} \subseteq X_1^{(p)} \cap X_2^{(p)} \cap \ldots X_n^{(p)}$. Thus for $Y^{(p)} = U^{(p)} \cap C_0$ we have $\varphi_i | Y^{(p)} = h_i | Y^{(p)}$ for $i = 1, \ldots, n$. Hence we obtain $\varphi | Y^{(p)} = f^{Y^{(p)}}(\varphi_1 | Y^{(p)}, \ldots, \varphi_n | Y^{(p)}) = f^{Y^{(p)}}(h_1 | Y^{(p)}, \ldots, h_n | Y^{(p)}) = f^{C_0'}(h_1, \ldots, h_n) | Y^{(p)} = h | Y^{(p)}$, where $h = f^{C_0'}(h_1, \ldots, h_n) \in Z$. Let C^* be the covering space such that $C_0^* = C_0'$ and $C_1^* = \left\{ U^{(p)} : p \in C_0' \right\}$. Then $C^* \vdash_r C'$ and $\varphi \in I_2(C^* \mid C_0 \leq C^*)(Z)$ and thus $\varphi \in Z_1$. Hence L preserves A and this finishes our proof of Theorem (2.8).

Let E and E' be any cloning closure operators. E' is said to be an enrichment of E provided $E_0 = E'_0$, $E_1 = E'_1$ and $E_2 \leq E'_2$ or for every morphism $C \leq C'$ and $Z \subseteq E_0^{C'_0}$ we have $E_2(C \leq C')(Z) \subseteq E'_2(C \leq C')(Z)$. If E' is an enrichment of E, then we write $E \leq_r E'$. The poset category $\langle CLO, \leq_r \rangle$, where CLO is the class of all cloning closure operators, will be denoted by CLO_r and called enrichmental poset category of cloning closure operators.

3. The clonings system of algebras

A cloning system of algebras is each functor $Q: Enr(AL) \longrightarrow CLO_r$ such that:

(1) for every algebra A Q(A) is a cloning closure operator which preserves the algebra A,

(2) for all algebras A and A' $Q(A)_1 = Q(A')_1$.

The covering poset category $Q^* = Q(A)_1$ from (2) is said to be the covering poset category of a cloning system Q of algebras. A selection over a cloning system Q of algebras is a mapping γ that relates to each algebra A a selection $\gamma(A)$ over the cloning closure operator Q(A). A selection of algebras is a mapping W that relates to each non-empty set X an algebra W(X) with $W(X)_0 = X$. By (2.3) we obtain:

(3.1) For each cloning system Q of algebras, each selection γ over Q and for every selection W of algebras the mapping H that relates to every algebra A an algebra $H(A) = Q(W(A_0))^{\gamma(W(A_0))}(A)$ is an enrichmental theory of algebras.

By easy verification we have the next fact.

(3.2) For each cloning system Q of algebras, each selection γ over Q and every enrichmental endofunctor H of algebras the mapping $m(Q,\gamma,H)$ that relates to each algebra A an algebra

$$m(Q,\gamma,H)(A) = Q(A)^{\gamma(A)}(H(A))$$

is an enrichmental endofunctor of algebras.

The endofunctor $m(Q,\gamma,H)$ for $H = id$ ($id(A) = A$) will be briefly denoted by $m(Q,\gamma)$. By (3.2) and (2.3) we obtain the following fact.

(3.3) If for a cloning system Q of algebras and for a selection γ over Q we have $Q(A) = Q(A')$ and $\gamma(A) = \gamma(A')$ provided $A_0 = A'_0$, then $m(Q,\gamma)$ is an enrichmental theory of algebras.

Let us consider the applications of (3.3).

E x a m p l e 1. To every algebra A we relate the interpolational operator $J(A) = I$ with $I_0 = A_0$. By (2.6) J is a cloning system of algebras and by (3.3) $m(J,c_w)$, where w is a mapping that relates to every non-empty set X a selection $w(X)$ over $I^{(X)}$ and c_w is such a selection over J that $c_w(A) = w(A_0)$ for all A, is an enrichmental theory of algebras.

E x a m p l e 2. To every algebra A we relate the localizational operator $l(A) = L$ with $L_0 = A_0$. By (2.8) the mapping l is a cloning system of algebras and by (3.3) $m(l,c_v)$, where v is a mapping that relates to every non-empty set X a selection $v(X)$ over $L^{(X)}$ and c_v is such a selection over l that $c_v(A) = v(A_0)$ for all A, is an enrichmental theory of algebras.

