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SOME FIXED POINT THEOREMS OF EXPANSION MAPPINGS

In a recent paper [3] some fixed point theorems of expansion mappings are proved. In this note using a combination of methods used in [1]-[3] other fixed point theorems of expansion mappings are proved.

Theorem 1. Let (X,d) be a complete metric space and $f: (X,d) \to (X,d)$ a surjective mapping. If there exist non-negative reals a,b,c with $a>0$ or $b>0$ and $a+b+c>1$ such that

$$(1)\quad d^2(fx,fy) \geq ad(x,fx)d(x,y)+bd(y,fy)d(x,y)+cd(x,fx)d(y,fy)$$

for each x,y in X with $x \neq y$, then f has a fixed point.

Proof. Let $x_0 \in X$. Since f is surjective, there exists an element x_1 satisfying $x_1 \in f^{-1}(x_0)$. In the same way we can take $x_n \in f^{-1}(x_{n-1})$, $n=2,3,\ldots$. If $x_m = x_{m-1}$ for some m, then x_m is a fixed point of f. Without loss of generality, we can suppose $x_{n-1} \neq x_n$ for every n.

If $b>0$ then from (1)

$$d^2(x_{n-1},x_n) = d^2(fx_n,fx_{n+1}) \geq ad(x_n,fx_n)d(x_n,x_{n+1}) +$$

$$+ bd(x_{n+1},fx_{n+1})d(x_n,x_{n+1}) + cd(x_n,fx_n)d(x_{n+1},fx_{n+1}) =$$

$$= ad(x_n,x_{n-1})d(x_n,x_{n+1}) + bd^2(x_{n+1},x_n) + cd(x_n,x_{n-1})d(x_{n+1},x_n).$$

Thus

$$bd^2(x_{n+1},x_n) + (a+c)d(x_{n+1},x_n)d(x_n,x_{n-1}) - d^2(x_{n-1},x_n) \leq 0$$

and $bt^2 + (a+c)t-1 \leq 0$, where $t = d(x_{n+1},x_n)/d(x_n,x_{n-1})$.

Let $g: [0,\infty) \to R$ be the function $g(t) = bt^2 + (a+c)t-1$. Then $g(0) = -1$ and $g(1) = a+b+c-1 > 0$ from the hypothesis. Let $k \in (0,1)$ be the root of the equation $g(t) = 0$, then $g(t) \leq 0$ for $t < k$ and thus

$$d(x_{n+1},x_n) \leq kd(x_n,x_{n-1}) \leq \dots \leq k^n d(x_1,x_0).$$

Then by a routine calculation one can show that $\{x_n\}$ is a Cauchy sequence and since X is complete we have $\lim x_n = x$ for some $x \in X$. Let $y \in f^{-1}(x)$. Then we have

$$d^2(x_n,x) = d^2(fx_{n+1},fy) \geq ad(x_{n+1},fx_{n+1})d(x_{n+1},y) +$$

$$+ bd(y,fy)d(x_{n+1},y) + cd(x_{n+1},fx_{n+1})d(y,fy) =$$

$$= ad(x_{n+1},x_n)d(x_{n+1},y) + bd(y,x)d(x_{n+1},y) + cd(x_{n+1},x_n)d(x,y)$$

and on letting n tend to infinity we have $0 \geq bd^2(x,y)$ which implies $y = x$. Since $x = f(y)$, then $x = f(x)$. So f has a fixed point.

The theorem may be proved in an analogous way if $a > 0$.

Remark. A fixed point of f is not unique in general. The identity mapping satisfies the condition of Theorem 1.

Theorem 2. Let (X,d) be a complete metric space and $f: (X,d) \to (X,d)$ a surjective mapping. If there exist non-negative reals a,b,c with $a+b+c > 2$ such that

$$(2) \quad d(fx,fy) \geq \frac{ad(x,fx)d(x,y)+bd(y,fy)d(x,y)+cd(x,fx)d(x,fy)}{d(x,fx)+d(y,fy)}$$

for each $x \neq y$ in X for which $d(x,fx)+d(y,fy) \neq 0$, then f has a fixed point.

Proof. It is similar to the proof of Theorem 1.

Theorem 3. Let (X,d) be a complete metric space and $f:(X,d) \to (X,d)$ a surjective continuous mappings. If there exists a real constant $k > 1$ such that

(3) $$d^2(fx,fy) \geq$$

$$\geq k \min\{d(x,fx)d(x,y);d(y,fy)d(x,y);d(x,fx)d(y,fy)\}$$

for any x,y in X, then f has a fixed point.

P r o o f . As in Theorem 1, there is a sequence $\{x_n\}$ with $x_{n-1} \neq x_n$ and $x_{n-1} = f(x_n)$. This implies

$$d^2(x_{n-1},x_n) = d^2(fx_n,fx_{n+1}) \geq k \min\{d(x_n,fx_n)d(x_n,x_{n+1});$$
$$d(x_{n+1},fx_{n+1})d(x_n,x_{n+1});d(x_n,fx_n)d(x_{n+1},fx_{n+1})\} =$$
$$= k \min\{d(x_n,x_{n-1})d(x_n,x_{n+1});d^2(x_{n+1},x_n);d(x_n,x_{n-1})d(x_n,x_{n+1})\} =$$
$$= kd(x_n,x_{n+1}) \min\{d(x_n,x_{n+1});d(x_n,x_{n-1})\}.$$

We have either

$$d^2(x_{n-1},x_n) \geq kd(x_{n+1},x_n)d(x_{n-1},x_n)$$

which implies

$$d(x_{n-1},x_n) \geq kd(x_{n+1},x_n) \geq \sqrt{k}\, d(x_{n+1},x_n)$$

or

$$d^2(x_{n-1},x_n) \geq kd^2(x_n,x_{n+1})$$

which implies

$$d(x_{n-1},x_n) \geq \sqrt{k}\, d(x_n,x_{n+1}).$$

Therefore

$$d(x_{n+1},x_n) \leq \frac{1}{\sqrt{k}} d(x_n,x_{n-1}) \leq \ldots \leq \left(\frac{1}{\sqrt{k}}\right)^n d(x_1,x_0).$$

It follows that $\{x_n\}$ is a Cauchy sequence and thus $\{x_n\}$ converges to a point $x \in X$. By the continuity of f, $f(x_n) =$

$= x_{n-1} \to x$ for $n \to \infty$. Hence $f(x) = x$, which means that x is a fixed point of f.

C o r o l l a r y . Let (X,d) be a complete metric space and $f:(X,d) \to (X,d)$ a surjective continuous mappings. If there exists a real constant $c > 1$ such that

$$d^2(fx,fy) \geq cd(x,fx)d(y,fy) \tag{4}$$

for any x,y in X, then f has a fixed point.

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