DEMONSTRATIO MATHEMATICA
Vol. XX No 1-2 1987

Bolesław Wojdyło

INTERPOLATION IN UNIVERSAL (PARTIAL) ALGEBRAS

1. Introduction

A classical notion of interpolation occurs on the ground of numerical methods as a some kind of approximation. Namely, if there is a finite table of real function f:

x	x_0	x_1	...	x_n
$f(x)$	y_0	y_1	...	y_n

where $x_i, y_i \in R$ and $x_i \neq x_j$ for $i \neq j$ $(i,j \in \{0,1,\ldots,n\})$, then it appears a problem of existence and uniqueness of a polynomial function φ of the degree less or equal to n, such that $\varphi(x_i) = y_i$ for $i = 0,1,\ldots,n$.

The next problem is to give an effective method to find such polynomial function φ (if it exists). It is very well known that exactly one such polynomial function φ exists in this case and there are given the simple (Lagrange, Newton) formulas which describe it.

The situation presented above is an example of so called local interpolation of unary functions by the unary polynomial functions over the field of real numbers R. We can say about global interpolation too, i.e. about substitution of "whole" functions by polynomial ones. Moreover, the problem of inter-

---

This paper is based on the lecture presented at the Conference on Universal Algebra held at the Pedagogical University in Opole (Jarnołtówek), May 23-25, 1985.

polation can be generalized in two directions: on n-ary functions ($n \in N$), or on other algebras than the field of real numbers (or even an arbitrary field).

Our purpose is a more general situation, namely interpolation in universal partial algebras. Before this we give a short survey of known facts concerning of interpolation in universal (total) algebras.

2. Interpolation in universal algebras

Let $\underline{A} = \langle A, (g_{\underline{A}}, g \in G)\rangle$ be a universal algebra of a finite type G. Denote by $\underline{F_k(A)}$ an algebra of type G such that $F_k(A) = \{f \mid f: A^k \longrightarrow A\}$ and for every $g \in G$, the operation $g_{\underline{F_k(A)}}$ is defined componentwise, i.e.:

for each $(f_\sigma, \sigma < n(g)) \in F_k(A)^{n(g)}$ $g_{\underline{F_k(A)}}(f_\sigma, \sigma < n(g)) = f$ such that for every $\underline{a} \in A^k$ $f(\underline{a}) = g_{\underline{A}}(f_\sigma(\underline{a}), \sigma < n(g))$.

A subalgebra $\underline{P_k(A)}$ of the algebra $\underline{F_k(A)}$, generated by all constant functions and all k-ary projections, is called the algebra of k-ary polynomial functions on A. An algebra A of type G is called k-polynomially complete (or A has a k-interpolation property), if $P_k(A) = F_k(A)$. If $\underline{A}$ is k-polynomially complete for every $k \in N$, then we say that $\underline{A}$ is polynomially complete (or $\underline{A}$ has an interpolation property). An algebra $\underline{A}$ of type G is called locally k-polynomially complete (or $\underline{A}$ has a local k-interpolation property), if for each $f \in F_k(A)$ and every finite subset $B \subseteq A^k$, there exists a polynomial function $\varphi \in P_k(A)$ such that $f|B = \varphi|B$ (cf. [8], [7]).

There are known the following results concerning the polynomial completeness of universal algebras.

T h e o r e m 1 ([8]). For any algebra $\underline{A}$ of type G it holds one from the following three cases:

(1) $\underline{A}$ is k-polynomially complete for no $k \in N$.

(2) $\underline{A}$ is only 1-polynomially complete.

(3) $\underline{A}$ is k-polynomially complete for every $k \in N$.

R e m a r k . We shall say ([8]) that algebras which have the property (1) (resp. (2) or (3)) are polynomially incomplete (resp. semicomplete or complete).