The Theorem (2.4) determines the algebraic cloning system al of algebras which does not fulfil the assumptions of (3.3).

Since a cloning system Q of algebras is a functor from Enr(AL) to CLO_r therefore if $A \leq_r A'$, then $Q(A) \leq_r Q(A')$ and thus every invariant space $\langle C,Z \rangle$ over $Q(A')$ is an invariant space over $Q(A)$ because $Z \subseteq Q(A)_C(Z) \subseteq Q(A')_C(Z) = Z$ or $Q(A)_C(Z) = Z$. Hence we have:

(3.4) If Q is a cloning system of algebras and for algebras A and A′ we have $A \leq_r A'$, then the category $Sp(Q(A'))$ of all invariant spaces over $Q(A')$ is a full subcategory of the category $Sp(Q(A))$.

If Q is a cloning system of algebras and V is a full subcategory of the poset category Q^* closed under the subspace operator $\bar{s}$, then V defines a cloning system $Q' = Q|V$ of algebras such that for every algebra A we have $Q'(A) = Q(A)|V$. Then Q' is called a subsystem of Q induced by V.

$Sp(Q'(A))$ is the full subcategory of $Sp(Q(A))$ defined by all invariant spaces $\langle C,Z \rangle$ with C in V. Every cloning system Q of algebras defines a comparison $\underset{Q}{\equiv}$ between algebras by the formula:

$$A \underset{Q}{\equiv} A' \text{ if and only if } A_0 = A'_0 \text{ and } Sp(Q(A)) = Sp(Q(A')).$$

If the comparison $\underset{Q}{\equiv}$ has an enrichmental hull, then we say that the enrichmental hull H of $\underset{Q}{\equiv}$ is an enrichmental hull of Q and we write $H = \bar{Q}$. Hence for an enrichmental theory H of algebras we have $H = \bar{Q}$ if and only if for all algebras A and A' $A \underset{Q}{\equiv} A'$ iff $H(A) = H(A')$.

If Q' is a subsystem of a cloning system Q of algebras and $A \underset{Q}{\equiv} A'$, then also $A \underset{Q'}{\equiv} A'$. Every cloning system Q of algebras such that $Q(A) = Q(A')$ provided $A_0 = A'_0$ has as an enrichmental hull the greatest enrichmental theory cm of algebras. Thus $\bar{J} = \bar{I} = cm$ by the examples 1 and 2. To determine the enrichmental hulls of the algebraic cloning system al of algebras and its subsystems let us consider the example 3 from the § 1. Every full subcategory V of the poset category $al^* = Set_s$ closed under the subspace operator $\bar{s}$ defines an enrichmental theory $Cs^{(V)}$ of algebras which relates to each algebra A the algebra $A' = Cs^{(V)}(A)$ such that $A'_0 = A_0$ and for all n, $A'_{1,n}$ is the set of all functions $f: A_0{}^n \longrightarrow A_0$ having the property (k) for all sets M in V (see § 1, example 3).

(3.5) T h e o r e m . The enrichmental theories Cs and $Cs^{(V)}$ of algebras are the enrichmental hulls of the cloning systems al and al|V of algebras, where V is any full subcategory of Set_s with $\bar{s}(V) = V$. Moreover, for every algebra A we have $A \underset{al}{\equiv} O(A) \underset{al}{\equiv} Cs(A)$ and $A \underset{al|V}{\equiv} O(A) \underset{al|V}{\equiv} Cs^{(V)}(A)$.

P r o o f . From the definitions of Cs and $Cs^{(V)}$ it follows that $Sp(al(A)) = \bigcup \left\{ \{M\} \times Su(Cs(A)^M) : M \in Ob\ Set \right\}$ and $Sp(al|V(A)) = \bigcup \left\{ \{M\}\ Su(Cs^{(V)}(A)^M) : M \in Ob\ V \right\}$ for every algebra A. Hence if $Cs(A) = Cs(A')$, then $Sp(al(A)) = Sp(al(A'))$

or $A \underset{al}{\equiv} A'$. If $A \underset{al}{\equiv} A'$, then from the definitions of $\underset{al}{\equiv}$ and Cs it follows that $Cs(A) = Cs(A')$. Thus $\overline{al} = Cs$. In analogical way is proved that $\overline{al|V} = Cs^{(V)}$. The rest is obvious.

From (3.5) for $V = \bar{s}(1)$, where $1 = \{\emptyset\}$ we obtain theorem of J. Schmidt [10] mentioned in § 1.

