DEMONSTRATIO MATHEMATICA
Vol. XIX No 2 1986

Stanisław Fudali

SIX-POINT PRIMITIVE FIXING SYSTEM IN A PLANE

Introduction

L. Fejes Tóth ([2],p.382) have noted that the maximal number of points in a primitive fixing system for a convex plane figure is 6. As an example of such a figure L.W. Danzer [1] gave a hexagon H with the parallel opposite sides (as $\mathrm{conv}(I^2 \cup (-I^2))$, where $I = \langle 0,1 \rangle \in R$). The points of the mentioned fixing system are the vertices of H (Fig.1).

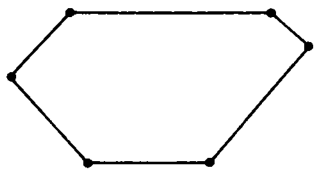

Fig.1

By a convex figure we will mean always in the sequel an non-empty open bounded convex set in a plane. It is easy to see that a six-point primitive fixing system does not exist for some plane figures. A simple example of a figure not having such a system is a triangle or a circle. Immediately the question arises what are the convex plane figures such that each of them has a six-point primitive fixing system. In the present paper it will be shown that only H is such a plane figure.

1. Preliminaries

Recall that a primitive fixing system for an open convex body $K \subset E^n$ is a set $A \subset \operatorname{bd} K$ of points $a_1, \ldots, a_p$ which has the following properties:

1°. For every translation T there exists a point $a_i \in A$ that the segment $\langle a_i, T(a_i) \rangle$ and the image $T(K)$ of K have the non-empty intersection, i.e.

(1) $$\langle a_i, T(a_i) \rangle \cap T(K) \neq \emptyset,$$

2°. For any p-1 points $a_{i_1}, \ldots, a_{i_{p-1}} \in A$ there exists a translation T' such that $\langle a_j, T'(a_j) \rangle \cap T'(K) = \emptyset$ for each $j \in \{i_1, \ldots, i_{p-1}\}$.

We will say that "K cannot be translated by any vector" if (1) holds. The set of all the primitive fixing systems for K will be denoted by $\operatorname{Fix}(K)$.

Consider an arbitrary convex plane figure G and let $F(G) \in \operatorname{Fix}(G)$. It is known [2] that $3 \leq \operatorname{card} F(G) \leq 6$. Moreover, in [2] we have

Remark F-T (Fejes Tóth). For any convex plane figure, different from a parallelogram, there exists a primitive fixing system consisting of three points, while for a parallelogram each one contains exactly four points.

Consider a point $q \in \operatorname{bd} G$. The union of all rays emanating from q and passing through a point $x \in \operatorname{int} G$ will be called the supporting cone of G at q (Fig.2) and will be denoted by $C_q(G)$. The union of the complements of the rays contained in $C_q(G)$ will be called the "dead" cone of G at q - denote it by $\hat{C}_q(G)$. It is evident that G cannot be translated by the vector $\overrightarrow{qy}$ if $q \in F(G)$ and $y \in \hat{C}_q(G)$.

Let b be an arbitrary point of the plane and $F(G) = \{a_1, \ldots, a_p\}$ ($p \leq 6$). We will say that all "dead" cones of G at points of F(G) are drawn to b if each of them is translated by the vector $\overrightarrow{a_i b}$ ($i \in \{1, \ldots, p\}$); denote this translation by T_i. After the drawing all "dead" cones to b the set $\bigcup_{i=1}^{p} T_i(\hat{C}_{a_i}(G))$ either covers or does not cover the plane. Hence

R e m a r k 1. A set $\{a_1,\dots,a_p\} \subset \mathrm{bd}\, G$ is a primitive fixing system for G iff for each j, k, l ($j \neq k \neq l \neq j$)

$$(2) \qquad T_j(\hat{C}_{a_j}(G)) \cap T_k(\hat{C}_{a_k}(G)) \cap T_l(\hat{C}_{a_l}(G)) = \emptyset$$

and

$$(3) \qquad \bigcup_{i=1}^{p} T_i(\hat{C}_{a_i}(G)) \text{ covers the plane containing } G.$$

R e m a r k 2. If G is different from a parallelogram, a primitive fixing system for G is not a minimal point system iff at least two of its points belong to $S_1 \cup S_2$ for a pair (S_1,S_2) of parallel supporting lines of G.