Theorem 1 holds for the local interpolation too (see [6]). From Theorem 1 follows that if an algebra $\underline{A}$ is 2-polynomially complete, then it is polynomially complete (cf. [9]). Let us note that if an algebra $\underline{A}$ is a local 1-polynomially complete, then it is a simple algebra ([8]). H. Kaiser ([7]) has proved on the base of results of W. Sierpiński ([9]) and J. Słupecki ([11]) the following theorem:

T h e o r e m 2 ([7]). Let $\underline{A}$ be a universal algebra which contains at least three elements and let $\underline{A}$ be locally 1-polynomially complete. If there is a surjective binary function over A which has the local interpolation property and depends on both variables, then A is locally polynomially complete.

2.1. Examples of polynomially complete algebras

Examples 1 - 4 cf. [8].

E x a m p l e 1. Only finite fields are the polynomially complete algebras in the variety of commutative rings with identity. All the other algebras of this variety are polynomially incomplete.

E x a m p l e 2. In the variety of lattices, every non-trivial algebra is polynomially incomplete.

E x a m p l e 3. In the variety of Boolean algebras, the algebra of order 2 is polynomially complete. All the other algebras are polynomially incomplete.

E x a m p l e 4. In the variety of groups, the finite non-abelian simple groups are polynomially complete, the group of order 2 is polynomially semicomplete, and all the other groups are polynomially incomplete.

The same as in Example 4 holds for the local polynomial completeness ([6]). Moreover H. Kaiser ([4]) has characterized locally polynomially complete rings:

E x a m p l e 5. A ring is a locally polynomially complete iff it is a simplo non-zero ring. A ring is only semilocally polynomially complete iff it is a ring of order 2.

From the interpolation point of view it is an interesting class of so called "bidual algebras", i.e. such algebras (with constant e) in which congruence relations are uniquely determined by its congruent classes of the element e. Investigation of bidual algebras was initiated by J. Słomiński ([10]). Examples of classes of bidual algebras are: groups, rings, almost-rings, loops, Boolean algebras.

The following theorems take place for the bidual algebras ([1]):

T h e o r e m 3. Bidual algebra $\underline{A}$ is a locally polynomially complete iff it is simple and exists 2-ary function q such that:

(1) q is not constant,

(2) q has local interpolating property,

(3) for every $a \in A$ holds $q(a,e) = q(e,a) = e$.

T h e o r e m 4. Bidual algebra $\underline{A}$ is locally polynomially complete iff a "discriminator" $d: A^3 \longrightarrow A$ defined as follows

$$d(x,y,z) = \begin{cases} x & \text{for} \quad x \neq y \\ z & \text{for} \quad x = y \end{cases}$$

is a locally polynomial function.

3. Interpolation in partial algebras

Let $\underline{A}$ be a partial algebra of type G (see [2] or [12]). Denote by $\underline{PF_k(A)}$ and $\underline{PP_k(A)}$ the algebras of type G of all k-ary partial functions and all k-ary partial polynomial functions on A respectively. Let us notice that these algebras are total algebras of type G. The problem of interpolation in partial algebras was not investigated up till now. This question is, in the case of partial algebras, more complicated than the problem of interpolation in the case of total algebras, namely:

1) A number of interpolated functions grows (essentially partial functions are added) and at the same time a number of polynomial functions (interpolating functions) grows small, e.g. in the case of discrete algebra, the only polynomial functions are: constant functions, projections and empty function.

2) It is possible to consider the partial projections or total ones.

3) There are a few sorts of validity of equations in partial algebras (farther on we shall consider four of them).

4) For any k-ary partial function $f: A^k \to A$ all finite subsets of A^k or only all finite subsets of dom f can be considered (in the case of local interpolation).

Let us recall considered in this paper four kinds of validity of equations in partial algebras ([2], [12]).