E x a m p l e 3. Let A be the algebra of all integers with the operations + and −. Moreover, let A' be the enrichment of A by adding the absolute value $|\cdot|$. Then $A \underset{al}{\equiv} A'$ and $O(A) \neq O(A')$, but $Cs(A) = Cs(A')$.

(3.6) If $O(A) = O(A')$, then $A \underset{al}{\equiv} A'$ and thus $Cs(A) = Cs(A')$.

P r o o f . By (3.5) $A \underset{al}{\equiv} O(A) = O(A') \underset{al}{\equiv} A'$ and thus $Cs(A) = Cs(A')$.

Let H and S be closed enrichmental endofunctors of algebras with $H \leq S$ (i.e. $H(A)_{1,n} \subseteq S(A)_{1,n}$ for all A and n). If Q is any cloning system of algebras, then an n-affine space over an algebra A under (Q,H,S) is an invariant space $\langle C,Z\rangle$ over Q(A) such that $C_0 = A_0{}^n$, $Z \geqslant S(A)_{1,n}$ and $Z = Q(A)_C(H(A)_{1,n})$. An algebra A is n-affine under (Q,H,S) if there is a n-affine space over A under (Q,H,S). An algebra A is affine under (Q,H,S) if A is n-affine under (Q,H,S) for all n. The n-affinity or affinity under (Q,H,cm) is called n-completion or completion under (Q,H). From the definitions it follows

(3.7) Every algebra A is affine under $(Q,H,m(Q,\gamma,H))$, where γ is any selection over Q.

An algebra A with $pr^n \subseteq A_{1,n}$ for all $n \neq 0$ is said to be a clone over a cloning system Q of algebras if there is a selection γ over Q such that $A = m(Q,\gamma)(A)$. If A is a clone over Q and $A = m(Q,\gamma|(A)$, then A is a γ-clone over Q. An algebra A is an algebraic clone if and only if A is a clone over al.

4. Interpolation-algebraic and localization-algebraic cloning systems of algebras

Let E and E' be two cloning closure operators such that $E_0 = E'_0$ and $E_1 = Set_s$. We say that E' preserves E provided for every morphism $C \leqslant C'$ in E_1 and for every $Z \subseteq E_0^{C'_0}$ which is $E_{C'_0}$-closed (i.e. $E_{C'_0}(Z) = Z$) the set $E'_2(C \leqslant C')(Z)$ is E_{C_0}-closed. Hence E' preserves an algebra A if and only if E' preserves al(A).

(4.1) Theorem. Let E and E' be two cloning closure operators such that $E_0 = E'_0$ and $E_1 = Set_s$. If E' preserves E, then a triple $E'' = \langle E''_0, E''_1, E''_2 \rangle$ such that $E''_0 = E'_0$, $E''_1 = E'_1$ and the map E''_2 is defined by the following formula

$$E''_2(C \leqslant C')(Z) = E'_2(C \leqslant C')(E_{C'_0}(Z))$$

is a cloning closure operator which preserves E. Moreover if $\langle C, Z \rangle$ is an object in $Sp(E'')$, then $\langle C_0, Z \rangle$ is an object in $Sp(E)$ and in this way a functor $F: Sp(E'') \longrightarrow Sp(E)$ is obtained.

Proof. Since E' preserves E the mapping E''_2 is a quasi-functor. Hence by easy verification we prove that E'' is a cloning closure operator which preserves E. The rest is obvious.

The cloning closure operator E'' given by (4.1) is denoted by $E' \triangle E$ and called a composition of E and E'. Now let us consider the applications of Theorem (4.1). By Theorems (2.6) and (2.8) the cloning closure operators $I^{(A_0)}$ and $L^{(A_0)}$ preserve al(A) and thus by (4.1) the formulas

$$Int(A) = I^{(A_0)} \triangle al(A) \quad \text{and} \quad Loc(A) = L^{(A_0)} \triangle al(A)$$

define the cloning closure operators preserving the algebra A and thus we obtain the interpolation-algebraic cloning system Int and the localization-algebraic cloning system Loc of algebras. Hence $Int^* = Cov_s$ and $Loc^* = OCov_s$. We prove that the systems Int and Loc and their subsystems have the enrichmental hulls. For this for each algebra A we define the algebras $A' = Cis(A)$ and $A'' = Cls(A)$. A′ or A″ is an algebra such that $A'_0 = A''_0 = A_0$ and for each natural number n an operation $f: A_0^n \longrightarrow A_0$ belongs to $A'_{1,n}$ or $A''_{1,n}$ if and only if for each covering space or for each open covering space C the following condition holds:

(u) if Z is a subuniverse of the algebra A^{C_0} and Z is $I_C^{(A_0)}$-closed or $L_C^{(A_0)}$-closed, then $\bar{f}(Z^n) \subseteq Z$, where $\bar{f} = f^{C_0}$.