R e m a r k 3. In a primitive fixing system for T two neighbour points of it cannot belong to $S_1 \cup S_2$ for a pair (S_1,S_2) of parallel supporting lines of G.

2. The six-point primitive fixing system

L. Fejes Tóth ([2], p.379) has remarked that for a smooth convex plane figure the greatest possible number of points forming a primitive fixing system equals four. Hence it follows from Remark F-T that a plane convex figure Q having a six-point primitive fixing system cannot be a smooth figure or a parallelogram. Thus Q must be a figure different from a parallelogram which contains the irregular points in its boundary. For this reason just such figures will be considered in the sequel.

Let G be a such plane convex figure and let $F(G) = \{a_0,\dots,a_p\}$ be a primitive fixing system for G. Denote by L_i ($i \in \{0,1,\dots,p\}$) the ray bounding the "dead" cone of G at a_i and intersecting $\hat{C}_{a_{i-1}}(G)$, and by L_i' - the one intersecting $\hat{C}_{a_{i+1}}(G)$ (the indices are taken modulo p+1). Then the rays $\hat{L}_i$ and $\hat{L}_i'$ being complements of the open L_i and L_i' to the straight lines, respectively, are the sides of $C_{a_i}(G)$ (Fig.2).

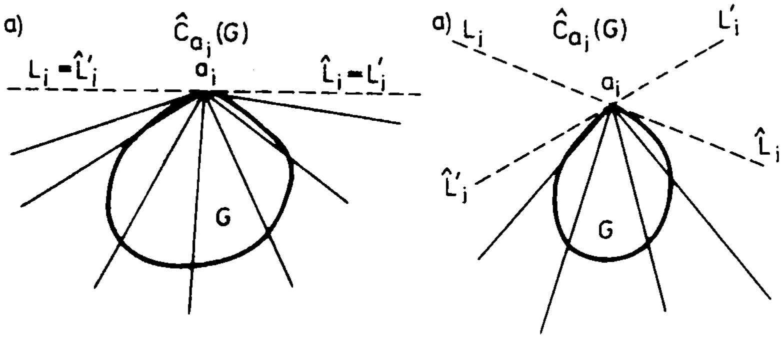

Fig.2

For each i, in view of Remark 1, the "dead" cone of G at a_i has the non-empty common part with the one only at a_{i-1} and at a_{i+1}. Hence

R e m a r k 4. If $F(G) = \{a_0,\dots,a_p\}$ is a primitive fixing system for G, then for each i the ray L'_i intersects L_{i+1} and does not intersect L_{i+2}.

This implies that $\hat{L}_i$ and $\hat{L}_{i+2}$ have a common point or are parallel one to another. Furthermore notice that for each i all points of int G belong to $C_{a_i}(G)$. This leads to

R e m a r k 5. The common part of all closures of supporting cones at the points of any F(G) contains G and all points of F(G), i.e. $G \cup F(G) \subset \bigcap_{a \in F(G)} \overline{C_a(G)}$ for every $F(G) \in \in \mathrm{Fix}(G)$.

Now we can prove

T h e o r e m . If a primitive fixing system of a convex figure consists of six points then the figure is a hexagon with parallel opposite sides, and the fixing points are the vertices of the hexagon.

P r o o f . Let G be any convex plane figure for which $F(G) = \{a_0,\dots,a_5\}$ is a primitive fixing system (Fig.3). Consider this system taking into account Remarks 1-5 and let L_i, L'_i, $\hat{L}_i$, $\hat{L}'_i$ have the same meaning as above.

