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GENERALIZED SOLUTIONS OF A GOURSAT PROBLEM FOR A POLYVIBRATING EQUATION OF D. MANGERON

Introduction

In papers [2] and [3] we examined a Goursat problem for the Mangeron's polyvibrating equation of order 2p $(p \geq 2)$ in a Banach space, with the boundary conditions of the problem given on 2p curves emanating from a common point, and found its classical solutions. In paper [4] we have given without proof some results concerning the existence and uniqueness of strong generalized solutions of the said Goursat problem. The aim of this paper is to prove these results.

The paper consists of three sections. In Section 1 we introduce some basic notions and formulate the assumptions. In Section 2 we prove several lemmas. Section 3 is devoted to the formulation and proof of the main result of the paper concerning the existence and uniqueness of strong generalized solutions of the Goursat problem.

1. Let $\Omega = \{(x,y) \in R^2 : 0 \leq x \leq A;\ 0 \leq y \leq B\}$, where A and B are finite positive numbers, and consider a system of 2p curves $(p \geq 2)$ given by the equations $y = f_i(x)$ and $x = h_i(y)$, respectively (where $f_i : \langle 0,A\rangle \to \langle 0,B\rangle$ and $h_i : \langle 0,B\rangle \to \langle 0,A\rangle$, $i=1,2,\ldots,p$, are continuous functions), passing through the origin $O(0,0)$ of the coordinates system and not intersecting elsewhere.

Denote by Y a Banach space with norm $\|\cdot\|$. The Goursat problem (G) considered in [2] and [3] consists in finding a function $u:\Omega \to Y$ that is a solution of the equation

$$(1) \qquad L^p u = c(x,y)$$

(where $L = \frac{\partial^2}{\partial x \partial y}$ and $c:\Omega \to Y$ is a given continuous function) in Ω *) and satisfies the boundary conditions

$$(2) \qquad \begin{cases} u[x,f_i(x)] = M_i(x) & \text{for } x \in \langle 0,A\rangle \\ u[h_i(y),y] = N_i(y) & \text{for } y \in \langle 0,B\rangle \end{cases}$$

$(i=1,2,\ldots,p)$, where $M_i:\langle 0,A\rangle \to Y$ and $N_i:\langle 0,B\rangle \to Y$ are given continuous functions. Such functions u will be called classical solutions of the Goursat problem (briefly: (c.s.) of the (G)-problem).

Now, we introduce the following definition of a generalized solution (briefly: a(g.s.) of the (G)-problem).

D e f i n i t i o n 1. A function $u:\Omega \to Y$ is called a (g.s.) of the (G)-problem if and only if there exists a sequence $\{u_m\}$ of functions $u_m:\Omega \to Y$ ($m=m_0, m_0+1,\ldots$, where m_0 is a positive integer) having the following properties:

1°. Each of the functions u_m is a (c.s.) of the corresponding Goursat problem (G_m) that is formulated analogously as the (G)-problem above with the replacement of f_i, h_i, c, M_i and N_i by f_{im}, h_{im}, c_m, M_{im} and N_{im}, respectively $(i=1,2,\ldots,p)$, where $f_{im}:\langle 0,A\rangle \to \langle 0,B\rangle$, $h_{im}:\langle 0,B\rangle \to \langle 0,A\rangle$, $c_m:\Omega \to Y$, $M_{im}:\langle 0,A\rangle \to Y$ and $N_{im}:\langle 0,B\rangle \to Y$ are continuous functions satisfying the relations **)

$c_m \to c$ in $\mathcal{L}$; $f_{im} \rightrightarrows f_i$; $h_{im} \rightrightarrows h_i$; $M_{im} \rightrightarrows M_i$; $N_{im} \rightrightarrows N_i$ $(i=1,2,\ldots,p)$ when $m \to \infty$.

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*) That is, possesses continuous derivatives $D^\beta u$ (where $D^\beta = \frac{\partial^{|\beta|}}{\partial x^{\beta_1} \partial y^{\beta_2}}$; $|\beta| = \beta_1+\beta_2$; $0 \le \beta_1, \beta_2 \le p$) in Ω and satisfies (1) at each point $(x,y) \in \Omega$.

**) $\mathcal{L} = \mathcal{L}(\Omega)$ is the space of Lebesgue-integrable functions, and the symbol $\rightrightarrows$ denotes the uniform convergence.

2°. The relation $u_m \rightrightarrows u$, when $m \to \infty$, holds good.

The purpose of this paper is to examine the generalized solutions of the (G)-problem.

We make the following assumptions.

I. The functions $f_i : \langle 0,A\rangle \to \langle 0,B\rangle$ and $h_i : \langle 0,B\rangle \to \langle 0,A\rangle$ $(i=1,2,\dots,p)$ are Hölder-continuous (the exponent $\beta \in (0,1\rangle$) and satisfy the following relations*)

$$(3) \qquad \left|\frac{f_i(x)}{x} - f_i^*\right| \leqslant \frac{f_i^*}{A}x; \quad \left|\frac{h_i(y)}{y} - h_i^*\right| \leqslant \frac{h_i^*}{B}y$$

$(x \in (0,A\rangle;\ y \in (0,B\rangle;\ i=1,2,\dots,p)$, where f_i^* and h_i^* are real numbers subject to the conditions

$$(4) \qquad \min(f_1^*,h_1^*) > 0;$$

$$(5) \qquad 0 < f_p^* \leqslant \min\left(a, \frac{4B}{A}\right); \quad 0 < h_p^* \leqslant \min\left(a, \frac{4A}{B}\right),$$

where

$$(6) \qquad a = \gamma_0 \cdot p^{-p/\varkappa_0};$$

$$(7) \qquad \min_{2\leqslant i\leqslant p} (f_i^*-f_{i-1}^*) > [p(1+\varepsilon)]^{-1}f_p^*; \quad \min_{2\leqslant i\leqslant p} (h_i^*-h_{i-1}^*) > [p(1+\varepsilon)]^{-1}h_p^*$$

with $\varkappa_0 \in (0,1)$, $\gamma_0 \in (0,1)$, ε being a real number such that

$$(8) \qquad 0 < \varepsilon < \gamma_0^{1/1-p} - 1.$$

Moreover, f_i and h_i $(i=1,2,\dots,p)$ satisfy the conditions mentioned at the beginning of this Section.

II. The functions $M_i : \langle 0,A\rangle \to Y$ and $N_i : \langle 0,B\rangle \to Y$ $(i=1,2,\dots,p)$ are Hölder-continuous (the exponent $\beta_{**} \in (0,1\rangle$) and satisfy the inequalities

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*) Evidently, as a consequence of (3), there exist the derivatives $f_i'(0_+)$ and $h_i'(0_+)$, and $f_i'(0_+) = f_i^*$; $h_i'(0_+) = h_i^*$ $(i=1,2,\dots,n)$.

$$(9)\qquad \|M_i(x)\| \leq Kx^{2p};\ \|N_i(y)\| \leq Ky^{2p},$$

where K is a positive constant.

III. The function $c:\Omega \to Y$ belongs to the space $\mathcal{X}$.

2. In this Section we prove several lemmas.

Lemma 1. There is a number δ_1 $(0<\delta_1 \leq \min(A,B,1))$ such that the inequalities

$$(10)\quad \begin{cases} f_1(x)>0;\ h_1(y)>0; \\ 0<f_p(x)\leq \min\left(a,\ \frac{4B}{A}\right)x;\ 0<h_p(y)\leq \min\left(a,\ \frac{4A}{B}\right)y; \\ f_i(x)-f_j(x)>\left[p(1+\varepsilon)\right]^{-1}f_p(x);\ h_i(y)-h_j(y)>\left[p(1+\varepsilon)\right]^{-1}h_p(y) \end{cases}$$

$(1\leq j<i\leq p)$;

$$(11)\quad \begin{cases} f_i^*(1-\varepsilon_0)x \leq f_i(x) \leq f_i^*(1+\varepsilon_0)x; \\ h_i^*(1-\varepsilon_0)y \leq h_i(y) \leq h_i^*(1+\varepsilon_0)y \end{cases}$$

$(i=1,2,\dots,p)$ hold good for $x\in(0,\delta_1)$ and $y\in(0,\delta_1)$, respectively, ε_0 being a real number subject to the inequalities

$$(12)\qquad 0<\varepsilon_0<1-\left[p^p\, g_0^{\varkappa_0/2}\right]^{1/8p-3+\varkappa_0},$$

where $g_0 = f_p^* h_p^*$.

Proof. The proof, being similar but easier for the remaining inequalities, will be given only for the last two of inequalities (10).