The mappings Cis and Cls are enrichmental theories of algebras. Let V be any full subcategory of Cov_s or $OCov_s$ closed under the subspace operator $\bar{s}$. In an analogical way for any algebra A we define the algebras $Cis^{(V)}(A)$ and $Cls^{(V)}(A)$ by using (u) only for covering spaces C in V. The mappings $Cis^{(V)}$ and $Cls^{(V)}$ are also enrichmental theories of algebras. From definitions we have:

(4.2) If Q is Int or Loc and $A \leq_r A'$ and moreover $\langle C, Z\rangle$ is an object in Sp(Q(A)), then $\langle C, Z\rangle$ is an object in Sp(Q(A')) if and only if $\langle C_0, Z\rangle$ is an object in Sp(al(A')).

(4.3) Theorem. The enrichmental theories Cis, Cls, $Cis^{(V)}$ and $Cls^{(W)}$ of algebras are the enrichmental hulls of the cloning systems Int, Loc, Int|V and Loc|W of algebras, where V and W are any full subcategories of Cov_s and $OCov_s$ closed under the subspace operator $\bar{s}$. Moreover, for all algebras A we have $A \equiv_{Int} O(A) \equiv_{Int} Cis(A)$, $A \equiv_{Int|V} O(A) \equiv_{Int|V} Cis^{(V)}(A)$, $A \equiv_{Loc|W} O(A) \equiv_{Loc|W} Cls^{(W)}(A)$ and $A \equiv_{Loc} O(A) \equiv_{Loc} Cls(A)$.

Proof. Let us denote by $Siu_C(A^{C_0})$ or $Slu_C(A^{C_0})$ the set of all $I_C^{(A_0)}$-closed or $L_C^{(A_0)}$-closed subuniverses of the algebra A^{C_0}. We put $ti(C) = \{C\} \times Siu_C(A^{C_0})$ and $tl(C) = \{C\} \times Slu_C(A^{C_0})$. Then we have $Sp(Int(A)) = \bigcup \{ti(C) : C \in Ob\ Cov_s\}$, $Sp(Int|V(A)) = \bigcup \{ti(C) : C \in Ob\ V\}$, $Sp(Loc(A)) = \bigcup \{tl(C) : C \in Ob\ OCov_s\}$ and $Sp(Loc|W(A)) = \bigcup \{tl(C) : C \in Ob\ W\}$. Hence by the definitions of theories Cis, $Cis^{(V)}$, Cls and $Cls^{(W)}$ we obtain the first part of Theorem (4.3). The rest is obvious. This finishs the proof of Theorem (4.3).

If α is a selection over a cloning closure operator E, then we say that a covering space C α-admits a set $Z \leq E_0^{C_0}$

provided for all n, all $\varphi_1,\varphi_2,\dots,\varphi_n \in Z$ the following condition holds:

(ad) for each $X \in C_1$ there is $Y \in \alpha(n)_1$ such that $\langle\varphi_1,\varphi_2,\dots,\varphi_n\rangle(X) \subseteq Y$. Since $S^{(A_0)} \leq_r \mathrm{Int}(A)$ the selections over $I^{(A_0)}$ and Int(A) are the same.

(4.4) Theorem. Let H be an enrichmental endofunctor of algebras closed under an algebra A. If α is a selection over Int(A), $\langle C,Z\rangle$ is an invariant space over Int(A) such that $\langle C_0,Z\rangle$ is an invariant space over al(H(A)) and the covering space C α-admits Z, then $\langle C,Z\rangle$ is an invariant space over Int(A′), where $A' = \mathrm{Int}(A)^{(\alpha)}(H(A))$.