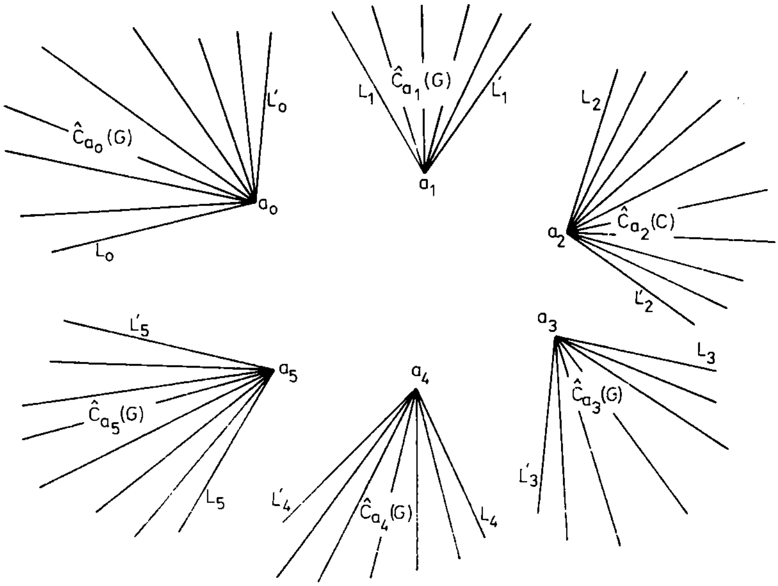

Fig.3

From (2) and from Remark 4 it follows that for each i an oriented angle $\widehat{L_i L'_{i+1}}$ with the sides L_i and L'_{i+1} (i.e. $\hat{C}_{a_i}(G) \cup \hat{C}_{a_{i+1}}(G)$) is not less than π (Fig.4), i.e.

$$(4) \qquad \widehat{L_i L'_{i+1}} \geq \pi,$$

and $\widehat{L'_{i+1} L_{i+3}} \geq 0$, which means that

$$(5) \qquad \widehat{L_i L_{i+3}} \geq \pi.$$

From (4) and (5) it follows that

$$\widehat{L_i L'_{i+4}} \geq 2\pi$$

(taking $i+3$ in (4) instead of i) which means that $\bigcup_{j=i}^{i+4} \hat{C}_{a_j}(G) \geq$ $\geq 2\pi$.

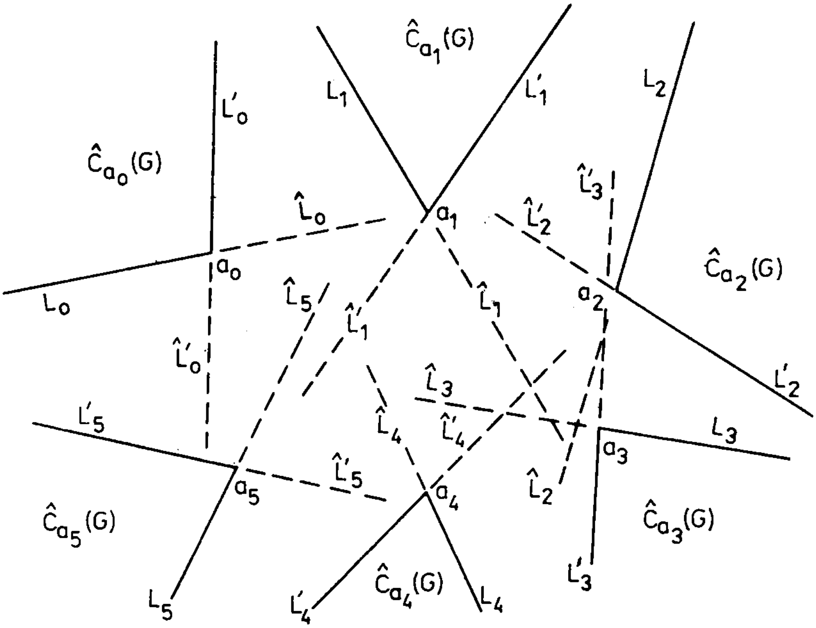

Fig.4

It is evident that $\widehat{L_iL'_{i+4}} = 2\pi$ implies $L_i \| L'_{i+4}$ and then, in view of Remarks 1 and 5, to fix G a sixth point a_5 is needed besides of $a_0,\dots,a_4$. On the other hand, if $\widehat{L_iL'_{i+4}} > 2\pi$, then L'_{i+4} intersects L_i; in this case to fix G it suffices to have 5 points; hence a primitive fixing system for G in this case consists of at most five points.