Let us observe that (3) implies the inequalities

$$f_\nu^* - \frac{f_\nu^*}{A}\,x \leq \frac{f_\nu(x)}{x} \leq f_\nu^* + \frac{f_\nu^*}{A}\,x$$

$(\nu=i,j$, where $1\leq j<i\leq p)$, whence and by (7) we have

$$\frac{f_i(x)}{x} - \frac{f_j(x)}{x} - [p(1+\varepsilon)]^{-1}\frac{f_p(x)}{x} \geq f_i^* - f_j^* - [p(1+\varepsilon)]^{-1}f_p^* +$$

$$- \left(f_i^* + f_j^* + [p(1+\varepsilon)]^{-1}f_p^*\right)\frac{x}{A} \geq \min_{2\leq i\leq p}(f_i^* - f_{i-1}^*) - [p(1+\varepsilon)]^{-1}f_p^* +$$

$$- \left(2 + [p(1+\varepsilon)]^{-1}\right)\frac{\delta_1}{A} f_p^* > 0$$

if $0 < \delta_1 \leq \delta_1^o = \frac{A}{2}\left\{2 + [p(1+\varepsilon)]^{-1}f_p^*\right\}^{-1}\cdot\left\{\min_{2\leq i\leq p}(f_i^* - f_{i-1}^*) - \right.$
$\left. - [p(1+\varepsilon)]^{-1}f_p^*\right\}$.

Thus, for $x \in (0,\delta_1)$, where $0 < \delta_1 < \min(A,B,1,\delta_1^o)$, the inequality

$$f_i(x) - f_j(x) > [p(1+\varepsilon)]^{-1}f_p(x)$$

holds good, q.e.d. The proof of the inequality $h_i(y) - h_j(y) > [p(1+\varepsilon)]^{-1}h_p(y)$ is analogous.

Now, let us consider the Bernstein polynomials

$$(13)\qquad f_{im}(x) = A^{-m}\sum_{k=1}^{m}\binom{m}{k} f_i\left(A\,\frac{k}{m}\right)x^k(A-x)^{m-k}$$

and

$$(14)\qquad h_{im}(y) = B^{-m}\sum_{k=1}^{m}\binom{m}{k} h_i\left(B\,\frac{k}{m}\right)y^k(B-y)^{m-k}$$

($i=1,2,\ldots,p$; $m \in \mathcal{N}$) approximating uniformly the functions f_i and h_i in the intervals $<0,A>$ and $<0,B>$, respectively.

Evidently, $0 \leq f_{im}(x) \leq B$ and $0 \leq h_{im}(y) \leq A$. We shall prove some lemmas concerning further properties of functions f_{im} and h_{im}.

L e m m a 2. The functions f_{im} and h_{im} ($i=1,2,\ldots,p$; $m \in \mathcal{N}$) are of class C^p. Their first-order derivatives attain at the point 0 the following values

$$(15)\qquad f_{im}^* := f_{im}'(0) = \frac{m}{A} f_i\left(\frac{A}{m}\right);\quad h_{im}^* := h_{im}'(0) = \frac{m}{B} h_i\left(\frac{B}{m}\right)$$

$(i=1,2,\dots,p;\ m\in\mathcal{N})$. There is a positive integer m_1 such that f^*_{im} and h^*_{im} satisfy the inequalities analogous to (4) - (7) for all $\mathcal{N}\ni m>m_1$ and $i=1,2,\dots,p$.

P r o o f . The first part of our thesis is evident. Relations (15) are obtained immediately by differentiation of (13) and (14). In order to verify the last part of the thesis it is enough to use relations (15) and Lemma 1.

R e m a r k 1. It follows from Lemma 2 that

$$g_{om} := f^*_{pm}h^*_{pm} < \gamma_o^2 \cdot p^{-2p/\varkappa_o}$$

for $m>m_1$, whence and by (8) we obtain the inequality

$$0<\varepsilon<\left(p^p\sqrt{g_{om}}\right)^{1/1-p}-1 \tag{16}$$

$(m>m_1)$. Let us observe that by (5) and (8) we also have the inequality

$$0<\varepsilon<\left(p^p\sqrt{g_o}\right)^{1/1-p}-1. \tag{16'}$$

L e m m a 3. There are a positive number δ_2 $(0<\delta_2<\min(A,B,1))$ and a positive integer m_2 such that the functions f_{im} and h_{im} $(i=1,2,\dots,p;\ m\in\mathcal{N})$ satisfy the inequalities (compare with (10) and (11))

$$\begin{cases} f_{1m}(x)>0;\ h_{1m}(y)>0; \\ 0<f_{pm}(x)\leqslant p^{-p/\varkappa_o}x;\ 0<h_{pm}(y)\leqslant p^{-p/\varkappa_o}y; \\ f_{im}(x)-f_{jm}(x)>\left[p(1+\varepsilon)\right]^{-1}f_{pm}(x);\ h_{im}(y)-h_{jm}(y)> \\ \qquad >\left[p(1+\varepsilon)\right]^{-1}h_{pm}(y) \end{cases} \tag{17}$$

$(1\leqslant j<i\leqslant p)$;

$$(18)\qquad \begin{cases} f_i^*(1 - \varepsilon_0)x \leqslant f_{im}(x) \leqslant f_i^*(1 + \varepsilon_0)x; \\ h_i(1 - \varepsilon_0)y \leqslant h_{im}(y) \leqslant h_i^*(1 + \varepsilon_0)y \end{cases}$$

$(i=1,2,\ldots,p)$ for $x \in (0,\delta_2)$ and $y \in (0,\delta_2)$, respectively, and $m > m_2$.

P r o o f . First of all we prove that

$$(19)\qquad \begin{cases} f_i^*\left(1 - \frac{1}{m} - \frac{x}{A}\right)x \leqslant f_{im}(x) \leqslant f_i^*\left(1 + \frac{1}{m} + \frac{x}{A}\right)x; \\ h_i^*\left(1 - \frac{1}{m} - \frac{y}{B}\right)y \leqslant h_{im}(y) \leqslant h_i^*\left(1 + \frac{1}{m} + \frac{y}{B}\right)y \end{cases}$$

$(x \in \langle 0,A\rangle;\ y \in \langle 0,B\rangle,\ m \in \mathcal{N};\ i=1,2,\ldots,p)$.

In fact, using inequality (3) and relation (13) we can write

$$f_{im}(x) \leqslant f_i^*\ A\left\{\frac{1}{m}\sum_{k=1}^{m} k\binom{m}{k}\left(\frac{x}{A}\right)^k\left(1 - \frac{x}{A}\right)^{m-k} + \right.$$

$$+ \frac{1}{m(m-1)}\sum_{k=1}^{m} k(k-1)\binom{m}{k}\left(\frac{x}{A}\right)^k\left(1 - \frac{x}{A}\right)^{m-k} +$$

$$\left. + \frac{1}{m^2}\sum_{k=1}^{m} k\binom{m}{k}\left(\frac{x}{A}\right)^k\left(1 - \frac{x}{A}\right)^{m-k}\right\},$$

whence, and by the well known equalities (see [5], p.150)

$$(20)\qquad x = \frac{1}{m}\sum_{k=1}^{m} k\binom{m}{k} x^k (1 - x)^{m-k}$$

and

$$(21)\qquad x^2 = \frac{1}{m(m-1)}\sum_{k=1}^{m} k(k-1)\binom{m}{k} x^k(1 - x)^{m-k},$$

we obtain

$$f_{im}(x) \leq f_i^* \left(1 + \frac{1}{m} + \frac{x}{A}\right)x .$$

In a similar way we get

$$f_i^*\left(1 - \frac{1}{m} - \frac{x}{A}\right)x \leq f_{im}(x)$$

and hence the first of inequalities (19) is proved. The proof of the second one is analogous.

We proceed to the proof of inequalities (17) and (18).

The first two of inequalities (17) are evidently true. In order to derive the third of the said inequalities, let us observe that by (5) and (19) we have

$$0 < f_{pm}(x) \leq \gamma_0 p^{-p/\varkappa_0} x + f_p^*\left(\frac{1}{m} + \frac{x}{A}\right)x \leq (1+\theta)\gamma_0 \cdot p^{-p/\varkappa_0} x$$

$(0 < \theta < \frac{1}{\gamma_0} - 1)$ provided that $m > \tilde{m}_2$; $x \in (0, \delta_2)$, where $\mathcal{N} \ni \tilde{m}_2$ is so large and $0 < \delta_2 < \min(A, B, 1)$ so small that $\frac{1}{m} + \frac{x}{A} \leq \leq \theta \cdot \gamma_0 \cdot p^{-p/\varkappa_0}$. As a consequence we get the required inequality $0 < f_{pm}(x) \leq p^{-p/\varkappa_0} x$. In a similar way we show that $0 < h_{pm}(y) \leq p^{-p/\varkappa_0} y$.