Proof. Since H is closed under A therefore $A' = I^{(\alpha)}(H(A))$, where I is the unique interpolational operator with $I_0 = A_0$. Hence for all n, $A'_{1,n} = I_{\alpha(n)}(H(A)_{1,n})$ and thus for every $f \in A'_{1,n}$ and every $Y \in \alpha(n)_1$ there is $h^{(Y)} \in H(A)_{1,n}$ with $f|Y = h^{(Y)}|Y$. By (4.2) it is sufficient to prove that $\langle C_0,Z\rangle$ is an object in Sp(al(A′)). For this let us consider any function $\varphi = f^{C_0}(\varphi_1,\varphi_2,\dots,\varphi_n)$, where $f \in A'_{1,n}$ and $\varphi_1,\varphi_2,\dots,\varphi_n \in Z$. The pair $\langle C,Z\rangle$ is an object in Sp(I). Hence $\varphi \in Z$ iff for every $X \in C_1$ there is $\psi \in Z$ with $\varphi|X = \psi|X$. Let $X \in C_1$. Since C α-admits Z there is, by (ad), $Y \in \alpha(n)_1$ with $\langle\varphi_1,\varphi_2,\dots,\varphi_n\rangle(X) \subseteq Y$. Thus $\varphi|X = f^X(\varphi_1|X,\varphi_2|X,\dots,\varphi_n|X) = \psi|X$ with $\psi = h^{(Y)}(\varphi_1,\varphi_2,\dots,\varphi_n) \in Z$, because $\langle C_0,Z\rangle$ is in Sp(al(H(A)). Hence $\varphi \in Z$ or $\langle C_0,Z\rangle$ is an object in Sp(al(A′)). This finishs our proof of Theorem (4.4).

Moreover, let us remark that $A' = m(\mathrm{Int},\gamma,H)(A)$, where A′ is the algebra in theorem (4.4) and γ is any selection over Int with $\gamma(A) = \alpha$.

(4.5) Theorem. Let H be an enrichmental endofunctor of algebras closed under an algebra A such that H(A) is an algebraical clone. Moreover let α be such a selection over Int(A) that for each n the covering space $\alpha(n)$ α-admits the set $H(A)_{1,n}$. Then the algebra

$$A' = Int(A)^{(\alpha)}(H(A))$$

is a clone over Int, i.e. A' is γ-clone over Int, where γ is any selection over Int with $\gamma(A') = \alpha$.

P r o o f . For all n, $A'_{1,n} = I^{(A_0)}_{\alpha(n)}(H(A)_{1,n})$. Hence the pair $\langle\alpha(n), A'_{1,n}\rangle$ fulfils the suppositions of Theorem (4.4). Indeed $\alpha(n)$ α-admits $A'_{1,n}$, because $\alpha(n)$ α-admits $H(A)_{1,n}$. But H(A) is an algebraical clone and thus $H(A)_{1,n}$ is a subuniverse of $F_n(H(A))$. Hence, by (2.6), $A'_{1,n}$ is a subuniverse of $F_n(H(A))$ or $\langle\alpha(n)_0, A'_{1,n}\rangle$ is an object in Sp(al(H(A))). Obviously $\langle\alpha(n), A'_{1,n}\rangle$ is an object in Sp(Int(A)). Hence, by Theorem (4.4), $\langle\alpha(n), A'_{1,n}\rangle$ is an object in Sp(Int(A')) for all n and thus $A' = Int(A')^{(\alpha)}(A')$ or A' is a γ-clone for any selection γ over Int with $\gamma(A') = \alpha$. α is a selection over Int(A') because $Int(A')_0 = A'_0 = A_0$. This finishs the proof of Theorem (4.5).

Let $\mathfrak{m}$ be any infinite cardinal number. Let us denote by $\alpha_{\mathfrak{m}}$ the selection over $I^{(A_0)}$ such that for all n $\alpha_{\mathfrak{m}}(n)_1$ is the set of all subsets X of $A_0{}^n$ with card(X)< $\mathfrak{m}$. The covering space $\alpha_{\mathfrak{m}}(n)$ $\alpha_{\mathfrak{m}}$-admits every set $Z \subseteq A_0{}^{A_0{}^n}$. Moreover every covering space C $\alpha_{\mathfrak{m}}$-admits any set $Z \subseteq A_0{}^{C_0}$. Hence by theorem (4.5) we obtain

(4.6) T h e o r e m . For every infinite cardinal number $\mathfrak{m}$ and for each enrichmental endofunctor H of algebras closed under an algebra A such that H(A) is an algebraical clone the algebra

$$A' = Int(A)^{(\alpha_{\mathfrak{m}})}H(A)$$

is a clone over Int, i.e. A' is a γ-clone over Int for any selection γ over Int with $\gamma(A') = \alpha_{\mathfrak{m}}$.

For a connexion to the last part of § 3 we give a remark.

