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REMARK ON MULTIPLICATIVE SYSTEMS OF FUNCTIONS

Let us start from the following definition.

Definition. A sequence (X_i) of random variables is called a multiplicative system (MS system), if $EX_{i_1}X_{i_2}\ldots X_{i_n} = 0$ for every sequence of different indices. If, moreover, $EX_{i_0}^2 X_{i_1}\ldots X_{i_n} = 0$ for every sequence of different indices, (X_i) is called a strongly multiplicative system (SMS system).

The multiplicative systems contain a wide class of random variables, for example a sequence of bounded martingale differences is a MS system, a sequence of independent random variables in L^2 with the expected value equal to zero is a SMS system.

In this note we shall use the notation $X = (X_i)_{i=1}^{\infty}$, $Y = (Y_i)_{i=1}^{\infty}$. P (resp. Q, P_n, Q_n) will denote the distribution of the sequence $(X_i)_{i=1}^{\infty}$ [resp. $(Y_i)_{i=1}^{\infty}$, $(X_i)_{i=1}^{n}$, $(Y_i)_{i=1}^{n}$].

Theorem. Let X be a MS system and Y be a bounded SMS system, a.e. for every n $|Y_n(\omega)| \leq K_n$ for some $K_n > 0$ and all ω. If $\sup|X_i| \leq \frac{D^2Y_i}{\sup|Y_i|}$ for $i = 1,2,\ldots$ (when $\sup|Y_i| = 0$ we assume that $\sup|X_i| = 0$), then there exist a probability space $(\Omega, \mathfrak{M}, M)$, a σ-algebra $\mathfrak{N} \subset \mathfrak{M}$ and a random vector Y' equidistributed with the random vector Y such that the random vectors X and $E(Y'|\mathfrak{N})$ are equidistributed.

This result is a generalization of the result from paper [2], which was obtained for finite sequences. Another generalizations are given in [3]. The theorem explains the structure of MS systems, which contain infinitely many bounded functions, for example an infinite lacunary trigonometric system can be represented as a conditional expectation of Rademacher system with respect to some σ-algebra. Using the previous result [2] we can obtain such representation only for finite systems.

L e m m a . Under the assumptions of the theorem for every continuous convex function $f: R^{\infty} \to R$, where topology on R^{∞} is generated by metric $\rho(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^n} |x_n - y_n|$, there holds $Ef(X) \leq Ef(Y)$.

P r o o f . Without loss of generality we may assume that Y_n, $n = 1,2,\dots$, are uniformly bounded by one. In the opposite case let $K_m > 0$ denote a bound for Y_m. We may consider sequences $X_n' = \frac{1}{K_n} X_n$, $Y_n' = \frac{1}{K_n} Y_n$, $n = 1,2,\dots$, for which the assumptions of this lemma are fulfilled and if the lemma is true for the sequences (X_n'), (Y_n'), then it is also true for the sequences (X_n) and (Y_n). So we take this additional assumption that Y_n are uniformly bounded by one. Taking $I = [-1,1]^{\infty}$ with the metric $\rho(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} |x_n - y_n|$ we obtain the compact, metric space in which the random vectors X and Y take their values. For a fixed arbitrary convex, continuous function $f: I \to R$ we define a sequence $(f_m)_{m=1}^{\infty}$ of convex continuous functions, $f_m: I \to R$, by formula $f_m(x_1,x_2,\dots) = f(x_1,x_2,\dots,x_m,0,0,\dots)$. The sequence (f_m) is uniformly convergent to f. Let us fix arbitrary $\varepsilon > 0$. Since f is uniformly continuous there exists $\delta > 0$ such that for every x,y satisfying $\rho(x,y) < \delta$ there holds $|f(x) - f(y)| < \varepsilon$. If we take N such that $\sum_{k=N}^{\infty} 2 \frac{1}{2^k} < \delta$, then for every $n \geq N$ and every $x \in I$ there holds $\rho(x,h_n(x)) < \delta$, where $h_n(x) = (x_1,x_2,\dots,x_n,0,0,\dots)$. So we have for $n \geq N$ and every $x \in I$ $|f(x) - f_n(x)| < \varepsilon$. This means that the sequence (f_m) is uniformly convergent to f and as a consequence we get

(1) $$\int_I f_m dP \xrightarrow[m\to\infty]{} \int_I f dP, \quad \int_I f_m dQ \xrightarrow[m\to\infty]{} \int_I f dQ.$$

By Corollary 1 of [2] we obtain $\int_{R^m} g dP_m \leq \int_{R^m} g dQ_m$ for every convex, continuous function $g: R^m \to R$. This implies $Ef(X_1,X_2,\ldots,X_m,0,0,\ldots) \leq Ef(Y_1,Y_2,\ldots,Y_n,0,0,\ldots)$. Hence, for $n \geq m$,

$$\int_I f_m dP = \int_I f_m dP_n \leq \int_I f_m dQ_n = \int_I f_m dQ.$$

This gives

(2) $$\int_I f_m dP \leq \int_I f_m dQ.$$

If we take m which tends to infinity, then, by (1), we receive from (2) the desired result $\int_I f dP \leq \int_I f dQ$. The lemma is proved.

Now the theorem follows from Edgar's result (Theorem 2.2 of [1]), which asserts that the theorem and the lemma are equivalent.

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