Now, let us note that in virtue of (19) we have

$$f_{im}(x) - f_{jm}(x) - \left[p(1+\varepsilon)\right]^{-1} f_{pm} \geq \min_{2 \leq i \leq p} (f_i^* - f_{i-1}^*) - \left[p(1+\varepsilon)\right]^{-1} f_p^* +$$
$$- f_p^*\left(2 + \left[p(1+\varepsilon)\right]^{-1}\right)\left(\frac{1}{m} + \frac{x}{A}\right)x$$

$(1 \leq j < i \leq p)$. Assuming that $m > m_2^*$; $x \in (0, \delta_2^*)$, where $m_2^* \in \mathcal{N}$ and δ_2^* $(0 < \delta_2^* < \min(A, B, 1))$ are chosen so that

$$f_p^*\left(2 + \left[p(1+\varepsilon)\right]^{-1}\right)\left(\frac{1}{m} + \frac{x}{A}\right)x \leq \frac{1}{2}\left(\min_{2 \leq i \leq p} (f_i^* - f_{i-1}^*) - \left[p(1+\varepsilon)\right]^{-1} f_p^*\right)$$

$(m > m_2^*;\ x \in (0,\delta_2^*))$, we can assert that

$$f_{im}(x) - f_{jm}(x) > \left[p(1+\varepsilon)\right]^{-1} f_{pm}(x).$$

The proof of the inequality $h_{im}(y)-h_{jm}(y) > \left[p(1+\varepsilon)\right]^{-1} h_{pm}(y)$ is analogous.

It still remains to prove the inequalities (18). Using (19), we have

$$f_i^* \left[1 - \left(\frac{1}{m} + \frac{x}{A}\right)\right] x \leqslant f_{im}(x) \leqslant f_i^* \left[1 + \left(\frac{1}{m} + \frac{x}{A}\right)\right] x,$$

$$h_i^* \left[1 - \left(\frac{1}{m} + \frac{y}{B}\right)\right] y \leqslant h_{im}(y) \leqslant h_i^* \left[1 + \left(\frac{1}{m} + \frac{y}{B}\right)\right] y,$$

and assuming that $m > m_2^{**}$; $x \in (0,\delta_2^{**})$, $y \in (0,\delta_2^{**})$, where $m_2^{**} \in \mathcal{N}$ and δ_2^{**} $(0<\delta_2^{**} < \min(A,B,1))$ are such that

$$\max\left[\left(\frac{1}{m} + \frac{x}{A}\right), \left(\frac{1}{m} + \frac{y}{B}\right)\right] < \varepsilon_0$$

(see (12)) for $m > m_2^{**}$; $x \in (0,\delta_2^{**})$; $y \in (0,\delta_2^{**})$, we can conclude that inequalities (18) hold good. Thus, Lemma 3 is valid if $m > m_2 = \max(\tilde{m}_2, m_2^*, m_2^{**})$ and $x \in (0,\delta_2)$; $y \in (0,\delta_2)$, where $0 < \delta_2 \leqslant \min(\tilde{\delta}_2, \delta_2^*, \delta_2^{**})$.

L e m m a 4. For each number δ_0, satisfying the condition $0 < \delta_0 < \min(A,B,1)$, there are a positive integer m_{δ_0} and positive numbers a_δ^0 and b_δ^0 such that

$$f_{im}(x) - f_{jm}(x) \geqslant a_\delta^0;\quad h_{im}(y) - h_{jm}(y)) \geqslant b_\delta^0 \tag{22}$$

for $1 \leqslant j < i \leqslant p$, $x \in \langle \delta_0, A\rangle$, $y \in \langle \delta_0, B\rangle$ and $\mathcal{N} \ni m > m_{\delta_0}$.

P r o o f . Set $\varepsilon_* = \frac{1}{4} \min\limits_{1\leqslant i\leqslant p} \inf\limits_{x\in\langle\delta_0,A\rangle} (f_i(x)-f_{i-1}(x))$. Evidently,

$$f_\nu(x) - \varepsilon_* \leqslant f_{\nu m}(x) \leqslant f_\nu(x) + \varepsilon_*$$

($\nu=1,2,\dots,p$) for $x\in\langle 0,A\rangle$ and $m>m_{\delta_0}$, m_{δ_0} being sufficiently large, whence

$$f_{im}(x)-f_{jm}(x)\geqslant f_i(x)-f_j(x)-2\varepsilon_*\geqslant a_\delta^0=$$

$$=\frac{1}{2}\min_{1\leqslant i\leqslant p}\ \inf_{x\in\langle\delta_0,A\rangle}\left(f_i(x)-f_{i-1}(x)\right)$$

($1\leqslant j<i\leqslant p$; $m>m_{\delta_0}$; $x\in\langle\delta_0,A\rangle$) and the proof of the first of inequalities (22) is completed. The proof of the second one is analogous.

L e m m a 5. There is a positive integer m_3 such that no two of the curves of equations $y=f_{im}(x)$ and $x=h_{jm}(y)$, respectively, ($i,j=1,2,\dots,p$) intersect in $\Omega\setminus\{0\}$ when $m>m_3$.

P r o o f . Let us consider two curves given by the equations $y=f_{im}(x)$ and $y=f_{jm}(x)$, respectively, where $1\leqslant j<i\leqslant p$ and $m\in\mathcal{N}$. Using formula (13), we have

$$f_{im}(x)-f_{jm}(x)=A^{-m}\sum_{k=1}^{m}\left[f_i\left(A\,\frac{k}{m}\right)-f_j\left(A\,\frac{k}{m}\right)\right]\binom{m}{k}x^k(A-x)^{m-k}$$

and as, by Assumption I, $f_i\left(A\,\frac{k}{m}\right)-f_j\left(A\,\frac{k}{m}\right)>0$ for $k=1,2,\dots,m$, we obtain the inequality $f_{im}(x)>f_{jm}(x)$ ($x\in(0,A\rangle$; $m\in\mathcal{N}$; $1\leqslant j<i\leqslant p$). A similar argument based on (14) shows that $h_{im}(y)>h_{jm}(y)$ ($y\in(0,A\rangle$; $m\in\mathcal{N}$; $1\leqslant j<i\leqslant p$).

In order to examine the case of two curves given by the equations $y=f_{im}(x)$ and $x=h_{jm}(y)$, respectively, (where $1\leqslant i,\ j\leqslant p$), let us observe that basing on (19) we obtain the inequalities

$$x-h_{jm}\circ f_{im}(x)\geqslant x-h_j^*\left(2+\frac{1}{m}\right)f_{im}(x)\geqslant x\left[1-\left(2+\frac{1}{m}\right)^2 f_i^*h_j^*\right],$$

whence

$$h_{jm}\circ f_{im}(x)\leqslant\left(2+\frac{1}{m}\right)^2 f_i^*h_j^*\cdot x.$$

Let $\theta_0 \in (4f_p^*h_p^*, 1)$ be arbitrarily fixed (by inequalities (5) we have $4f_p^*h_p^* < 1$). It is easily seen that (see (10))
$\left(2+\frac{1}{m}\right)^2 f_i^*h_i^* \leqslant \theta_0$ for $m > m_3 \geqslant \dfrac{\sqrt{f_p^*h_p^*}}{\sqrt{\theta_0} - 2\sqrt{f_p^*h_p^*}}$, and hence we have

$$h_{jm} \circ f_{im}(x) \leqslant \theta_0 \cdot x \tag{24}$$

$(m > m_3;\ i,j=1,2,\ldots,p;\ x \in (0,A\rangle)$. It follows from inequality (24) that the curves $y = f_{im}(x)$ and $x = h_{jm}(y)$ $(m > m_3;$ $i,j=1,2,\ldots,p)$ do not intersect for $x \in (0,A\rangle$; $y \in (0,B\rangle$, and the proof is completed.

Let us introduce the following notation

$$(25)\quad \begin{cases} z^m_{\vec{k}(2s)}(x) = h_{k_{2s}m} \circ f_{k_{2s-1}m} \circ z^m_{\vec{k}(2s-2)}(x);\ \ \overset{*}{z}{}^m_{\vec{k}(2s-1)}(x) = \\ \qquad = f_{k_{2s-1}m} \circ z^m_{\vec{k}(2s-2)}(x) \\ \tilde{z}^m_{\vec{k}(2s)}(y) = f_{k_{2s}m} \circ h_{k_{2s-1}m} \circ \tilde{z}^m_{\vec{k}(2s-2)}(y);\ \ \overset{**}{z}{}^m_{\vec{k}(2s-1)}(y) = \\ \qquad = h_{\vec{k}(2s-1)m} \circ \tilde{z}^m_{\vec{k}(2s-2)}(y) \end{cases}$$

for $s=2,3,\ldots$;

$$(25')\quad \begin{cases} z^m_{\vec{k}(2)}(x) = h_{k_2m} \circ f_{k_1m}(x);\ \ \overset{*}{z}{}^m_{\vec{k}(1)}(x) = f_{k_1m}(x) \\ \tilde{z}^m_{\vec{k}(2)}(y) = f_{k_2m} \circ h_{k_1m}(y);\ \ \overset{**}{z}{}^m_{\vec{k}(1)}(y) = h_{k_1m}(y) \end{cases}$$

$(x \in \langle 0,A\rangle;\ y \in \langle 0,B\rangle;\ m \in \mathcal{N})$, where $\vec{k}_{(r)} = (k_1,k_2,\ldots,k_r)$ for $r \in \mathcal{N}$, and k_i $(i=1,2,\ldots)$ are positive integers not exceeding p.