Denote by ${}_{\underline{A}}p$ an α-ary operation induced in a partial algebra $\underline{A}$ by a term $p \in P(\alpha,G)$, where $\underline{P(\alpha,G)}$ is a Peano-algebra of type G with basis $X = \{x_\lambda \mid \lambda<\alpha\}$. Let $\ulcorner p = q \urcorner$, $p,q \in P(\alpha,G)$, be an equation. We say that this equation is:

- weakly valid in $\underline{A}$, iff for all $\underline{a} \in A^\alpha$ we have: ${}_{\underline{A}}p(\underline{a}) = {}_{\underline{A}}q(\underline{a})$ provided $\underline{a} \in \mathrm{dom}_{\underline{A}}p \cap \mathrm{dom}_{\underline{A}}q$;
- valid in $\underline{A}$, iff $\mathrm{dom}_{\underline{A}}p = \mathrm{dom}_{\underline{A}}q$ and ${}_{\underline{A}}p(\underline{a}) = {}_{\underline{A}}q(\underline{a})$ for $\underline{a} \in \mathrm{dom}_{\underline{A}}p$ $(=\mathrm{dom}_{\underline{A}}q)$;
- strongly valid in $\underline{A}$, iff $p \sim_{\underline{a}} q$ for each $\underline{a} \in A^\alpha$, where $\sim_{\underline{a}}$ is the least congruence relation $\sim$ of Peano-algebra $\underline{P(\alpha,G)}$ such that for all $r,s \in P(\alpha,G)$ we have: $r \sim s$ if $\underline{a} \in \mathrm{dom}_{\underline{A}}r \cap \mathrm{dom}_{\underline{A}}s$ and ${}_{\underline{A}}r(\underline{a}) = {}_{\underline{A}}s(\underline{a})$;
- existentially valid in $\underline{A}$, iff $\mathrm{dom}_{\underline{A}}p = \mathrm{dom}_{\underline{A}}q = A^\alpha$ and ${}_{\underline{A}}p(\underline{a}) = {}_{\underline{A}}q(\underline{a})$ for all $\underline{a} \in A^\alpha$.

With respect to on 2) - 4) we recive 16 cases of local interpolation for partial algebras (see Table 1).

Table 1

Total projections							
$B \subseteq A^k$				$B \subseteq \mathrm{dom}\, f$			
w.v.	v.	s.v.	e.v.	w.v.	v.	s.v.	e.v.
I_t	II_t	III_t	IV_t	V_t	VI_t	VII_t	$VIII_t$
Partial projections							
$B \subseteq A^k$				$B \subseteq \mathrm{dom}\, f$			
w.v.	v.	s.v.	e.v.	w.v.	v.	s.v.	e.v.
I	II	III	IV	V	VI	VII	VIII

Denotations:

w.v. - weak validity of equations,
v. - validity of equations,
s.v. - strong validity of equations,
e.v. - existence validity of equations.

Let us notice that the following implications take place:

(i) Because of e.v. $\Rightarrow$ s.v. $\Rightarrow$ v. $\Rightarrow$ w.v. we have:
$IV_t \Rightarrow III_t \Rightarrow II_t \Rightarrow I_t$, $VIII_t \Rightarrow VII_t \Rightarrow VI_t \Rightarrow V_t$,
$IV \Rightarrow III \Rightarrow II \Rightarrow I$, $VIII \Rightarrow VII \Rightarrow VI \Rightarrow V$.

(ii) Because of $\mathrm{dom}\, f \subseteq A^k$ we receive:
$IV_t \Rightarrow VIII_t$, $IV \Rightarrow VIII$, $II_t \Rightarrow VI_t$, $II \Rightarrow VI$,
$III_t \Rightarrow VII_t$, $III \Rightarrow VII$, $I_t \Rightarrow V_t$, $I \Rightarrow V$.

(iii) Because total projections are partial projections, we have:
$I_t \Rightarrow I$, $II_t \Rightarrow II$, $III_t \Rightarrow III$, $IV_t \Rightarrow IV$,
$V_t \Rightarrow V$, $VI_t \Rightarrow VI$, $VII_t \Rightarrow VII$, $VIII_t \Rightarrow VIII$.