R e m a r k . The determination of a characterization of algebras A being n-affine or affine under a given triple ~~(Q,H,S) is~~ a open problem. Some partial solutions of this problem for special cases: Q is al or Int|Fin (where Fin is the full subcategory of Cov_s defined by all C such that C_1 is the set of all finite subsets of C_0) and $(H,S) \in \{(p,Co), (p,cm), (O,cm)\}$ are given in papers [1,2,4,6,7].

5. The smooth cloning system of topological algebras

A topological algebra is a pair $A = \langle A_0, A_1 \rangle$ such that $A_0 = \langle A_{00}, A_{01} \rangle$ is a topological space and A_1 is a function that relates to each natural number n a set $A_{1,n}$ of n-ary continuous operations $f: A_{00}{}^n \longrightarrow A_{00}$ over the topological space A_0 (i.e. f is a morphism in the category Top from the product space $A_0{}^n$ to A_0).

Every topological algebra A determines an algebra U(A) such that $U(A)_0 = A_{00}$ and $U(A)_1 = A_1$. A morphism for a topological algebra A to a topological algebra A' is such a morphism $h = \langle h_0, h_1 \rangle$ from U(A) to U(A') in AL that h_0 is a morphism in Top from A_0 to A'_0. Hence we obtain a category TAL of topological algebras such that U is a forgetful functor from TAL to AL. Puting for topological algebras A and A' $A \leq_r A'$ if and only if $A_0 = A'_0$ and $A_{1,n} \subseteq A'_{1,n}$ for all n we have a poset category $Enr(TAL) = \langle Ob\ TAL, \leq_r \rangle$ which may be considered as a subcategory of TAL. The endofunctors of Enr(TAL) of the forms H such that $A \leq_r H(A)$ for all A are called enrichmental endofunctors of topological algebras. The monads of Enr(TAL) i.e. the enrichmental endofunctors H of topological algebras with $H^2 = H$ are said to be the enrichmental theories of topological algebras. For instance we have the enrichmental theories of topological algebras O and p such that O(A) and p(A) are the topological algebras with $O(A)_0 = A_0$, $O(A)_1 = O(U(A))_1$, $p(A)_0 = A_0$ and $p(A)_1 = p(U(A))_1$ for all topological algebras A.

A cloning system of topological algebras is a functor Q from the category Enr(TAL) to the category CLO_r such that:

(1) for all A Q(A) preserves U(A) ,

(2) for all A and A' $Q(A)_1 = Q(A')_1$,

(3) for all A, all selections α over Q(A) and all n we have if Z is a set of continuous n-ary operations over A_0, then the set $Q(A)_{\alpha(n)}(Z)$ is also a set of continuous n-ary operations over the topological space A_0.

Hence if Q is a cloning system of topological algebras and α is a selection over Q(A), then $Q(A)^{(\alpha)}$ determines a monad of the category $Enr_{A_0}(TAL)$ which is the full subcategory of Enr(TAL) defined by all topological algebras A' with $A'_0 = A_0$. For any topological algebra A' with $A'_0 = A_0$ we have $A'' = Q(A)^{(\alpha)}(A')$ if $A''_0 = A_0$ and $A''_{1,n} = Q(A)_{\alpha(n)}(A'_{1,n})$ for all n. A topological algebra A is a clone over Q if $A = Q(A)^{(\alpha)}(A)|$ for a selection α over Q(A) and $pr^n \subseteq A_{1,n}$ for all $n \neq 0$.

A cloning system Q of topological algebras defines a comparison $\underset{Q}{\equiv}$ of topological algebras such that

$$A \underset{Q}{\equiv} A' \quad \text{if and only if} \quad A_0 = A'_0 \quad \text{and} \quad Sp(Q(A)) = Sp(Q(A')).$$

If $\underset{Q}{\equiv}$ is a kernel of an enrichmental theory H of topological algebras, then H is called an enrichmental hull of Q.

For a topological algebra A, a set M and a set $Z \subseteq A_{00}{}^M$ let us denote by $in_A(Z)$ the least topological space (with respect to the enrichmental relation $\vdash_{\overline{r}}$) C with $C_0 = M$ such that every mapping $f \in Z$ is a morphism in Top from C to A_0. Then we have the following facts:

$$(5.1) \qquad in_A(Z) = in_A(Sg_{U(A)M}(Z)),$$

$$(5.2) \qquad in_A(Z|M') = in_A(Z)|M' = in_A(Z_1), \quad \text{where} \quad M' \subseteq M;$$

$$Z_1 = L_2(in_A(Z)|M' \leq in_A(Z))(Z)$$

and L is the localizational operator with $L_0 = A_{00}$.

Using (5.1) and (5.2) we prove the next theorem.