The following lemma is valid.

L e m m a 6. The sequences $\left\{z^{m}_{\vec{k}(2n)}\right\}$, $\left\{\overset{*}{z}^{m}_{\vec{k}(2n-1)}\right\}$, $\left\{\tilde{z}^{m}_{\vec{k}(2n)}\right\}$ and $\left\{\overset{**}{z}^{m}_{\vec{k}(2n-1)}\right\}$ tend to zero, when $n \to \infty$, uniformly with respect to x,y and m, on the sets $\langle 0,A\rangle$ and $\langle 0,B\rangle$, respectively.

P r o o f . Evidently, it is enough to consider $\left\{z^{m}_{\vec{k}(2n)}\right\}$ and $\left\{\tilde{z}^{m}_{\vec{k}(2n)}\right\}$. The proof, being similar for the other sequence, will be given only for the sequence $\left\{z^{m}_{\vec{k}(2n)}\right\}$.

Let us assume that $m \geqslant m_3$. By inequality (24) we have the following estimate

$$z^{m}_{\vec{k}(2n)}(x) \leqslant \theta_0^n \cdot A \tag{26}$$

and hence the considered sequence $\left\{z^{m}_{\vec{k}(2n)}\right\}$ tends to zero (uniformly with respect to x and m), as required.

L e m m a 7. For each $\tilde{\varepsilon} > 0$ there is a positive integer $m_{\tilde{\varepsilon}}$ such that the inequalities *)

$$\left|z^{m}_{\vec{k}(2n)}(x) - z_{\vec{k}(2n)}(x)\right| < n\tilde{\varepsilon}; \quad \left|\tilde{z}^{m}_{\vec{k}(2n)}(y) - \tilde{z}_{\vec{k}(2n)}(y)\right| < n\tilde{\varepsilon} \tag{27}$$

hold good for $n \in \mathcal{N}$, $\mathcal{N} \ni m > m_{\tilde{\varepsilon}}$, $x \in \langle 0,A\rangle$ and $y \in \langle 0,B\rangle$, respectively.

P r o o f . In order to prove the first of relations (27), let us observe that

$$\left|z^{m}_{\vec{k}(2n)}(x) - z_{\vec{k}(2n)}(x)\right| \leqslant \sum_{\nu=1}^{2n-1} r^{\nu}_{nm}(x)\left|z^{m}_{\vec{k}(2n)}(x) - h_{k_{2n}} \circ \overset{*}{z}^{m}_{\vec{k}(2n-1)}(x)\right|,$$

*) $z_{\vec{k}(2n)}(x),\dots,\overset{**}{z}_{\vec{k}(2n-1)}(y)$ are defined by relations (25), (25′) with the replacement of $f_{\nu m}$ and $h_{\nu m}$ ($\nu = 1,2,\dots,p$; $m \in \mathcal{N}$) by f_ν and h_ν, respectively.

where

$$r^{\nu}_{nm}(x) = \left| h_{k_{2n}} \circ f_{k_{2n-1}} \circ h_{k_{2n-2}} \circ \dots \circ g_{k_\nu m} \circ \mathfrak{z}_{\vec{k}(\nu-1)}^{m}(x) - \right.$$

$$\left. - h_{k_{2n}} \circ f_{k_{2n-1}} \circ h_{k_{2n-2}} \circ \dots \circ g_{k_\nu} \circ \mathfrak{z}_{\vec{k}(\nu-1)}^{m}(x) \right|$$

with

$$g_{k_\nu m} = \begin{cases} h_{k_\nu m}, & \nu \text{ even} \\ f_{k_\nu m}, & \nu \text{ odd} \end{cases}; \quad g_{k_\nu} = \begin{cases} h_{k_\nu}, & \nu \text{ even} \\ f_{k_\nu}, & \nu \text{ odd}; \end{cases}$$

and

$$\mathfrak{z}_{\vec{k}(\nu-1)}^{m} = \begin{cases} z_{\vec{k}(\nu-1)}^{m}, & \nu \text{ odd} \\ \overset{*}{z}{}_{\vec{k}(\nu-1)}^{m}, & \nu \text{ even} \end{cases}; \quad \mathfrak{z}_{\vec{k}(0)}^{m}(x) = x.$$

Basing on the relations $f_{im} \rightrightarrows f_i$; $h_{im} \rightrightarrows h_i$ ($m \to \infty$, $i=1,2,\dots,p$) and using the uniform continuity of the functions f_i and h_i ($i=1,2,\dots,p$), we can assert that for each positive number $\tilde{\varepsilon}$ there is a positive integer $m_{\tilde{\varepsilon}}$ such that $r^{\nu}_{nm} < \tilde{\varepsilon}$ ($\nu=1,2,\dots,n$), and hence

$$r_{nm} < n\tilde{\varepsilon}$$

($m > m_{\tilde{\varepsilon}}$; $x \in \langle 0,A\rangle$; $y \in \langle 0,B\rangle$) which was to be proved. The proof of the second of relations (27) is analogous.

We proceed to the examination of the "truncated" Bernstein polynomials

$$M_{im}(x) = A^{-m} \sum_{k=2p}^{m} \binom{m}{k} M_i\left(A\,\frac{k}{m}\right) x^k (A-x)^{m-k} \tag{28}$$

($x \in \langle 0,A\rangle$) and

$$(29)\qquad N_{im}(y) = B^{-m}\sum_{k=2p}^{m}\binom{m}{k} N_i\left(B\,\frac{k}{m}\right) y^k(B-y)^{m-k}$$

($y \in \langle 0,B\rangle$), where $i=1,2,\dots,p$, $\mathcal{N} \ni m \geqslant 2p$.

Evidently, $M_{im}:\langle 0,A\rangle \to Y$ and $N_{im}:\langle 0,B\rangle \to Y$, $i=1,2,\dots,p$, $\mathcal{N} \ni m \geqslant 2p$.

The following lemmas are valid.

L e m m a 8. The relations

$$(30)\qquad M_{im} \rightrightarrows M_i \quad \text{on } \langle 0,A\rangle;\quad N_{im} \rightrightarrows N_i \quad \text{on } \langle 0,B\rangle$$

($m \to \infty$) hold good.

P r o o f . Using (28), we get

$$(31)\ \left\|M_i(x) - M_{im}(x)\right\| \leqslant A^{-m}\sum_{k=0}^{m}\binom{m}{k}\left\|M_i(x) - M_i\left(A\,\frac{k}{m}\right)\right\| \cdot$$

$$\cdot\ x^k(A-x)^{m-k} + A^{-m}\sum_{k=0}^{2p-1}\binom{m}{k}\left\|M_i\left(A\,\frac{k}{m}\right)\right\| x^k(A-x)^{m-k}.$$

Let us denote the terms on the right hand side of (31) by $e_{1m}(x)$ and $e_{2m}(x)$, respectively.

It is well known (see [5], p.150) that for each $\varepsilon_* > 0$ there is a positive integer m'_{ε_*} such that $e_1(x) < \frac{\varepsilon_*}{2}$ when $m > m'_{\varepsilon_*}$ ($x \in \langle 0,A\rangle$; $i=1,2,\dots,p$).

For the second term, $e_{2m}(x)$, we have (see (9))

$$e_{2m}(x) \leqslant K\left(\frac{2Ap}{m}\right)^{2p} A^{-m}\cdot\sum_{k=0}^{m}\binom{m}{k} x^k(A-x)^{m-k} = K\left(\frac{2Ap}{m}\right)^{2p}$$

and hence

$$e_{2m}(x) < \frac{\varepsilon_*}{2}$$

when $m > m''_{\varepsilon_*}$ (where m''_{ε} is a positive integer), $x \in \langle 0,A\rangle$ and $i=1,2,\dots,p$.

Upon joining the above-obtained results, we can assert that $M_{im} \rightrightarrows M_i$ on $<0,A>$ when $m \to \infty$. In a similar way we show that $N_{im} \rightrightarrows N_i$ on $<0,B>$ when $m \to \infty$. Thus the proof of Lemma 8 is completed.