Here $K \Rightarrow L$ means: if some theorem about interpolation property holds in the case K then this theorem holds in the case L too, where $K, L \in \{I_t, \ldots, VIII_t, I, \ldots, VIII\}$.

The case I is trivial, namely it is easy to see that the following theorem takes place:

T h e o r e m 5. Any partial algebra of type G is in the case I locally polynomially p-complete[1].

P r o o f . Let $\underline{A}$ be a partial algebra of type G and let f: $A^k \longrightarrow A$ be an arbitrary k-ary partial function. Let us notice that "empty" k-ary projection $e_k^{\emptyset}$ is a such polynomial function that for any finite subset $B \subseteq A^k$ holds $f|B = e_k^{\emptyset}$.

T h e o r e m 6. If a total algebra $\underline{A}$ of type G is locally polynomially complete, then it is, in the case II, locally polynomially p-complete.

P r o o f . Let f: $A^k \longrightarrow A$. An extension f' of f to total function exists (of course not unique). By assumption f' has a local interpolation property. Let $B \subseteq A^k$ be a finite subset. Then there exists polynomial function φ such that $f'|B = \varphi|B$. By substitution of projections occurring in the φ by suitable partial projections we recive a partial polynomial function $\tilde{\varphi}$ such that $\mathrm{dom}\, f \cap B = \mathrm{dom}\, \tilde{\varphi} \cap B$ and for any $\underline{a} \in \mathrm{dom}\, \tilde{\varphi}$ holds: $\tilde{\varphi}(\underline{a}) = \varphi(\underline{a})$, i.e. $f|B = \tilde{\varphi}|B$.

C o r o l l a r y 6.1. Theorem 6 holds in the following cases: I, II, V, VI, VIII, VI_t, $VIII_t$, VII, VII_t.

P r o o f . See (i) and (ii). Moreover let us notice that for $B \subseteq \mathrm{dom}\, f$ equation $f|B = \varphi|B$ is valid iff it is existentially valid. Projections used in definition of φ are total on B. ∎

Now we quote two theorems (Theorem 7 and Theorem 8) which base on the results given by W. Sierpiński [9] and J. Słupecki [11]. These theorems are generalizations in the case of partial algebras of corresponding theorems given by H. Kaiser [7] for total algebras.

T h e o r e m 7. Let $n \geq 2$. If partial algebra $\underline{A}$ of type G is, in the case II_t, n-locally[2] 2-polynomially p-com-

---

1) A polynomial p-completeness denotes the polyncmial completeness for partial algebras.

2) It means that we consider only such subsets $B \subseteq A^k$ which cardinalities are equal to n, where $n \in N$.

plète, then $\underline{A}$ is, in this case, n-locally polynomially p-complete.

P r o o f . (It is a modification of the proof given by H. Kaiser [7], Proof of Theorem 1). Let f be a partial k-ary function on A, $k \in N$. Let 0 be a fixed element of A. We define three 2-ary functions $\square, \circ, \Delta$ as follows:

$$x \square y = \begin{cases} 0 & \text{if } x = y \\ \neq 0 & \text{if } x \neq y \end{cases}$$

$$x \circ y = \begin{cases} 0 & \text{if } x \neq 0 \\ y & \text{if } x = 0 \end{cases}$$

$$x \Delta y = \begin{cases} x & \text{if } y = 0 \\ y & \text{if } x = 0 \\ \neq 0 & \text{if } x \neq 0 \text{ and } y \neq 0. \end{cases}$$

Let $B = \{\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n\} \in A^k$ be a such subset that $\underline{a}_i \in \operatorname{dom} f$ for $i \in \{1,\ldots,r\}$ and $\underline{a}_j \notin \operatorname{dom} f$ for $j \in \{r+1,\ldots,n\}$. Now we define the functions f_i, $1 \leq i \leq n$, as follows:

$$f_i(\underline{x}) = \begin{cases} c_i & \text{if } \underline{x} = \underline{a}_i \text{ and } 1 \leq i \leq r \text{ and } c_i = f(\underline{a}_i) \\ 0 & \text{if } \underline{x} = \underline{a}_j \text{ and } i \neq j \text{ and } 1 \leq i,j \leq r \\ \text{undefined elsewhere.} \end{cases}$$