(5.3) T h e o r e m . We have a cloning system

$$D : Enr(TAL) \longrightarrow CLO_r$$

of topological algebras that relates to a topological algebra A a cloning closure operator D(A) such that $D(A)_0 = A_{00}$, $D(A)_1 = Set_s$ and

$$D(A)_2(M' \subseteq M)(Z) = L_2(in_A(Z)|M' \leq in_A(Z)(Sg_{U(A)M}(Z))$$

where L is the localizational operator with $L_0 = A_{00}$.

Let us observe that an invariant space over D(A) is a pair $\langle M,Z\rangle$ such that $Z \subseteq A_{00}{}^M$, the pair $\langle in_A(Z),Z\rangle$ is an invariant space over Loc(U(A)) and $\langle M,Z\rangle$ is an invariant space over al(U(A)). Hence it follows that Sp(D(A)) is a full coreflective subcategory of Sp(al(U(A))). Moreover, we have the following fact:

(5.4) The mapping

$$\langle M,Z\rangle \longmapsto in_A(Z)$$

defines a functor $\overline{in}_A$: Sp(D(A)) ⟶ Top having a left adjoint. The unit and counit of this adjunction are pointwise bimorphisms.

P r o o f . If $f:\langle M,Z\rangle \longrightarrow \langle M',Z'\rangle$ is a morphism in Sp(D(A)), then f is a morphism in Top from $in_A(Z)$ to $in_A(Z')$ since, for all h in Z', $h \circ f \in Z$ and thus $h \circ f$ is a continuous mapping from $in_A(Z)$ to A_0. Hence $\overline{in}_A$ is a functor. For each topological space C the pair $\langle C_0,Z\rangle$, where Z is the set of all morphisms in Top from C to A_0, is an object in Sp(D(A)). Puting $F^*(C) = \langle C_0,Z\rangle$ we obtain a left adjoint functor to $\overline{in}_A$. The remain part is obvious.

By (5.4) every morphism in Sp(D(A)) is continuous and any isomorphism in Sp(D(A)) is an homeomorphism. Therefore the morphisms and isomorphisms in Sp(D(A)) are called smooth mappings and diffeomorphisms over the topological algebra A and D is said to be a smooth cloning system of topological algebras.

If A is a topological algebra and M a set, then a subuniverse Z of $U(A)^M$ is said to be closed under localization provided $Z = L_{in_A(Z)}(Z)$, where L is the localizational operator with $L_0 = A_{00}$. The set of all subuniverses of $U(A)^M$ closed under the localization is denoted by $Slcu(U(A)^M)$ and the product $\{M\} \times Slcu(U(A)^M)$ by $t_A(M)$. With each topological algebra A we relate a topological algebra $A' = Cts(A)$ such that $A'_0 = A_0$ and, for all n, $A'_{1,n}$ is the set of all continuous n-ary operations $f: A_{00}{}^n \longrightarrow A_{00}$ over A_0 having the property that for all set M holds (1c):

(1c) if Z is a subuniverse of $U(A)^M$ closed under the localization, then $f^M(Z^n) \subseteq Z$. Hence Cts is an enrichmental theory of topological algebras. For every full subcategory V of Set_s closed under the operator $\bar{s}$ we obtain an enrichmental theory $Cts^{(V)}$ of topological algebras by using the condition (1c) only for M in V. Moreover, we have a cloning system $Q = D|V$ of topological algebras such that $Q(A) = D(A)|V$ for all topological algebras A. Let us observe that $Sp(D(A)) = \bigcup \{t_A(M) : M \in Ob\ Set_s\}$ and $Sp(D|V(A)) = \bigcup \{t_A(M) : M \in Ob\ V\}$. Hence and from the definitions of Cts and $Cts^{(V)}$ it follows that Cts and $Cts^{(V)}$ are the enrichmental hulls of D and $D|V$ and thus we have proved the next theorem.

(5.5) T h e o r e m . The enrichmental theories Cts and $Cts^{(V)}$ are the enrichmental hulls of the cloning systems D and D/V of topological algebras. Moreover, for each topological algebra A we have: $A \underset{D}{\equiv} O(A) \underset{D}{\equiv} Cts(A)$ and $A \underset{D|V}{\equiv} O(A) \underset{D|V}{\equiv} Cts^{(V)}(A)$.