L e m m a 9. The functions M_{im} and N_{im} (see (28) and (29)) are of class C^p. There are positive constants $C_{m\nu}$ and $\tilde{C}_{m\nu}$ (depending on m and ν) such that the inequalities

$$(32) \qquad \begin{cases} \left\| M_{im}^{(\nu)}(x) \right\| \leqslant C_{m\nu} x^{2p-\nu} \\ \left\| N_{im}(y) \right\| \leqslant \tilde{C}_{m\nu} y^{2p-\nu} \end{cases}$$

($x \in <0,A>$; $y \in <0,B>$; $i,\nu = 1,2,\dots,p$) are satisfied.

P r o o f . The first part of the thesis is evident (M_{im} and N_{im} are in fact of class C^∞). In order to prove the first of inequalities (32), let us observe that

$$M_{im}^{(\nu)}(x) = A^{-m} \sum_{k=2p}^{m} \binom{m}{k} M_i\left(A\,\frac{k}{m}\right) \sum_{\alpha=0}^{\min(\nu,m-k)} \binom{\nu}{\alpha} (-1)^\alpha \frac{k!}{(k-\nu+\alpha)!} \cdot$$

$$\cdot \frac{(m-k)!}{(m-k-\alpha)!} x^{k-\nu+\alpha}(A-x)^{m-k-\alpha}$$

($\nu = 1,2,\dots,p$), whence and by (9) we obtain

$$\left\| M_{im}^{(\nu)}(x) \right\| \leqslant \text{const}\; x^{2p-\nu} A^{2p-m} \sum_{k=2p}^{m} \binom{m}{k} \sum_{\alpha=0}^{\min(\nu,m-k)} \nu! \binom{k}{\nu-\alpha} \cdot$$

$$\cdot \binom{m-k}{\alpha} x^{k-2p+\alpha}(A-x)^{m-k-\alpha}$$

and as a consequence we get the first of inequalities (32), as required. The proof of the second of the said inequalities is analogous.

In the sequel (see the proof of Theorem 1 in Section 3) we shall need estimates of $M_{im}(x)$ and $N_{im}(y)$ ($i=1,2,\dots,p$) uniform with respect to m. We have the following lemma

L e m m a 10. The inequalities

$$(33)\qquad \begin{cases} \|M_{im}(x)\| \leq K(2p)^{2p-1}x^{2p} \\ \|N_{im}(y)\| \leq K(2p)^{2p-1}y^{2p} \end{cases}$$

($x \in <0,A>$; $y \in <0,B>$; $i=1,2,\dots,p$) are valid.

P r o o f . According to (28) and (9), we have

$$\|M_{im}(x)\| \leq KA^{2p}\, m^{-2p} \sum_{k=2p}^{m} \binom{m}{k} k^{2p} \left(\frac{x}{A}\right)^{k} \left(1 - \frac{x}{A}\right)^{m-k}$$

and hence, basing on the inequalities

$$m^{-2p} \leq \left[m(m-1)\dots(m-2p+1)\right]^{-1}$$

and *)

$$k^{2p} \leq (2p)^{2p-1} \cdot k(k-1)\dots(k-2p+1)$$

($2p \leq k \leq m$), we obtain

$$(34)\qquad \|M_{im}(x)\| \leq KA^{2p}(2p)^{2p-1}\cdot\left[m(m-1)\dots(m-2p+1)\right]^{-1} \cdot$$

$$\cdot \sum_{k=2p-1}^{m} k(k-1)\dots(k-2p+1) \binom{m}{k} \left(\frac{x}{A}\right)^{k} \left(1 - \frac{x}{A}\right)^{m-k}.$$

In order to estimate the expression on the right hand side of (34), we use equality (20) (with the replacement of m by $(m-2p+1)$) and multiplying it by $\left(\frac{x}{A}\right)^{2p-1}$, we obtain

$$(35)\qquad \left(\frac{x}{A}\right)^{2p} = (m-2p+1)^{-1}\cdot \sum_{k=0}^{m-2p+1} k\binom{m-2p+1}{k}\left(\frac{x}{A}\right)^{k+2p-1}\left(1-\frac{x}{A}\right)^{m-(k+2p-1)}.$$

---

*) $k \geq 2p$ implies $k \leq 2p(k-i)$ for $i=1,2,\dots,2p-1$.

As the equality

$$\binom{m-2p+1}{k} = (k+1)\dots(k+2p-1)\left[m(m-1)\dots(m-2p+2)\right]^{-1}\binom{m}{k+2p-1}$$

$(0 \leqslant k \leqslant m-2p+1)$ holds good, relation (35) takes the form

$$\left(\frac{x}{A}\right)^{2p} = \left[m(m-1)\dots(m-2p+1)\right]^{-1} \cdot \sum_{k=0}^{m-2p+1} k(k+1)\dots(k+2p-1) \cdot$$

$$\cdot \binom{m}{k+2p-1} \cdot \left(\frac{x}{A}\right)^{k+2p-1}\left(1-\frac{x}{A}\right)^{m-(k+2p-1)},$$

that is

$$(36) \quad \left(\frac{x}{A}\right)^{2p} = \left[m(m-1)\dots(m-2p+1)\right]^{-1} \cdot \sum_{k=2p-1}^{m} k(k-1)\dots(k-2p+1) \cdot$$

$$\cdot \binom{m}{k}\left(\frac{x}{A}\right)^{k}\left(1-\frac{x}{A}\right)^{m-k}.$$

By (34) and (36), we have

$$\|M_{im}(x)\| \leqslant KA^{2p}(2p)^{2p-1}\left(\frac{x}{A}\right)^{2p} = K(2p)^{2p-1}x^{2p}$$

which completes the proof of the first of inequalities (33). The proof of the second one is analogous.

Finally, let us consider the Bernstein polynomials

$$(37) \quad c_m(x,y) = (AB)^{-m}\sum_{\alpha,\beta=0}^{m}\binom{m}{\alpha}\binom{m}{\beta}\tilde{c}\left(A\frac{\alpha}{m}, B\frac{\beta}{m}\right)x^{\alpha}(A-x)^{m-\alpha} \cdot$$

$$\cdot y^{\beta}(B-y)^{m-\beta}$$

($(x,y)\in\Omega$; $m\in\mathcal{N}$), where $\tilde{c}$ is a function of class $C(\Omega)$ approximating in $\mathcal{L}$ the function c.

Evidently, the functions $c_m : \Omega \to Y$ $(m=1,2,\dots)$ are continuous. It is also well known that $c_m \to c$ in $\mathcal{L}$ when $m\to\infty$.

In the sequel (see p.21) we shall need the functions

$$(38) \quad R_m(x,y) = \left[(p-1)!\right]^{-2}\int_0^x\left\{\int_0^y \left[(x-\xi)(y-\eta)\right]^{p-1}c_m(\xi,\eta)\,d\eta\right\}d\xi,$$

c_m being given by (37). It follows directly from the aforesaid properties of c_m $(m=1,2,\dots)$ that

$$(38') \qquad R_m \rightrightarrows R \quad \text{in } \Omega \text{ when } m \to \infty$$

where R is given by a formula analogous to (38) with the replacement of c_m by c.

As a result of the considerations performed in this Section, we can formulate the following Corollary.

C o r o l l a r y 1 (see Lemmas 2, 3, 5 and 9). If $m > m_0$, where $\mathcal{N} \ni m_0 \geq \max(2p, \max_{1\leq i\leq 3} m_i)$, then the functions $f_{im}, h_{im}, M_{im}, N_{im}$ $(i=1,2,\dots,p)$ and c_m satisfy the assumptions of papers [2] and [3] (see [3], Assumptions I-III).

3. Let us introduce the following notation*)

$$(39) \qquad \omega_r(x) = \Big[\prod_{\substack{\beta=1\\ \beta\neq r}}^{p} (f_\beta(x) - f_r(x))\Big]^{-1},$$

$$(40) \qquad e_r^1(x) = \prod_{\substack{\beta=1\\ \beta\neq r}}^{p} f_\beta(x),$$

$$(41) \qquad e_r^\alpha(x) = \sum_{\substack{1\leq\beta_1<\beta_2<\dots<\beta_{p-\alpha}\leq p\\ (\beta_k\neq r \text{ for } k=1,2,\dots,p-\alpha)}} f_{\beta_1}(x)\dots f_{\beta_{p-\alpha}}(x)$$

for $\alpha=2,3,\dots,p-1$ (when $p\geq 3$);

$$(42) \qquad e_r^p(x) = 1$$

$(x \in (0,A\rangle;\ r=1,2,\dots,p)$;

---

*) The functions $\tilde{\omega}_r$ and $\tilde{e}_r$ are given by formulas analogous to (39)-(42), with the replacement of f_r by h_r and x by y, respectively.