Let us notice that:

$$f_i(x_1, x_2, \ldots, x_k) = \begin{cases} ((\ldots((x_1 \square a_{i1}) \Delta (x_2 \square a_{i2}))\Delta \ldots)\Delta(x_k \square a_{ik}))\circ c_i \\ \text{undefined for } \underline{x} \notin \{\underline{a}_1, \ldots, \underline{a}_r\}. \end{cases}$$

Denote by $g = f_1 \vee f_2 \vee \ldots \vee f_r$, where:

$$x \vee y = \begin{cases} x & \text{if } y = 0 \\ y & \text{if } x = 0 \\ \text{undefined in all other cases} \end{cases}$$

So we have $f|B = g|B$.

The function g has the n-interpolation property, because it is a composition of 2-ary functions and the partial algebra $\underline{A}$ is n-locally 2-polynomially p-complete. Hence there exists a polynomial function φ such that $g|B = \varphi|B$, i.e. $f|B = \varphi|B$.

C o r o l l a r y 7.1. Theorem 7 holds in the following cases: I, II, V, VI, I_t, II_t, V_t, VI_t.

P r o o f . See (i) - (iii).

T h e o r e m 8 (case II_t). Let $\underline{A}$ be a partial algebra of type G with at least three elements set A and let $\underline{A}$ be locally 1-polynomially p-complete. If there exists, defined on A, surjective binare partial function which depends on both variables and having local partial interpolation property, then $\underline{A}$ is locally polynomially p-complete.

P r o o f . A spacious proof of this theorem is a modification of the proof given by H. Kaiser for total algebras [7], Proof of Theorem 2.

C o r o l l a r y 8.1. Theorem 8 holds in the following cases: I, II, V, VI, I_t, II_t, V_t, VI_t.

P r o o f . See (i) - (iii).

REFERENCES

[1] E. B e u t l e r , H. K a i s e r , G. M a t t h i e s s e n , J. T i m m : Biduale Algebren, Preprint, Universitat Bremen, 1979, 1-160.

[2] P. B u r m e i s t e r : Partial algebras - survey of a unifying approach towards a two-valued model theory for partial algebras, Algebra Universalis, 15 (1982) 306-358.

[3] G. G r ä t z e r : Universal Algebra, Springer-Verlag, 1979.

[4] H.K. K a i s e r : Uber das Interpolationsproblem in nichtkommutativen Ringen, Acta Sc. Math. 36 (3-4) (1974) 271-273.

[5] H.K. K a i s e r : A class of locally complete universal algebras, J. London Math. Soc. (2), 9, 1974.

[6] H.K. K a i s e r : Uber local polynomvollstandige universale Algebren, Abh. Math. Sem. Hamburg 43 (1975) 158-165.

[7] H.K. K a i s e r : A remark on algebras with the interpolation property, Preprint, TU Wien, 1978, 1-7.

[8] H. L a u s c h , W. N ö b a u e r : Algebra of Polynomials, North-Holland, 1973.

[9] W. S i e r p i ń s k i : Sur les fonctions de plusieurs variables, Fund. Math. 33 (1945) 169-175.

[10] J. S ł o m i ń s k i : On the determining of the form of congruences in abstract algebras with equationally definable constant elements, Fund. Math. 48 (1960) 325--341.

[11] J. S ł u p e c k i : Kryterium pełności wielowartościowych systemów logiki zdań, C. R. Soc. Sci. Varsovie, 32 (1939) 102-110.

[12] B. W o j d y ł o : Categories of quasi-algebras, Preprint 2, UMK Toruń, 1972, 1-60.

INSTITUTE OF MATHEMATICS, NICOLAUS COPERNICUS UNIVERSITY
IN TORUŃ, 87-100 TORUŃ, POLAND
Received July 3, 1985.