Now we prove

(5.6) T h e o r e m . Let H be an enrichmental endofunctor of algebras closed under U(A), where A is a topological algebra. Moreover, let $\langle M,Z \rangle$ be an object in Sp(D(A)) and in Sp(al(H(U(A)))). Then we have:

I. If X is a open subset in the product space $A_0{}^n$ and $\varphi_1, \varphi_2, \ldots, \varphi_n$ are such functions in Z that $f = \langle \varphi_1, \varphi_2, \ldots, \varphi_n \rangle : M \longrightarrow X$, then $h \circ f \in Z$ for all $h \in L_2(A_0{}^n | X \leqslant \leqslant A_0{}^n(H(U(A))_{1,n})$, where L is the localizational operator with $L_0 = A_{00}$.

II. The pair $\langle M,Z\rangle$ is a object in $Sp(al(A'))$, where

$$A' = Loc(U(A))^{(\alpha_A)}(H(U(A))) = L^{(\alpha_A)}(H(U(A)))$$

and α_A is a selection over $Loc(A)$ and thus over L such that $\alpha_A(n) = A_0{}^n$ for all n.

P r o o f . Let $H_n = H(U(A))_{1,n}$. There is $C^* \vdash_{\overline{r}} A_0{}^n$ such that $h \in I_2(C^*|X \leqslant C^*)(H_n)$. Let $p \in M$. Then $q = f(p) \in X$ and there is open subset $X^{(p)}$ in $A_0{}^n$ such that $q \in X^{(p)} \in C_1^*$ and thus $Y^{(p)} = X \cap X^{(p)}$ belongs to $C^*|X_1$ and it is also open in $A_0{}^n$. Hence there is $d^{(p)}$ in H_n such that $h|Y^{(p)} = d^{(p)}|Y^{(p)}$. Since $f: in_A(Z) \longrightarrow A_0{}^n$ is a morphism in Top the set $X_1^{(p)} = f^{-1}(Y^{(p)})$ is open in $in_A(Z)$. Thus we have obtained a covering space $C \vdash_{\overline{r}} In_A(Z)$ with $C_0 = M$ and $C_1 = \{X_1^{(p)} : p \in M\}$ such that $h \circ f|X^{(p)} \circ f|X^{(p)}$ for all $p \in M$, where $d^{(p)} \circ f \in Z$ because $\langle M,Z\rangle$ is an object in $Sp(al(H(U(A))))$. Hence $h \circ f \in I_C(Z)$ and thus $h \circ f \in L_{in_A(Z)}(Z) = Z$ because $\langle M,Z\rangle$ is an object in $Sp(D(A))$. This finishs the proof of part I. The part II follows from the part I for $X = A_{00}{}^n$.

(5.7) T h e o r e m . Let H be an enrichmental endofunctor of algebras closed under $U(A)$, where A is a topological algebra, such that $H(U(A))$ is an algebraical clone with continuous fundamental operations over the topological space A_0. Then the algebra A' from 5.6.II is a clone over Loc, i.e. A' is γ-clone over Loc for any selection γ over Loc such that $\gamma(A') = \alpha_A$. Moreover, the topological algebra A'' such that $A''_0 = A_0$ and $A''_1 = A'_1$ is a clone over D.

P r o o f . Let $A^* = H(U(A))$ and let $M = A_{00}{}^n$. Then $A'_{1,n} = L_{A_0{}^n}(A^*_{1,n})$. Since A^* is an algebraical clone $A^*_{1,n}$ is a subuniverse of A^{*M} and thus, by (2.8), $A'_{1,n}$ is a subuniverse of A^{*M} or $\langle M, A'_{1,n}\rangle$ is an object in $Sp(al(A^*))$. But

$A^*_{1,n}$ is a set of continuous n-ary operations over A_0 and thus $L_{in_A(A'_{1,n})}(A'_{1,n}) = A'_{1,n}$ or $\langle M, A'_{1,n}\rangle$ is an object in Sp(D(A)). Hence, by 5.6.II, $\langle M, A'_{1,n}\rangle$ is an object in Sp(al(A′)) and by (4.2), the pair $\langle A_0{}^n, A'_{1,n}\rangle$ is an object in Sp(Loc(A′)) for all n or $A' = Loc(A')^{(\alpha_A)}(A')$, i.e. A′ is a clone over Loc. Moreover, by the above considerations $A'' = D(A'')^{(\alpha)}(A'')$, where α is the unique selection over D(A″) and $\alpha(n) = A_{00}{}^n$, or A″ is a clone over D.

Let us observe that for instance the enrichmental theories O and p of algebras fulfil the suppositions concerning H of the theorems (4.6), (5.6) and (5.7) for all algebras or topological algebras A.

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