(43) $$G^{\beta}_{sr}(x) = (-1)^{\beta}\omega_r(x)e^{\beta}_r(x)x^{s-1};$$

$$\tilde{G}^{\beta}_{sr}(y) = (-1)\beta\tilde{\omega}_r(y)\tilde{e}^{\beta}_r(y)y^{s-1}$$

$(x \in (0,A>;\ y \in (0,B>;\ s,r,\beta=1,2,\dots,p)$;

(44) $$\mathcal{A}^{\alpha}_{\vec{\nu}_{(n)}\vec{k}_{(n)}}(x) = G^{\alpha}_{\nu_1 k_1}(x)\left(\prod_{j=2}^{\left[\frac{n+1}{2}\right]} G^{\nu_{2j-2}}_{\nu_{2j-1}k_{2j-1}} \circ z_{\vec{k}_{(2j-2)}}(x)\right)\cdot$$

$$\cdot\left(\prod_{j=2}^{\left[\frac{n+2}{2}\right]} \tilde{G}^{\nu_{2j-3}}_{\nu_{2j-2}k_{2j-2}} \circ \overset{*}{z}_{\vec{k}_{(2j-3)}}(x)\right)$$

$(x \in (0,A>;\ \vec{\nu}_{(n)}=(\nu_1,\dots,\nu_n);\ \vec{k}_{(n)}=(k_1,\dots,k_n)$ with $1 \leqslant \nu_i$, $k_i \leqslant p;\ i=1,2,\dots,n)$

(45) $$\tilde{\mathcal{A}}^{\alpha}_{\vec{\nu}_{(n)}\vec{k}_{(n)}}(y) = \tilde{G}^{\alpha}_{\nu_1 k_1}(y)\left(\prod_{j=2}^{\left[\frac{n+1}{2}\right]} \tilde{G}^{\nu_{2j-2}}_{\nu_{2j-1}k_{2j-1}} \circ \tilde{z}_{\vec{k}_{(2j-2)}}(y)\right)\cdot$$

$$\cdot\left(\prod_{j=2}^{\left[\frac{n+2}{2}\right]} G^{\nu_{2j-3}}_{\nu_{2j-2}k_{2j-2}} \circ \overset{**}{z}_{\vec{k}_{(2j-3)}}(y)\right)$$

$(y \in (0,B>)$, where $[a]$ denotes the greatest integer not exceeding a;

(46) $$V^{\alpha}(x) = (-1)^{\alpha-1}\sum_{r=1}^{p} \omega_r(x)e^{\alpha}_r(x)(M_r(x) - R[x,f_r(x)])$$

and

(47) $$\tilde{V}^{\alpha}(y) = (-1)^{-1}\sum_{r=1}^{p} \tilde{\omega}_r(y)\tilde{e}^{\alpha}_r(y)(N_r(y) - R[h_r(y),y]).$$

We are going to prove the following theorem.

Theorem 1. If Assumptions I-III are satisfied then there is a (g.s.) of the (G)-problem given by the formula

$$(48)\qquad u(x,y) = R(x,y) + \sum_{\alpha=1}^{p}\left[y^{\alpha-1}\varphi_{\alpha}(x) + x^{\alpha-1}\psi_{\alpha}(y)\right]$$

($(x,y)\in\Omega$), where φ_α and ψ_α ($\alpha=1,2,\dots,p$) are defined by

$$(49)\qquad \varphi_\alpha(x) = V^{\alpha}(x) + \sum_{n=1}^{\infty}\ \sum_{\nu_1,\dots,\nu_n=1}^{p}\ \sum_{k_1,\dots,k_n=1}^{p} \cdot$$

$$\cdot\, \mathcal{A}^{\alpha}_{\vec{\nu}_{(n)}\vec{k}_{(n)}}(x)F^{\nu_n,n}\circ \mathfrak{z}_{\vec{k}_{(n)}}(x)$$

($x\in(0,A\rangle$); $\varphi_\alpha(0) = 0$ and

$$(50)\qquad \psi_\alpha(y) = \tilde{V}^{\alpha}(y) + \sum_{n=1}^{\infty}\ \sum_{\nu_1,\dots,\nu_n=1}^{p}\ \sum_{k_1,\dots,k_n=1}^{p} \cdot$$

$$\cdot\, \tilde{\mathcal{A}}^{\alpha}_{\vec{\nu}_{(n)}\vec{k}_{(n)}}(y)\tilde{F}^{\nu_n,n}\circ \mathfrak{z}_{\vec{k}_{(n)}}(y)$$

($y\in(0,B\rangle$); $\psi_\alpha(0) = 0$, respectively, with

$$(51)\qquad \tilde{F}^{\nu_n,n} = \begin{cases} V^{\nu n} & \text{when } n \text{ is even} \\ \tilde{V}^{\nu n} & \text{when } n \text{ is odd;} \end{cases}$$

$$(52)\qquad \tilde{F}^{\nu_n,n} = \begin{cases} \tilde{V}^{\nu n} & \text{when } n \text{ is even} \\ V^{\nu n} & \text{when } n \text{ is odd;} \end{cases}$$

$$(53)\qquad \mathfrak{z}_{\vec{k}_{(n)}} = \begin{cases} z_{\vec{k}_{(n)}} & \text{when } n \text{ is even} \\ \overset{*}{z}_{\vec{k}_{(n)}} & \text{when } n \text{ is odd;} \end{cases}$$

$$(54)\qquad \tilde{\mathfrak{z}}_{\vec{k}(n)} = \begin{cases} \tilde{z}_{\vec{k}(n)} & \text{when } n \text{ is even} \\ \overset{**}{z}_{\vec{k}(n)} & \text{when } n \text{ is odd.} \end{cases}$$

If, moreover, there is a positive integer m_0' such that for all $\mathcal{N} \ni m > m_0'$ the following conditions concerning the (G_m)-problems are satisfied:

1^0. The functions c_m are equibounded;

2^0. The functions M_{im} and N_{im} $(i=1,2,\ldots,p)$ are Hölder-continuous (uniformly with respect to m) and satisfy inequalities of the form (9) with the same coefficient K;

3^0. The functions f_{im} and h_{im} $(i=1,2,\ldots,p)$ are Hölder-continuous (uniformly with respect to m), and have the properties expressed by Lemmas 2-7;

4^0. The functions $\varphi_{\alpha m}$ and $\psi_{\alpha m}$ $(\alpha=1,2,\ldots,p)$ appearing in the formula

$$(55)\quad u_m(x,y) = R_m(x,y) + \sum_{\alpha=1}^{p} \left[y^{\alpha-1}\varphi_{\alpha m}(x) + x^{\alpha-1}\psi_{\alpha m}(y)\right]$$

for the (c.s.) of the (G_m)-problems (see [3]) satisfy the inequalities

$$(56)\qquad \begin{cases} \|\varphi_{\alpha m}(x)\| \leqslant \text{const } x^{2p-\alpha+\tilde{\varkappa}_0} \\ \|\psi_{\alpha m}(y)\| \leqslant \text{const } y^{2p-\alpha+\tilde{\varkappa}_0} \end{cases}$$

$(x \in (0,A\rangle;\ y \in (0,B\rangle;\ \tilde{\varkappa}_0 \in (0,1\rangle)$, then the (g.s.) is unique.

P r o o f . Let us consider a sequence $\{u_m\}$ of functions given by (55), where $R_m(x,y)$ is defined by (38) with c_m given by (37), $\varphi_{\alpha m}(x)$ and $\psi_{\alpha m}(y)$ are the sums of the series (49) and (50) [in which f_i, h_i, M_i, N_i $(i=1,2,\ldots,p)$ and c are replaced by f_{im}, h_{im}, M_{im}, N_{im} and c_m, respectively (see (13), (14), (28), (29) and (37))[*)]] for $x \in (0,A>$; $y \in (0,B>$, while $\varphi_{\alpha m}(0) = \psi_{\alpha m}(0) := 0$. Let $\{(G_m)\}$ be the sequence of the Goursat problems (see p.2) corresponding to the aforesaid functions f_{im}, h_{im}, M_{im}, N_{im} and c_m. Based on Corollary 1 and Lemmas 3 and 6 above, and on the Theorem in [3], we can assert that for each $\mathcal{N} \ni m > m_0$ the function u_m is a (c.s.) of the (G_m)-problem. Moreover (see Section 2), $f_{im} \rightrightarrows f_i$; $h_{im} \rightrightarrows h_i$; $M_{im} \rightrightarrows M_i$; $N_{im} \rightrightarrows N_i$ $(i=1,2,\dots,p)$ and $c_m \to c$ in α when $m \to \infty$. Thus, according to Definition 1, in order to prove the existence of a (g.s.) of the (G)-problem in the form (48) it is enough to show that $u_m \rightrightarrows u$ in Ω when $m \to \infty$.