Remark that $\widehat{L_iL'_{i+4}} = 2\pi$ holds only in the case when $\widehat{L_iL'_{i+1}} = \pi$, $\widehat{L'_{i+1}L_{i+3}} = 0$ and $\widehat{L_{i+3}L'_{i+4}} = \pi$, which means that L_i and L'_{i+1} are contained in a straight line, L'_{i+1} and L_{i+3} are parallel and L_{i+3} and L'_{i+1} are contained in some other straight line. Therefore for each i the straight line M_i, containing L_i and L'_{i+1}, is parallel to M_{i+3}. Since in the considered case the points a_i and a_{i+1} belong to M_i for each i we see that G is a hexagon, each side of which is parallel to the opposite one.

3. Some open questions and problems

The maximal number of points in a primitive fixing system for 3-dimensional body is 14 ([2], [3]). Such a system exists for a rhombic dodecaedron [1] (Fig.5).

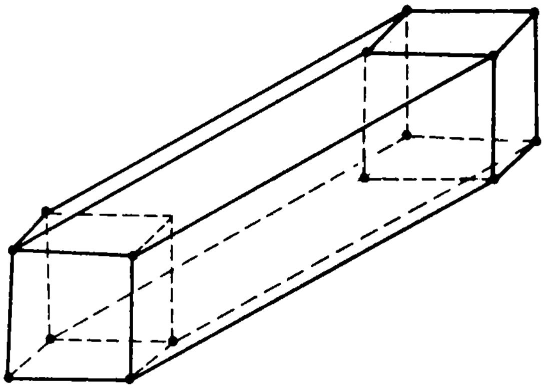

Fig.5

A. Is a rhombic dodecaedron the unique body in E^3 for which there exists a primitive fixing system consisting of 14 points?

Generally, it was conjectured [1], [3] that a primitive fixing system of an n-dimensional convex body consists of at most $2(2^n-1)$ points. Such a system exists for $\mathrm{conv}(I^n \cup (-I^n))$.

B. Is $\mathrm{conv}(I^n \cup (-I^n))$ the unique body in E^n for which there exists a primitive fixing system consisting of $2(2^n-1)$ points?

Any primitive fixing system for a triangle consists of exactly three points.

C. Characterize all convex plane figures for which any primitive fixing system consists of exactly three points.

A primitive fixing system for a circle consits of three or four points.

D. Find all convex plane figures for which there exists a primitive fixing system consisting of three points, there is another one consisting of four points, and there is no one consisting of more than four points.

A primitive fixing system for the regular pentagon consists of three, four or five points (Fig.6).

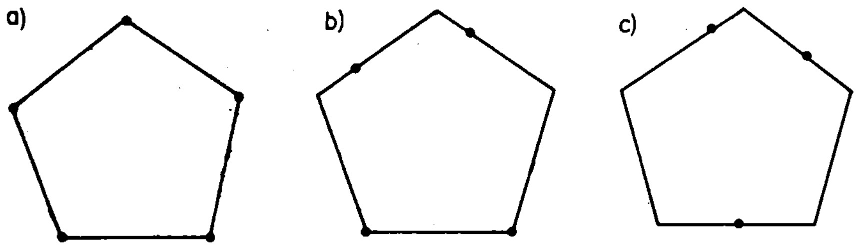

Fig.6

E. Find all convex plane figures having this attribute, i.e. for which there are primitive fixing systems consisting of three, four and five points only.

Let m(Q) be the maximal number of points in a primitive fixing system F(Q) for an n-dimensional convex body $Q \subset E^n$ (it is $m(Q) = 2(2^n-1)$, if the conjecture mentioned above is thrue). Then $n+1 \leq \text{card } F(Q) \leq m(Q)$.

F. Let $p > n$. Characterize a body $Q \subset E^n$ for which $m(Q) \geq p$ and there exist $F_1(Q),\dots,F_{p-n}(Q) \in \text{Fix}(Q)$ such that card $F_i(Q) = n+i$.

REFERENCES

[1] L.W. Danzer: Review of [2], Math. Reviews 26 (1963) 569-570.

[2] L. Fejes Tóth: On primitive polyhedra, Acta Math. Acad. Sci. Hungar. 13 (1962) 379-382.

[3] B. Grünbaum: Fixing systems and inner illumination, Acta Math. Acad. Sci. Hungar. 15 (1964) 161-163.

INSTITUTE OF MATHEMATICS, UNIVERSITY OF WROCŁAW,
50-384 WROCŁAW, POLAND
Received February 14, 1985.