To this purpose let us observe that (see (48) and (55))

$$(57)\quad \|u_m(x,y) - u(x,y)\| \leqslant \|R_m(x,y) - R(x,y)\| + \sum_{\alpha=1}^{p} \left[y^{\alpha-1}\|\varphi_{\alpha m}(x) - \varphi_\alpha(x)\| + x^{\alpha-1}\|\psi_{\alpha m}(y) - \psi_\alpha(y)\|\right]$$

$((x,y) \in (0,A>\times(0,B>)$, where (see (49))

$$(58)\quad \|\varphi_{\alpha m}(x)-\varphi_\alpha(x)\| \leqslant \sum_{n=1}^{\infty} \left[\tilde{a}_n^{\alpha m}(x)+\hat{a}_n^{\alpha m}(x)\right] + \|V^{\alpha m}(x)-V^{\alpha}(x)\|$$

$(x \in (0,A>)$ with

$$(59)\quad \tilde{a}_n^{\alpha m}(x) = \sum_{\nu_1,\dots,\nu_n=1}^{p} \sum_{k_1,\dots,k_n=1}^{p} \left|\mathcal{A}^{\alpha m}_{\vec{\nu}_{(n)}\vec{k}_{(n)}}(x)-\mathcal{A}^{\alpha}_{\vec{\nu}_{(n)}\vec{k}_{(n)}}(x)\right| \cdot \left\|F^{\nu_n,n,m} \circ \mathfrak{z}_{\vec{k}_{(n)}}(x)\right\|$$

---

*) The relevant functions ω_r, e_r^α, $\mathcal{A}^\alpha_{\vec{\nu}_{(n)}\vec{k}_{(n)}}$, ..., V^α and $\tilde{V}^\alpha$ will now be denoted by ω_r^m, $e_r^{\alpha m}$, $\mathcal{A}^{\alpha m}_{\vec{\nu}_{(n)}\vec{k}_{(n)}}$, ..., $V^{\alpha m}$ and $\tilde{V}^{\alpha m}$ respectively.

and

$$(60)\quad \hat{a}_n^{\alpha m}(x) = \sum_{\nu_1,\dots,\nu_n=1}^{p} \sum_{k_1,\dots,k_n=1}^{p} \cdot$$

$$\cdot \left| \mathcal{A}^{\alpha m}_{\vec{\nu}_{(n)}\vec{k}_{(n)}}(x) \right| \left\| F^{\nu_n,n,m} \circ \mathfrak{z}^m_{\vec{k}_{(n)}}(x) - F^{\nu_n,n} \circ \mathfrak{z}_{\vec{k}_{(n)}}(x) \right\|,$$

the expression $\|\psi_{\alpha m}(y) - \psi_\alpha(y)\|$, $(y \in (0,B>)$ being estimated in a way analogous to (58) (see formula (50))*). In further reasoning we will assume that $m > m_0 = \max(2p, m_1, m_2, m_3)$ (see Lemmas 2, 3 and 5).

In the examination of $\tilde{a}_n^{\alpha m}(x)$ (see (59)) we distinguish the following two cases (see [3], p.264): 1^0. $x \in (0,\delta)$, where $0 < \delta \leqslant \min(\delta_1, \delta_2)$ is a sufficiently small positive number and 2^0. $x \in <\delta, A>$; $n > N_0$, N_0 being a positive integer so large that the relations $\mathfrak{z}^m_{\vec{k}_{(n)}}(x) \in (0,\delta)$ and $\mathfrak{z}_{\vec{k}_{(n)}}(x) \in (0,\delta)$ hold good for $n \geqslant N_0$ and $x \in <\delta, A>$ (see Lemma 6 above and Lemma 4 in [3]).

Let η be a fixed positive number and $\theta \in (0, \min(\beta_*, \beta_{**}))$. We first consider the case 1^0. Based on relations (39) and analogous relations for ω_r^m (see the footnote on p.22), and using Lemmas 1 and 3, we have**)

$$(61)\quad \left|\omega_r^m(x) - \omega_r(x)\right| \leqslant 2(p-1)\left[\max_{1 \leqslant \beta \leqslant p} \sup_{<0,A>} \left|f_{\beta m}(x) - f_\beta(x)\right|^\theta \cdot\right.$$

$$\cdot \left[p(1+\varepsilon)\right]^{p-1+\theta}\left[(1-\varepsilon_0)\overset{*}{f}_p x\right]^{1-p-\theta} \leqslant 2(p-1)\eta^\theta \left[p(1+\varepsilon)\right]^{p-1+\theta} \cdot$$

$$\cdot \left[(1-\varepsilon_0)\overset{*}{f}_p \cdot x\right]^{1-p-\theta}$$

$(r = 1, 2, \dots, p)$.

---

*) In the sequel we examine only the expression appearing on the right hand side of (58). The argument for that estimating $\|\psi_{\alpha m}(y) - \psi_\alpha(y)\|$ is analogous.

**) Here and in the sequel we assume that $m > m_\eta$, where m_η is a sufficiently large positive integer.

In a similar way we get (see (40)-(42))

$$(62)\quad \left|e_r^{\alpha m}(x)-e_r^{\alpha}(x)\right| \leqslant 2(p-1)C_*\eta^{\theta}\left[(1+\varepsilon_0)\overset{*}{f}_p x\right]^{p-\alpha-\theta}\cdot(1-\delta_\alpha^p)$$

$(r,\alpha=1,2,\ldots,p)$, where $C_* = p^{p-2}$ and δ_α^p is the Kronecker delta.

Finally (see Lemma 7), the inequality

$$(63)\quad \left|\left(\mathfrak{z}_{\vec{k}_{(r)}}^{m}(x)\right)^{\nu-1} - \left(\mathfrak{z}_{\vec{k}_{(r)}}(x)\right)^{\nu-1}\right| \leqslant 2(\nu-1)\cdot r\cdot\eta^{\theta}\cdot$$

$$\cdot\left[\max\left(\mathfrak{z}_{\vec{k}_{(r)}}^{m}(x),\ \mathfrak{z}_{\vec{k}_{(r)}}(x)\right)\right]^{\nu-1-\theta}$$

$(r=1,2,\ldots;\ \nu=1,2,\ldots,p)$ is valid.

Using (43), (61)-(63), (53), Lemmas 1 and 3, and the inequalities

$$(64)\quad (1-\varepsilon_0)^{4s}\, z_{\vec{k}_{(2s)}}(x) \leqslant z_{\vec{k}_{(2s)}}^{m}(x) \leqslant (1-\varepsilon_0)^{-4s}\, z_{\vec{k}_{(2s)}}(x)$$

$(x\in(0,\delta);\ s=1,2,\ldots)$ that immediately result from (25), (25′), (11) and (18), we obtain (see relations (32) in [3])

$$(65)\quad \left|G_{\nu_{2s+1}k_{2s+1}}^{\nu_{2s}m}\circ z_{\vec{k}_{(2s)}}^{m}(x) - G_{\nu_{2s+1}k_{2s+1}}^{\nu_{2s}}\circ z_{\vec{k}_{2s}}(x)\right| \leqslant$$

$$\leqslant \mathrm{const}\cdot s\cdot\eta^{\theta}\cdot(1-\varepsilon_0)^{-4ps}\, C_*\left[p(1+\varepsilon)\right]^{p-1+\theta}\left[(1-\varepsilon_0)\overset{*}{f}_p z_{\vec{k}_{(2s)}}(x)\right]^{1-\nu_{2s}-\theta}\cdot$$

$$\cdot\left[z_{\vec{k}_{(2s)}}(x)\right]^{\nu_{2s+1}-1}$$

$(x\in(0,\delta);\ s=1,2,\ldots)$, where $1\leqslant \nu_{2s},\nu_{2s+1},k_{2s+1}\leqslant p$ and const is a positive constant not depending on s.

By a similar argument we get

$$(66)\quad \left|\tilde{G}_{\nu_{2s}k_{2s}}^{\nu_{2s-1}m}\circ \overset{*}{z}_{\vec{k}_{(2s-1)}}^{m}(x) - \tilde{G}_{\nu_{2s}k_{2s}}^{\nu_{2s-1}}\circ \overset{*}{z}_{\vec{k}_{(2s-1)}}(x)\right| \leqslant$$

$$\leqslant \mathrm{const}\cdot s\cdot\eta^{\theta}\cdot(1-\varepsilon_0)^{-2p(2s-1)}\cdot C_*\left[p(1+\varepsilon)\right]^{p-1+\theta}.$$

$$\cdot \left[(1-\varepsilon_0)\overset{*}{h}_p \overset{*}{z}_{\vec{k}_{(2s-1)}}(x)\right]^{1-\vartheta_{2s-1}-\theta} \left[\overset{*}{z}_{\vec{k}_{(2s-1)}}(x)\right]^{\vartheta_{2s-1}}$$

$(x \in (0,\delta);\ s=1,2,\ldots)$.

In the sequel we will need the following inequality

$$(67)\qquad \left|\prod_{i=\gamma}^{\delta} a_i - \prod_{i=\gamma}^{\delta} \bar{a}_i\right| \leq \sum_{i=\gamma}^{\delta} |a_i-\bar{a}_i| \prod_{j=\gamma}^{i-1}|\bar{a}_j| \prod_{j=i+1}^{\delta}|a_j|$$

$(\gamma, \delta \in \mathcal{N},\ \gamma < \delta)$, whose proof is straightforward.

Basing on (59), (64)-(67) and inequalities (32) and (34) in [3], and repeating the argument used in the proof of relation (35) in [3], we obtain the following estimates

$$(68)\qquad \left\|\tilde{a}_n^{\alpha m}(x)\right\| \leq \text{const } n^2\eta^\theta\cdot(1-\varepsilon_0)^{-4pn}\cdot\left[p^p(1+\varepsilon)^{p-1}\sqrt{g_0}\right]^n \cdot$$
$$\cdot \left[p^p\, g_0^{\varkappa_0/2}(1-\varepsilon_0)^{3-4p-\varkappa_0}\right]^n x^{2p-\alpha+1-\theta} \leq$$
$$\leq \text{const } n^2\eta^\theta\left[p^p(1+\varepsilon)^{p-1}\sqrt{g_0}\right]^n \cdot$$
$$\cdot \left[p^p g_0^{\varkappa_0/2}(1-\varepsilon_0)^{3-8p-\varkappa_0}\right]^n \cdot x^{2p-\alpha+1-\theta}$$

where const is a positive constant independent of n.

It follows from the choice of the parameters ε and ε_0 (see pp.3, 4 and 6) that

$$p^p(1+\varepsilon)^{p-1}\sqrt{g_0} < 1 \quad \text{and} \quad p^p g_0^{\varkappa_0/2}(1-\varepsilon_0)^{3-8p-\varkappa_0} < 1$$

whence and by (68) we have

$$(69)\qquad \left\|\tilde{a}_n^{\alpha m}(x)\right\| \leq \text{const } n^2 q^n\cdot\eta^\theta\cdot x^{2p-\alpha+1-\theta}$$

where $0 < q < 1$; $x \in (0,\delta)$.

In the case 2^0 (see p.23) we use an argument similar to that in the proof of relation (37) in [3] and we obtain an inequality of the same type as (69). Thus, (69) holds good for $x \in (0,A\rangle$.

In a similar way, based on the inequality

$$\left\| F^{\nu_n,n,m} \circ \mathfrak{z}_{\vec{k}(n)}^{m}(x) - F^{\nu_n,n} \circ \mathfrak{z}_{\vec{k}(n)}(x) \right\| \leqslant$$

$$\leqslant \operatorname{const} n \eta^{\theta} (1-\varepsilon_0)^{-2n(2p-\nu_n+1)} \cdot \left(\mathfrak{z}_{\vec{k}(n)}(x)\right)^{2p-\nu_n+1-\theta}$$

we get

(70) $$\left\| \hat{a}_n^{\alpha m}(x) \right\| \leqslant \operatorname{const}\ n\ q^n \eta^{\theta}\ x^{2p-\alpha+1-\theta}$$

$(x \in (0,A>)$.

From (58), (69), (70), the inequality

(71) $$\left\| V^{\alpha m}(x) - V^{\alpha}(x) \right\| \leqslant \operatorname{const}\ \eta^{\theta}\ x^{2p-\alpha+1-\theta}$$

$(x \in (0,A>)$ and the relations $\varphi_\alpha(0) = \varphi_{\alpha m}(0) = 0$, it follows that

(72) $$\left\| \varphi_{\alpha m}(x) - \varphi_\alpha(x) \right\| \leqslant \operatorname{const}\ \eta^{\theta}\ x^{2p-\alpha+1-\theta} \leqslant \operatorname{const}\ \eta^{\theta}$$

for $x \in \langle 0,A \rangle$; $\alpha = 1,2,\dots,p$; $m > m_\eta$.

By a similar argument we show that

(73) $$\left\| \psi_{\alpha m}(y) - \psi_\alpha(y) \right\| \leqslant \operatorname{const}\ \eta^{\theta}\ y^{2p-\alpha+1-\theta} \leqslant \operatorname{const}\ \eta^{\theta}$$

for $y \in \langle 0,B \rangle$; $\alpha = 1,2,\dots,p$; $m > m_\eta$.

Using (57), (38′), (72) and (73) we easily conclude that for each positive number η_0 there is a positive integer m_{η_0} such that

(74) $$\sup_{\Omega} \left\| u_m(x,y) - u(x,y) \right\| < \eta_0$$

when $m > m_{\eta_0}$. This completes the proof of the existence of a (g.s.) of the (G)-problem (see Definition 1).

In order to prove the uniqueness of the (g.s.) (under the additional assumptions 1^0–4^0 formulated on p.21) it suffices to show that if $\left(\{f_{im}^s\}, \{h_{im}^s\}, \{M_{im}^s\}, \{N_{im}^s\}, \{c_m\}\right)$ and $\{u_m^s\}$ $(s=1,2)$ satisfy the conditions of Definition 1 and the said assumptions 1^o-4^o, then the corresponding (g.s.) u^s (s=1,2) of the (G)-problem coincide in Ω. Let us observe that

$$(75)\quad \|u^2(x,y) - u^1(x,y)\| \leq \|u^2(x,y) - u^2_m(x,y)\| + \\ + \|u^1(x,y) - u^1_m(x,y)\| + \|u^2_m(x,y) - u^1_m(x,y)\|$$

($(x,y)\in\Omega$), where for each $\eta_* > 0$ the inequality

$$(76)\quad \|u^2(x,y) - u^2_m(x,y)\| + \|u^1(x,y) - u^1_m(x,y)\| < \frac{\eta_*}{2}$$

($(x,y)\in\Omega$) holds when $m > m'_{\eta_*}$, m'_{η_*} being a sufficiently large positive integer. Furthermore, in virtue of the present assumptions and the Theorem in [3], we can assert that the functions u^s_m (s=1,2) are of the form (55) where R_m, $\varphi_{\alpha m}$ and $\psi_{\alpha m}$ ($\alpha=1,2,\ldots,p$) are given by formulas (38), (49) and (50) with f_i, h_i, V^α, $\mathcal{A}^\alpha_{\vec{\jmath}_{(n)}\vec{k}_{(n)}},\ldots,\tilde{\jmath}_{\vec{k}_{(n)}}$ replaced by f^s_{im}, h^s_{im}, $V^{\alpha s}$, $\mathcal{A}^{\alpha m s}_{\vec{\jmath}_{(m)}\vec{k}_{(n)}},\ldots,\tilde{\jmath}^{ms}_{\vec{k}_{(n)}}$ (s=1,2), respectively. Using an argument similar to that applied in the proof of (74), we obtain the following inequality

$$\|u^2_m(x,y) - u^1_m(x,y)\| \leq \text{const}\ \Big\{\max <\max_{1\leq i\leq p}\Big[\sup_{<0,A>}\|M^2_{im}(x) + \\ - M^1_{im}(x)\|,\ \sup_{<0,B>}\|N^2_{im}(y) - N^1_{im}(y)\|,\ \sup_{<0,A>}|f^2_{im}(x) - f^1_{im}(x)|, \\ \sup_{<0,B>}|h^2_{im}(y) - h^1_{im}(y)|\Big],\ \sup_{\Omega}\|R^2_m(x,y) - R^1_m(x,y)\|>\Big\}^\theta$$

($(x,y)\in\Omega$; $\theta\in(0,1)$) and since the sequences $\{f^2_{im} - f^1_{im}\}$, $\{h^2_{im} - h^1_{im}\}$, $\{M^2_{im} - M^1_{im}\}$, $\{N^2_{im} - N^1_{im}\}$ (i=1,2,...,p) and $\{R^2_m - R^1_m\}$ tend uniformly to zero when $m\to\infty$, we can assert that

$$(77)\quad \|u^2_m(x,y) - u^1_m(x,y)\| < \frac{\eta_*}{2}$$

$((x,y) \in \Omega)$ for $m > m''_{\eta_*}$, where m''_{η_*} is a sufficiently large positive integer.

From (75)-(77) it follows that

$$\sup_{\Omega} \| u^2(x,y) - u^1(x,y) \| < \eta_*$$

when $m > \max(m'_{\eta_*}, m''_{\eta_*})$, which completes the proof of Theorem 1.

R e m a r k 2. It follows from the results obtained in paper [3] (see [3], Theorem) that if Assumptions I-III of the present paper are replaced by Assumptions I-III of paper [3], then the (g.s.) of the (G)-problem given by formulas (49)-(54) above is a (c.s.) of this problem.

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